ORIGINAL PAPER

ON KILLING MAGNETIC FIELDS OF MAGNETIC PSEUDO NULL CURVES IN $\mathbb{R}_1^3$

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Abstract. *In present paper, we study the properties of the Killing magnetic field of a pseudo null magnetic curve in* $\mathbb{R}_1^3$ *. We investigate for the* $\boldsymbol{T}$, $\boldsymbol{N}$ *and* $\boldsymbol{B}$*-magnetic cases of a pseudo null curve. We find the conditions for the Killing magnetic field to be pseudo null and derive the Frenet formulas for this field. Finally, we obtain the conditions for the transformation of the pseudo null curve to its pseudo null Killing magnetic field with equal torsions.*

Keywords: *killing magnetic field; pseudo null magnetic curve; Minkowski 3-space.*

1. INTRODUCTION

The basics of electrodynamics, optics and electric circuits is constructed by a set of partial differential equations of Maxwell. Maxwell's second equation is $\nabla \cdot \boldsymbol{B} = 0$ which says the magnetic field $\boldsymbol{B}$ is a divergence-free vector field. Landau-Hall problem which investigates the motion of a charged particle on a Riemannian surface under the effect of a constant and static magnetic field is an interesting research area. Free of any electric field, a particle moves with velocity vector $v(t)$ satisfying the Lorentz force law [1],

$$\mathcal{F}_M = \frac{dP(t)}{dt} = \frac{q}{c} v(t) \times \boldsymbol{B},$$

where m is the mass, q is the charge and $P = \frac{\varepsilon}{c^2} v$ is the momentum of the particle, $\mathcal{F}_M$ is the magnetic force on the particle, c shows the light speed and $\varepsilon = mc^2 (1 - \frac{\|v\|^2}{c^2})^{-\frac{1}{2}}$ is its energy [2].

A curve γ on a Riemannian manifold is called a magnetic curve, if γ' satisfies the following differential equation,

$$\nabla_{\gamma'} \gamma' = \phi(\gamma'),$$

where ∇ is the Riemannian connection and ϕ is Lorentz force. Magnetic curves are the trajectories of a particle which is under the charge of a magnetic field F. When $F = 0$, one gets $\nabla_{\gamma'} \gamma' = 0$, geodesics of the Riemannan manifold coincides with the magnetic curves. This shows that charged particles move along geodesics, free of electric and magnetic fields. So, magnetic curves can be considered as a generalization of geodesics [2].

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In Euclidean 3-space, Frenet vector fields of this charged particle is affected by the magnetic field F. When the tangent vector field $\boldsymbol{T}$ is affected by that magnetic field F, the Lorentz force defined by

$$\phi(\boