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STRONGLY HYPERBOLIC TYPE CONVEXITY AND SOME NEW INEQUALITIES

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Abstract. *In this study, we introduce and study the concept of strongly hyperbolic type convexity functions and their some algebraic properties. We obtain Hermite-Hadamard type inequalities for the strongly hyperbolic type convex functions. After that, by using an identity, we get some inequalities for strongly hyperbolic type convex functions. In addition, we compare the results obtained with both Hölder, Hölder-İşcan inequalities and power-mean, improved power-mean integral inequalities.*

Keywords: *Convex function; hyperbolic type convexity; strongly hyperbolic type convexity; Hölder integral inequality; Hölder-İşcan integral inequality.*

1. INTRODUCTION

Convexity theory provides powerful principles and techniques to study a wide class of problems in both pure and applied mathematics. In recent years, the theory of inequalities and convex functions, and therefore the definition of convex function, has been the subject of continuous innovative research. Mathematical inequalities and convexity theory play a key role in understanding a range of problems in various fields of mathematics and the other branches of sciences such as economics and engineering. Many articles have been written by a number of mathematicians on convex functions and inequalities for their different classes, using, for example, the last articles [1-6] and the references in these papers.

A function $f: I \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in 0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$.

Let $f: I \to \mathbb{R}$ be a convex function. Then the following inequalities hold

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

for all $a, b \in I$ with $a < b$. Both inequalities hold in the reversed direction if the function f is concave. This double inequality is well known as the Hermite-Hadamard inequality [7]. Until

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now, many inequalities have been established for convex functions and their types, but it must be admitted that the most famous inequality is the Hermite-Hadamard integral inequality because of its rich geometric significance and applications. Note that some of the classical inequalities for means can be derived from Hermite-Hadamard integral inequalities for appropriate particular selections of the mapping f.

Definition 1. [8] Let $h: J \to \mathbb{R}$ be a non-negative function, $h \not\equiv 0$. We say that $f: I \to \mathbb{R}$ is an h-convex function, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$, $\alpha \in (0,1)$ we have

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + h(1-\alpha)f(y).$$

If this inequality is reversed, then f is said to be h-concave, i.e. $f \in SV(h, I)$. It is clear that, if we choose $h(\alpha) = \alpha$ and $h(\alpha) = 1$, then the h-convexity reduces to convexity and definition of P-function, respectively. Readers can look at [8,9] for studies on h-convexity.

Definition 2. [10] Let $I \subset \mathbb{R}$ be an interval and c be a positive number. A function $f: I \subset \mathbb{R} \to \mathbb{R}$ is called strongly convex with modulus c if

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) - ct(1-t)(b-a)^2$$

for all $a, b \in I$ and $t \in [0,1]$.

The definition of the strongly convex functions have been introduced by Polyak [10], and the strongly convex functions play an important role in optimization theory, variational inequalities, mathematical economics, nonlinear programming, approximation theory and other branches of pure and applied mathematics. Since strongly convexity is a strengthening of the notion of convexity theory, some properties of strongly convex functions are just stronger versions of known properties of convex functions. Many properties and applications about strongly convex functions can be found in the literature. For more information on strongly convex functions, see [11-14] and references therein.

Definition 3. [15] The function $f: I \subset \mathbb{R} \to \mathbb{R}$ is called hyperbolic type convex function if for every $a, b \in I$ and $t \in [0,1]$,

$$f(ta + (1-t)b) \leq \left(\frac{\sinh t}{\sinh 1}\right) f(a) + \left(\frac{\sinh 1 - \sinh t}{\sinh 1}\right) f(b).$$

In [15], the authors obtained the following Hermite-Hadamard type inequalities for the hyperbolic type convex functions:

Theorem 1. [15] Let $f: [a, b] \to \mathbb{R}$ be a hyperbolic type convex function. If $a < b$ and $f \in L[a, b]$, then the following Hermite-Hadamard type inequalities hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \left(\frac{\cosh 1 - 1}{\sinh 1}\right) f(a) + \left(\frac{e-1}{e\sinh 1}\right) f(b). \tag{2}$$

Theorem 2. (Hölder-İşcan integral inequality [16]) Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on interval $[a, b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a, b]$, then

$$\int_a^b |f(x)g(x)|dx \le \frac{1}{b-a}\left\{\left(\int_a^b (b-x)|f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)|g(x)|^q dx\right)^{\frac{1}{q}} + \left(\int_a^b (x-a)|f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)|g(x)|^q dx\right)^{\frac{1}{q}}\right\}. \tag{3}$$

Theorem 3. (Improved power-mean integral inequality [17]) Let $q \ge 1$. If f and g are real functions defined on interval $[a,b]$ and if $|f|$, $|f||g|^q$ are integrable functions on $[a,b]$, then

$$\int_a^b |f(x)g(x)|dx \le \frac{1}{b-a}\left\{\left(\int_a^b (b-x)|f(x)|dx\right)^{1-\frac{1}{q}}\left(\int_a^b (b-x)|f(x)||g(x)|^q dx\right)^{\frac{1}{q}} + \left(\int_a^b (x-a)|f(x)|dx\right)^{1-\frac{1}{q}}\left(\int_a^b (x-a)|f(x)||g(x)|^q dx\right)^{\frac{1}{q}}\right\}.$$

In recent years, in most of the studies on strongly hyperbolic type convex functions, after algebraic properties are given, Hermite-Hadamard type integral inequalities are obtained with the help of any identity. It will be seen that there is no comparison between the inequalities obtained in these articles. However, it should be noted that in this paper we obtained inequalities that give better approximation by using both Hölder-İşcan and improved power-mean integral inequalities. In the articles in the references of this study, inequalities that give a better approximation were not obtained. For more information on Hölder-İscan integral inequalities, see [18-21] and references therein.

This article is organized as follows. In Chapter 2, we introduce a new concept, which is called strongly hyperbolic type convexity, and we give by setting some algebraic properties of them. In Chapter 3, we obtain the Hermite-Hadamard inequality for strongly hyperbolic-type convex functions. In Chapter 4, by using an identity we establish new estimates that refine Hermite-Hadamard inequality for strongly hyperbolic type convex functions. Then, we compare the results obtained with both Hölder, Hölder-İşcan inequalities and power-mean, improved power-mean integral inequalities.

2. MAIN RESULTS FOR THE STRONGLY HYPERBOLIC TYPE CONVEXITY

In this section, we introduce a new concept, which is called strongly hyperbolic type convexity, and we give by setting some algebraic properties for the strongly hyperbolic type convex functions, as follows:

Definition 4. Let $I \subset \mathbb{R}$ be an interval and c be a positive number. The function $f: I \subset \mathbb{R} \to \mathbb{R}$ is called strongly hyperbolic type convex function with modulus c if for every $a, b \in I$ and $t \in [0,1]$,

$$f(ta+(1-t)b) \le \left(\frac{\sinh t}{\sinh 1}\right) f(a) + \left(\frac{\sinh 1 - \sinh t}{\sinh 1}\right) f(b) - ct(1-t)(b-a)^2. \tag{4}$$

We will denote the class of all strongly hyperbolic type convex functions on I by $SHC(I)$. In the following theorems some algebraic properties of strongly hyperbolic type convex functions are given.

Theorem 4. Let $\mathrm{f}, \mathrm{g}:[\mathrm{a}, \mathrm{b}] \rightarrow \mathbb{R}$. If f and g are strongly hyperbolic type convex functions, then $\mathrm{f}+\mathrm{g}$ is strongly hyperbolic type convex function and for $\mathrm{k} \in \mathbb{R}(\mathrm{k} \geq 1)$ kf is strongly hyperbolic type convex function.

Proof:
(i) Let f, g be strongly hyperbolic type convex functions and $x, y \in J$ be arbitrary, then

$$
\begin{aligned}
&(f+g)(tx+(1-t)y) \\
&= f(tx+(1-t)y)+g(tx+(1-t)y) \\
&\leq \left(\frac{\sinh t}{\sinh 1}\right) f(x)+\left(\frac{\sinh 1-\sinh t}{\sinh 1}\right) f(y)-ct(1-t)(y-x)^2 \\
&+\left(\frac{\sinh t}{\sinh 1}\right) g(x)+\left(\frac{\sinh 1-\sinh t}{\sinh 1}\right) g(y)-ct(1-t)(y-x)^2 \\
&\leq \left(\frac{\sinh t}{\sinh 1}\right) f(x)+\left(\frac{\sinh 1-\sinh t}{\sinh 1}\right) f(y) \\
&+\left(\frac{\sinh t}{\sinh 1}\right) g(x)+\left(\frac{\sinh 1-\sinh t}{\sinh 1}\right) g(y)-ct(1-t)(y-x)^2 \\
&=\left(\frac{\sinh t}{\sinh 1}\right)[f(x)+g(x)]+\left(\frac{\sinh 1-\sinh t}{\sinh 1}\right)[f(y)+g(y)]-ct(1-t)(y-x)^2 .
\end{aligned}
$$

(ii) Let f be strongly hyperbolic type convex function, $x, y \in J$ be arbitrary and $k \in \mathbb{R}(k \geq 0)$, *then*

$$
\begin{aligned}
(kf)(tx+(1-t)y) &\leq k\left[\left(\frac{\sinh t}{\sinh 1}\right) f(x)+\left(\frac{\sinh 1-\sinh t}{\sinh 1}\right) f(y)-ct(1-t)(y-x)^2\right] \\
&=\left(\frac{\sinh t}{\sinh 1}\right)[kf(x)]+\left(\frac{\sinh 1-\sinh t}{\sinh 1}\right)[kf(y)]-kct(1-t)(y-x)^2 \\
&\leq\left(\frac{\sinh t}{\sinh 1}\right)[kf(x)]+\left(\frac{\sinh 1-\sinh t}{\sinh 1}\right)[kf(y)]-ct(1-t)(y-x)^2 .
\end{aligned}
$$

Theorem 5. Let $\mathrm{b}>0$ and $\mathrm{f}_\alpha:[\mathrm{a}, \mathrm{b}] \rightarrow \mathbb{R}$ be an arbitrary family of the strongly hyperbolic type convex functions and let $\mathrm{f}(\mathrm{x})=\sup_\alpha \mathrm{f}_\alpha(\mathrm{x})$. If $\mathrm{J}=\{\mathrm{u} \in[\mathrm{a}, \mathrm{b}]: \mathrm{f}(\mathrm{u})<\infty\}$ is nonempty, then J is an interval and f is a strongly hyperbolic type convex function on interval J.

Proof: Let $t \in[0,1]$ and $x, y \in J$ be arbitrary. Then

$$
\begin{aligned}
f(tx+(1-t)y) &= \sup_\alpha f_\alpha(tx+(1-t)y) \\
&\leq \sup_\alpha\left[\left(\frac{\sinh t}{\sinh 1}\right) f_\alpha(x)+\left(\frac{\sinh 1-\sinh t}{\sinh 1}\right) f_\alpha(y)-ct(1-t)(y-x)^2\right] \\
&\leq\left(\frac{\sinh t}{\sinh 1}\right) \sup_\alpha f_\alpha(x)+\left(\frac{\sinh 1-\sinh t}{\sinh 1}\right) \sup_\alpha f_\alpha(y)-ct(1-t)(y-x)^2 \\
&=\left(\frac{\sinh t}{\sinh 1}\right) f(x)+\left(\frac{\sinh 1-\sinh t}{\sinh 1}\right) f(y)-ct(1-t)(y-x)^2 \\
&<\infty .
\end{aligned}
$$

This shows simultaneously that J is an interval, since it contains every point between any two of its points, and that f is a strongly hyperbolic type convex function on interval J. This completes the proof of theorem.

3. HERMITE-HADAMARD INEQUALITIES FOR THE STRONGLY HYPERBOLIC TYPE CONVEX FUNCTIONS

In this section, we obtain the Hermite-Hadamard integral inequalities for strongly hyperbolic type convex functions. Also, we will denote the space of (Lebesgue) integrable functions on interval $[a,b]$ by $L[a,b]$.

Theorem 6. Let $f:[a,b]\to\mathbb{R}$ be a strongly hyperbolic type convex function with modulus c. If $a<b$ and $f\in L[a,b]$, then the following Hermite-Hadamard type inequalities hold:

$$f\left(\frac{a+b}{2}\right)+\frac{c}{12}(b-a)^2\le\frac{1}{b-a}\int_a^b f(x)dx\le\left(\frac{\cosh1-1}{\sinh1}\right)f(a)+\left(\frac{e-1}{e\sinh1}\right)f(b)-\frac{c}{6}(b-a)^2. \quad (5)$$

Proof: From the propery of the strongly hyperbolic type convex function of the function f, we obtain

$$\begin{aligned}f\left(\frac{a+b}{2}\right)&=f\left(\frac{[ta+(1-t)b]+[(1-t)a+tb]}{2}\right)\\&=f\left(\frac{1}{2}[ta+(1-t)b]+\frac{1}{2}[(1-t)a+tb]\right)\\&\le\left(\frac{\sinh\frac{1}{2}}{\sinh1}\right)f(ta+(1-t)b)+\left(\frac{\sinh1-\sinh\frac{1}{2}}{\sinh1}\right)f((1-t)a+tb)-\frac{c}{4}(2t-1)^2(b-a)^2.\end{aligned}$$

By taking integral in the last inequality with respect to $t\in[0,1]$, we deduce that

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\\&\le\left(\frac{\sinh\frac{1}{2}}{\sinh1}\right)\int_0^1 f(ta+(1-t)b)dt+\left(\frac{\sinh1-\sinh\frac{1}{2}}{\sinh1}\right)\int_0^1 f((1-t)a+tb)dt\\&\quad-\frac{c}{4}(b-a)^2\int_0^1(2t-1)^2dt\\&=\left(\frac{\sinh\frac{1}{2}}{\sinh1}\right)\frac{1}{b-a}\int_a^b f(x)dx+\left(\frac{\sinh1-\sinh\frac{1}{2}}{\sinh1}\right)\frac{1}{b-a}\int_a^b f(x)dx-\frac{c}{12}(b-a)^2\\&=\frac{1}{b-a}\int_a^b f(x)dx-\frac{c}{12}(b-a)^2.\end{aligned}$$

By using the property of the strongly hyperbolic type convex function f, if the variable is changed as $x=ta+(1-t)b$, then

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)dx &= \int_0^1 f(ta+(1-t)b)dt\\ &\leq \int_0^1 \left[\left(\frac{\sinh t}{\sinh 1}\right)f(a)+\left(\frac{\sinh 1-\sinh t}{\sinh 1}\right)f(b)-ct(1-t)(b-a)^2\right]dt\\ &= \frac{f(a)}{\sinh 1}\int_0^1 \sinh t dt+\frac{f(b)}{\sinh 1}\int_0^1(\sinh 1-\sinh t)dt-c(b-a)^2\int_0^1 t(1-t)dt\\ &= \left(\frac{\cosh 1-1}{\sinh 1}\right)f(a)+\left(\frac{e-1}{e\sinh 1}\right)f(b)-\frac{c}{6}(b-a)^2\end{aligned}$$

where

$$\int_0^1 \sinh t dt=\cosh 1-1,\quad \int_0^1(\sinh 1-\sinh t)dt=\frac{e-1}{e},\quad \int_0^1 t(1-t)dt=\frac{1}{6}.$$

This completes the proof of theorem.

Corollary 1. If we take $c=0$ in the inequality (5), we get the following inequality:

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\left(\frac{\cosh 1-1}{\sinh 1}\right)f(a)+\left(\frac{e-1}{e\sinh 1}\right)f(b)$$

This inequality coincides with the inequality (2) in [15].

4. NEW INEQUALITIES FOR THE STRONGLY HYPERBOLIC TYPE CONVEX FUNCTIONS

In this section we establish new estimates that refine Hermite-Hadamard inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is strongly hyperbolic type convex function. Dragomir and Agarwal [22] used the following lemma:

Lemma 1. Let $f: I^\circ \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{b-a}{2}\int_0^1(1-2t)f'(ta+(1-t)b)dt$$

Theorem 7. Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function, $a, b \in I$, $a < b$ and $f' \in L[a,b]$. If $|f'|$ is the strongly hyperbolic type convex function with modulus c on interval $[a,b]$, then the following inequality holds for $t \in [0,1]$,

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ &\leq \frac{b-a}{2}\frac{|f'(a)|}{\sinh 1}\left(\frac{2(e-1)}{\sqrt{e}}-\frac{e^2-3}{2e}-1\right)+\frac{b-a}{2}\frac{|f'(b)|}{\sinh 1}\left(\frac{3e}{4}-\frac{7}{4e}+\frac{2}{\sqrt{e}}-2\sqrt{e}+1\right)\\ &\quad -c\frac{(b-a)^3}{32}.\end{aligned} \tag{6}$$

Proof: Because the function $|f'|$ is the strongly hyperbolic type convex function, we can write the below inequality,

$$|f'(ta+(1-t)b)| \leq \left(\frac{\sinh t}{\sinh 1}\right)|f(a)| + \left(\frac{\sinh 1-\sinh t}{\sinh 1}\right)|f(b)| - ct(1-t)(b-a)^2.$$

By using Lemma 1, we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \left|\frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt\right|$$
$$\leq \frac{b-a}{2}\int_0^1 |1-2t|\left[\left(\frac{\sinh t}{\sinh 1}\right)|f'(a)| + \left(\frac{\sinh 1-\sinh t}{\sinh 1}\right)|f'(b)| - ct(1-t)(b-a)^2\right]dt$$
$$\leq \frac{b-a}{2}\left(\begin{array}{l}\frac{|f'(a)|}{\sinh 1}\int_0^1 |1-2t|\sinh t dt + \frac{|f'(b)|}{\sinh 1}\int_0^1 |1-2t|(\sinh 1-\sinh t)dt \\ -c(b-a)^2\int_0^1 |1-2t|t(1-t)dt\end{array}\right)$$
$$= \frac{b-a}{2}\frac{|f'(a)|}{\sinh 1}\left(\frac{2(e-1)}{\sqrt{e}} - \frac{e^2-3}{2e} - 1\right) + \frac{b-a}{2}\frac{|f'(b)|}{\sinh 1}\left(\frac{3e}{4} - \frac{7}{4e} + \frac{2}{\sqrt{e}} - 2\sqrt{e} + 1\right) - c\frac{(b-a)^3}{32},$$

where

$$\int_0^1 |1-2t|\sinh t dt = \frac{2(e-1)}{\sqrt{e}} - \frac{e^2-3}{2e} - 1,$$
$$\int_0^1 |1-2t|(\sinh 1-\sinh t) = \frac{3e}{4} - \frac{7}{4e} + \frac{2}{\sqrt{e}} - 2\sqrt{e} + 1$$
$$\int_0^1 |1-2t|t(1-t) = \frac{1}{16}.$$

This completes the proof of theorem.

Corollary 2. If we take $c = 0$ in the inequality (6), we get the following inequality*:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{2}\frac{|f'(a)|}{\sinh 1}\left(\frac{2(e-1)}{\sqrt{e}} - \frac{e^2-3}{2e} - 1\right) + \frac{b-a}{2}\frac{|f'(b)|}{\sinh 1}\left(\frac{3e}{4} - \frac{7}{4e} + \frac{2}{\sqrt{e}} - 2\sqrt{e} + 1\right).$$

This inequality coincides with the inequality in [15]. In the following two theorems, we get the new integral inequalites by using the Hölder integral inequality.

Theorem 8. Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function, $a, b \in I$, $a < b$, $q > 1, \frac{1}{p} + \frac{1}{q} = 1$ and $f' \in L[a,b]$. If $|f'|^q$ is a strongly hyperbolic type convex function with modulus c on interval $[a,b]$, then the following inequality holds for $t \in [0,1]$,

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^q}{\sinh 1}\left(\frac{(e-1)^2}{2e}\right)+\frac{|f'(b)|^q}{\sinh 1}\left(\frac{e-1}{e}\right)-\frac{c}{6}(b-a)^2\right]^{\frac{1}{q}}. \tag{7}$$

Proof: Since $|f'|^q$ is a strongly hyperbolic type convex function, we can write the following inequality,

$$|f'(ta+(1-t)b)|^q \leq \left(\frac{\sinh t}{\sinh 1}\right)|f(a)|^q+\left(\frac{\sinh 1-\sinh t}{\sinh 1}\right)|f(b)|^q-ct(1-t)(b-a)^2.$$

If we use Lemma 1 and Hölder's integral inequality in the proper line,

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{2}\int_0^1 |1-2t||f'(ta+(1-t)b)|dt$$
$$\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\int_0^1\left[\left(\frac{\sinh t}{\sinh 1}\right)|f'(a)|^q+\left(\frac{\sinh 1-\sinh t}{\sinh 1}\right)|f'(b)|^q-ct(1-t)(b-a)^2\right]dt\right)^{\frac{1}{q}}$$
$$=\frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^q}{\sinh 1}\int_0^1 \sinh t dt+\frac{|f'(b)|^q}{\sinh 1}\int_0^1(\sinh 1-\sinh t)dt-c(b-a)^2\int_0^1 t(1-t)dt\right)^{\frac{1}{q}}$$
$$=\frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\frac{|f'(a)|^q}{\sinh 1}\left(\frac{(e-1)^2}{2e}\right)+\frac{|f'(b)|^q}{\sinh 1}\left(\frac{e-1}{e}\right)-\frac{c}{6}(b-a)^2\right]^{\frac{1}{q}},$$

where

$$\int_0^1 |1-2t|^p dt=\frac{1}{p+1},$$
$$\int_0^1 \sinh t=\cosh 1-1=\frac{(e-1)^2}{2e},$$
$$\int_0^1(\sinh 1-\sinh t)dt=\sinh 1-\cosh 1+1=\frac{e-1}{e}$$
$$\int_0^1 t(1-t)dt=\frac{1}{6}.$$

This completes the proof of theorem.

Theorem 9. Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function, $a, b \in I$, $a < b$, $q \geq 1$ and $f' \in L[a,b]$. If $|f'|^q$ is a hyperbolic type convex function with modulus c on the interval $[a,b]$, then the following inequality holds for $t \in [0,1]$,

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$
$$\times\left[\frac{|f'(a)|^q}{\sinh 1}\left(\frac{2(e-1)}{\sqrt{e}}-\frac{e^2-3}{2e}-1\right)+\frac{|f'(b)|^q}{\sinh 1}\left(\frac{3e}{4}-\frac{7}{4e}+\frac{2}{\sqrt{e}}-2\sqrt{e}+1\right)\right. \tag{8}$$
$$\left.-\frac{c}{16}(b-a)^2\right]^{\frac{1}{q}}.$$

Proof: Firstly, we assume that $q>1$. From Lemma 1, Hölder integral inequality and the property of the strongly hyperbolic type convex function of $|f'|^q$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{2}\int_0^1 |1-2t||f'(ta+(1-t)b)|dt \\
&\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t||f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}} \\
&= \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\left[\frac{\sinh t}{\sinh 1}|f'(a)|^q+\frac{\sinh 1-\sinh t}{\sinh 1}|f'(b)|^q-ct(1-t)(b-a)^2\right]dt\right)^{\frac{1}{q}} \\
&= \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\begin{array}{l}\frac{|f'(a)|^q}{\sinh 1}\int_0^1 |1-2t|\sinh t dt+\frac{|f'(b)|^q}{\sinh 1}\int_0^1 |1-2t|(\sinh 1-\sinh t)dt \\ -c(b-a)^2\int_0^1 t(1-t)|1-2t|dt\end{array}\right]^{\frac{1}{q}} \\
&= \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\begin{array}{l}\frac{|f'(a)|^q}{\sinh 1}\left(\frac{2(e-1)}{\sqrt{e}}-\frac{e^2-3}{2e}-1\right)+\frac{|f'(b)|^q}{\sinh 1}\left(\frac{3e}{4}-\frac{7}{4e}+\frac{2}{\sqrt{e}}-2\sqrt{e}+1\right) \\ -\frac{c}{16}(b-a)^2\end{array}\right]^{\frac{1}{q}}
\end{aligned} \tag{9}$$

where

$$\int_0^1 |1-2t|dt=\frac{1}{2}$$
$$\int_0^1 |1-2t|\sinh t dt=\frac{2(e-1)}{\sqrt{e}}-\frac{e^2-3}{2e}-1$$
$$\int_0^1 |1-2t|(\sinh 1-\sinh t)dt=\frac{3e}{4}-\frac{7}{4e}+\frac{2}{\sqrt{e}}-2\sqrt{e}+1.$$

Secondly, let $q=1$. We use the estimates from the proof of Theorem 7, which also follow step by step the above estimates. This completes the proof of theorem.

Corollary 3. Under the assumption of Theorem 9 with $\mathrm{q}=1$ and $\mathrm{c}=0$, we get the conclusion of Theorem 7 as follow:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{2}\left[\frac{|f'(a)|}{\sinh 1}\left(\frac{2(e-1)}{\sqrt{e}}-\frac{e^2-3}{2e}-1\right)+\frac{|f'(b)|}{\sinh 1}\left(\frac{3e}{4}-\frac{7}{4e}+\frac{2}{\sqrt{e}}-2\sqrt{e}+1\right)\right].$$

Corollary 4. Under the assumption of Theorem 9 with $\mathrm{c}=0$, we get the following inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$
$$\times\left[\frac{|f'(a)|^q}{\sinh 1}\left(\frac{2(e-1)}{\sqrt{e}}-\frac{e^2-3}{2e}-1\right)+\frac{|f'(b)|^q}{\sinh 1}\left(\frac{3e}{4}-\frac{7}{4e}+\frac{2}{\sqrt{e}}-2\sqrt{e}+1\right)\right]^{\frac{1}{q}}.$$

This inequality coincides with the inequality in [15].

Corollary 5. Under the assumption of Theorem 9 with $q = 1$, we get the following inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}$$
$$\times\left[\frac{|f'(a)|}{\sinh 1}\left(\frac{2(e-1)}{\sqrt{e}}-\frac{e^2-3}{2e}-1\right)+\frac{|f'(b)|}{\sinh 1}\left(\frac{3e}{4}-\frac{7}{4e}+\frac{2}{\sqrt{e}}-2\sqrt{e}+1\right)-\frac{c}{16}(b-a)^2\right].$$

In the following theorem, we get new integral inequality by using Hölder-İşcan integral inequality.

Theorem 10. Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function, $a, b \in I$, $a < b$, $q > 1, \frac{1}{p}+\frac{1}{q} = 1$ and $f' \in L[a,b]$. If $|f'|^q$ is a strongly hyperbolic convex function with modulus c on interval $[a,b]$, then the following inequality holds for $t \in [0,1]$,

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq \frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^q}{\sinh 1}\left(\frac{e^2-2e-1}{2e}\right)+\frac{|f'(b)|^q}{\sinh 1}\left(\frac{-e^2+4e+1}{4e}\right)\right. \\ &\qquad \left.-\frac{c}{12}(b-a)^2\right]^{\frac{1}{q}} \\ &+\frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^q}{\sinh 1}\left(\frac{1}{e}\right)+\frac{|f'(b)|^q}{\sinh 1}\left(\frac{e^2-5}{4e}\right)-\frac{c}{12}(b-a)^2\right]^{\frac{1}{q}}.\end{aligned} \tag{10}$$

Proof: Because $|f'|^q$ is a strongly hyperbolic convex function, we can write the following,

$$|f'(ta+(1-t)b)|^q \leq \left(\frac{\sinh t}{\sinh 1}\right)|f'(a)|^q+\left(\frac{\sinh 1-\sinh t}{\sinh 1}\right)|f'(b)|^q-ct(1-t)(b-a)^2.$$

If we use Lemma 1 and apply Hölder-İşcan integral inequality in proper line,

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{2}\int_0^1 |1-2t||f'(ta+(1-t)b)|dt$$

$$\leq \frac{b-a}{2}\left(\int_0^1 (1-t)|1-2t|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 (1-t)|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{2}\left(\int_0^1 t|1-2t|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 t|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{b-a}{2}\left(\int_0^1 (1-t)|1-2t|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 (1-t)\left[\begin{matrix}\frac{\sinh t}{\sinh 1}|f'(a)|^q + \frac{\sinh 1 - \sinh t}{\sinh 1}|f'(b)|^q \\ -ct(1-t)(b-a)^2\end{matrix}\right] dt\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{2}\left(\int_0^1 t|1-2t|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 t\left[\begin{matrix}\frac{\sinh t}{\sinh 1}|f'(a)|^q + \frac{\sinh 1 - \sinh t}{\sinh 1}|f'(b)|^q \\ -ct(1-t)(b-a)^2\end{matrix}\right] dt\right)^{\frac{1}{q}}$$
$$= \frac{b-a}{2}\left(\int_0^1 (1-t)|1-2t|^p dt\right)^{\frac{1}{p}} \left(\frac{|f'(a)|^q}{\sinh 1}\int_0^1 (1-t)\sinh t dt\right.$$
$$\left.+\frac{|f'(b)|^q}{\sinh 1}\int_0^1 (1-t)(\sinh 1-\sinh t)dt - c(b-a)^2\int_0^1 t(1-t)^2 dt\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{2}\left(\int_0^1 t|1-2t|^p dt\right)^{\frac{1}{p}} \left(\begin{matrix}\frac{|f'(a)|^q}{\sinh 1}\int_0^1 t\sinh t dt + \frac{|f'(b)|^q}{\sinh 1}\int_0^1 t(\sinh 1-\sinh t)dt \\ -c(b-a)^2\int_0^1 t^2(1-t)dt\end{matrix}\right)^{\frac{1}{q}}$$
$$= \frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^q}{\sinh 1}\left(\frac{e^2-2e-1}{2e}\right) + \frac{|f'(b)|^q}{\sinh 1}\left(\frac{-e^2+4e+1}{4e}\right) - \frac{c}{12}(b-a)^2\right]^{\frac{1}{q}}$$
$$+\frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^q}{\sinh 1}\frac{1}{e} + \frac{|f'(b)|^q}{\sinh 1}\frac{e^2-5}{4e} - \frac{c}{12}(b-a)^2\right]^{\frac{1}{q}}$$

where

$$\int_0^1 (1-t)|1-2t|^p dt = \int_0^1 t|1-2t|^p dt = \frac{1}{2(p+1)}$$

and

$$\int_0^1 (1-t)\sinh t dt = \frac{e^2-2e-1}{2e}$$
$$\int_0^1 (1-t)(\sinh 1-\sinh t)dt = \frac{-e^2+4e+1}{4e}$$
$$\int_0^1 t\sinh t dt = \frac{1}{e}$$
$$\int_0^1 t(\sinh 1-\sinh t)dt = \frac{e^2-5}{4e}$$
$$\int_0^1 t(1-t)^2 dt = \int_0^1 t^2(1-t)dt = \frac{1}{12}.$$

This completes the proof of theorem.

Corollary 6. Under the assumption of Theorem 10 with c = 0, we get the following inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^q}{\sinh 1}\left(\frac{e^2-2e-1}{2e}\right)+\frac{|f'(b)|^q}{\sinh 1}\left(\frac{-e^2+4e+1}{4e}\right)\right]^{\frac{1}{q}}$$
$$+\frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^q}{\sinh 1}\left(\frac{1}{e}\right)+\frac{|f'(b)|^q}{\sinh 1}\left(\frac{e^2-5}{4e}\right)\right]^{\frac{1}{q}}.$$

This inequality coincides with the inequality in [15].

Corollary 7. Under the assumption of Theorem 10 with q = 1 and c = 0, we get the following inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq \frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|}{\sinh 1}\left(\frac{(e-1)^2}{2e}\right)+\frac{|f'(b)|}{\sinh 1}\left(\frac{e-1}{e}\right)\right].$$

Corollary 8. Under the assumption of Theorem 10 with q = 1 and c = 0, we get the following inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq \frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|}{\sinh 1}\left(\frac{(e-1)^2}{2e}\right)+\frac{|f'(b)|}{\sinh 1}\left(\frac{e-1}{e}\right)-\frac{c}{6}(b-a)^2\right].$$

Remark 1. The inequality (10) gives better result than the inequality (7). Let us show that

$$\frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^q}{\sinh 1}\left(\frac{e^2-2e-1}{2e}\right)+\frac{|f'(b)|^q}{\sinh 1}\left(\frac{-e^2+4e+1}{4e}\right)-\frac{c}{12}(b-a)^2\right]^{\frac{1}{q}}$$
$$+\frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^q}{\sinh 1}\left(\frac{1}{e}\right)+\frac{|f'(b)|^q}{\sinh 1}\left(\frac{e^2-5}{4e}\right)-\frac{c}{12}(b-a)^2\right]^{\frac{1}{q}}$$
$$\leq \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^q}{\sinh 1}\left(\frac{(e-1)^2}{2e}\right)+\frac{|f'(b)|^q}{\sinh 1}\left(\frac{e-1}{e}\right)-\frac{c}{6}(b-a)^2\right]^{\frac{1}{q}}.$$

Since $h:[0,\infty)\to\mathbb{R}$, $h(x)=x^s, 0<s\leq 1$, is a concave function, for all $u,v\geq 0$ we have

$$h\left(\frac{u+v}{2}\right)=\left(\frac{u+v}{2}\right)^s\geq\frac{h(u)+h(v)}{2}=\frac{u^s+v^s}{2}.$$

From here, we get

$$\frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^q}{\sinh 1}\frac{e^2-2e-1}{2e}+\frac{|f'(b)|^q}{\sinh 1}\frac{-e^2+4e+1}{4e}-\frac{c}{12}(b-a)^2\right]^{\frac{1}{q}}$$

$$+\frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^q}{\sinh 1}\frac{1}{e}+\frac{|f'(b)|^q}{\sinh 1}\frac{e^2-5}{4e}-\frac{c}{12}(b-a)^2\right]^{\frac{1}{q}}$$
$$\leq 2\frac{b-a}{2}\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{1}{2}\frac{|f'(a)|^q}{\sinh 1}\frac{(e-1)^2}{2e}+\frac{1}{2}\frac{|f'(b)|^q}{\sinh 1}\frac{e-1}{e}-\frac{1}{2}\frac{c}{6}(b-a)^2\right]^{\frac{1}{q}}$$
$$=\frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^q}{\sinh 1}\left(\frac{(e-1)^2}{2e}\right)+\frac{|f'(b)|^q}{\sinh 1}\left(\frac{e-1}{e}\right)-\frac{c}{6}(b-a)^2\right]^{\frac{1}{q}}.$$

This completes the proof of the remark. In the following theorem, we get new integral inequality using improved power-mean integral inequality.

Theorem 11. Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function, $a, b \in I$, $a < b$, $q \geq 1$ and $f' \in L[a,b]$. If $|f'|^q$ is a strongly hyperbolic type convex function with modulus c on the interval $[a,b]$, then the following inequality holds for $t \in [0,1]$.

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left[\left(\frac{|f'(a)|^q}{\sinh 1}D_1+\frac{|f'(b)|^q}{\sinh 1}D_2-\frac{c}{32}(b-a)^2\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{|f'(a)|^q}{\sinh 1}D_3+\frac{|f'(b)|^q}{\sinh 1}D_4-\frac{c}{32}(b-a)^2\right)^{\frac{1}{q}}\right] \quad (11)$$

where

$$D_1:=\int_0^1(1-t)|1-2t|\sinh t\,dt=3e^{-\frac{1}{2}}-\frac{3}{2}e-\frac{5}{2}e^{-1}+5e^{\frac{1}{2}}-5,$$
$$D_2:=\int_0^1(1-t)|1-2t|(\sinh 1-\sinh t)dt=\frac{19}{8}e^{-1}+\frac{13}{8}e-3e^{-\frac{1}{2}}-5e^{\frac{1}{2}}+5,$$
$$D_3:=\int_0^1 t|1-2t|\sinh t\,dt=4e^{-1}+e-5e^{-\frac{1}{2}}-3e^{\frac{1}{2}}+4,$$
$$D_4:=\int_0^1 t|1-2t|(\sinh 1-\sinh t)dt=5e^{-\frac{1}{2}}-\frac{7}{8}e-\frac{33}{8}e^{-1}+3e^{\frac{1}{2}}-4.$$

Proof: Firstly, we assume first that $q>1$. From Lemma 1, improved power-mean integral inequality and the property of the hyperbolic type convex function of $|f'|^q$, we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{2}\int_0^1|1-2t||f'(ta+(1-t)b)|dt$$
$$\leq \frac{b-a}{2}\left(\int_0^1(1-t)|1-2t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)|1-2t||f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{2}\left(\int_0^1 t|1-2t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t|1-2t||f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{b-a}{2}\left(\int_0^1 (1-t)|1-2t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)|1-2t|\begin{bmatrix}\frac{\sinh t}{\sinh 1}|f'(a)|^q+\frac{\sinh 1-\sinh t}{\sinh 1}|f'(b)|^q\\ -ct(1-t)(b-a)^2\end{bmatrix}\right)^{\frac{1}{q}}$$

$$+\frac{b-a}{2}\left(\int_0^1 t|1-2t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t|1-2t|\begin{bmatrix}\frac{\sinh t}{\sinh 1}|f'(a)|^q+\frac{\sinh 1-\sinh t}{\sinh 1}|f'(b)|^q\\ -ct(1-t)(b-a)^2\end{bmatrix}\right)^{\frac{1}{q}}$$

$$=\frac{b-a}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\begin{pmatrix}\frac{|f'(a)|^q}{\sinh 1}\int_0^1 (1-t)|1-2t|\sinh t dt+\frac{|f'(b)|^q}{\sinh 1}\int_0^1 (1-t)|1-2t|(\sinh 1-\sinh t)dt\\ -c(b-a)^2\int_0^1 t(1-t)^2|1-2t|dt\end{pmatrix}^{\frac{1}{q}}$$

$$+\frac{b-a}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\begin{pmatrix}\frac{|f'(a)|^q}{\sinh 1}\int_0^1 t|1-2t|\sinh t dt+\frac{|f'(b)|^q}{\sinh 1}\int_0^1 t|1-2t|(\sinh 1-\sinh t)dt\\ -c(b-a)^2\int_0^1 t^2(1-t)|1-2t|dt\end{pmatrix}^{\frac{1}{q}}$$

$$=\frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\begin{bmatrix}\left(\frac{|f'(a)|^q}{\sinh 1}D_1+\frac{|f'(b)|^q}{\sinh 1}D_2-\frac{c}{32}(b-a)^2\right)^{\frac{1}{q}}\\ +\left(\frac{|f'(a)|^q}{\sinh 1}D_3+\frac{|f'(b)|^q}{\sinh 1}D_4-\frac{c}{32}(b-a)^2\right)^{\frac{1}{q}}\end{bmatrix},$$

where

$$\int_0^1 (1-t)|1-2t|dt=\int_0^1 t|1-2t|dt=\frac{1}{4}$$

$$\int_0^1 t(1-t)^2|1-2t|dt=\int_0^1 t^2(1-t)|1-2t|dt=\frac{1}{32}.$$

Secondly, let $q=1$. Then we use the estimates from the proof of Theorem 11, which also follow step by step the above estimates. This completes the proof of theorem.

Corollary 9. Under the assumption of Theorem 11 with $c=0$, we get the following inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left[\left(\frac{|f'(a)|^q}{\sinh 1}D_1+\frac{|f'(b)|^q}{\sinh 1}D_2\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q}{\sinh 1}D_3+\frac{|f'(b)|^q}{\sinh 1}D_4\right)^{\frac{1}{q}}\right]$$

This inequality coincides with the inequality in [15].

Corollary 10. Under the assumption of Theorem 10 with $q=1$ and $c=0$, we get the following inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{2}\left[\left(\frac{|f'(a)|}{\sinh 1}D_1+\frac{|f'(b)|}{\sinh 1}D_2\right)+\left(\frac{|f'(a)|}{\sinh 1}D_3+\frac{|f'(b)|}{\sinh 1}D_4\right)\right].$$

Corollary 11. Under the assumption of Theorem 10 with $q=1$, we get the following inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le \frac{b-a}{2}\left[\left(\frac{|f'(a)|}{\sinh 1}D_1+\frac{|f'(b)|}{\sinh 1}D_2-\frac{c}{32}(b-a)^2\right)+\left(\frac{|f'(a)|}{\sinh 1}D_3+\frac{|f'(b)|}{\sinh 1}D_4-\frac{c}{32}(b-a)^2\right)\right].$$

Remark 2. The inequality (11) gives better result than the inequality (8). Let us show that

$$\frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left[\begin{array}{l}\left(\frac{|f'(a)|^q}{\sinh 1}D_1+\frac{|f'(b)|^q}{\sinh 1}D_2-\frac{c}{32}(b-a)^2\right)^{\frac{1}{q}}\\ +\left(\frac{|f'(a)|^q}{\sinh 1}D_3+\frac{|f'(b)|^q}{\sinh 1}D_4-\frac{c}{32}(b-a)^2\right)^{\frac{1}{q}}\end{array}\right]$$
$$\le \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\begin{array}{l}\frac{|f'(a)|^q}{\sinh 1}\left(\frac{2(e-1)}{\sqrt{e}}-\frac{e^2-3}{2e}-1\right)\\ +\frac{|f'(b)|^q}{\sinh 1}\left(\frac{3e}{4}-\frac{7}{4e}+\frac{2}{\sqrt{e}}-2\sqrt{e}+1\right)-\frac{c}{16}(b-a)^2\end{array}\right]^{\frac{1}{q}}.$$

If we use the concavity of the function $h:[0,\infty)\to\mathbb{R}$, $h(x)=x^{\lambda}, 0<\lambda\le 1$, we get

$$\frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left[\begin{array}{l}\left(\frac{|f'(a)|^q}{\sinh 1}D_1+\frac{|f'(b)|^q}{\sinh 1}D_2-\frac{c}{32}(b-a)^2\right)^{\frac{1}{q}}\\ +\left(\frac{|f'(a)|^q}{\sinh 1}D_3+\frac{|f'(b)|^q}{\sinh 1}D_4-\frac{c}{32}(b-a)^2\right)^{\frac{1}{q}}\end{array}\right]$$
$$\le 2\frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}2-\frac{2}{q}}\left[\begin{array}{l}\frac{1}{2}\left(\frac{|f'(a)|^q}{\sinh 1}D_1+\frac{|f'(b)|^q}{\sinh 1}D_2-\frac{c}{32}(b-a)^2\right)\\ +\frac{1}{2}\left(\frac{|f'(a)|^q}{\sinh 1}D_3+\frac{|f'(b)|^q}{\sinh 1}D_4-\frac{c}{32}(b-a)^2\right)\end{array}\right]^{\frac{1}{q}}$$
$$=2\frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left[\frac{1}{2}\frac{|f'(a)|^q}{\sinh 1}(D_1+D_3)+\frac{1}{2}\frac{|f'(b)|^q}{\sinh 1}(D_2+D_4)-\frac{1}{2}\frac{c}{32}(b-a)^2\right]^{\frac{1}{q}}$$
$$=\frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\begin{array}{l}\frac{|f'(a)|^q}{\sinh 1}\left(\frac{2(e-1)}{\sqrt{e}}-\frac{e^2-3}{2e}-1\right)\\ +\frac{|f'(b)|^q}{\sinh 1}\left(\frac{3e}{4}-\frac{7}{4e}+\frac{2}{\sqrt{e}}-2\sqrt{e}+1\right)-\frac{c}{16}(b-a)^2\end{array}\right]^{\frac{1}{q}},$$

where

$$D_1+D_3=\frac{2(e-1)}{\sqrt{e}}-\frac{e^2-3}{2e}-1$$

$$D_2 + D_4 = \frac{3e}{4} - \frac{7}{4e} + \frac{2}{\sqrt{e}} - 2\sqrt{e} + 1$$

which completes the proof of remark.

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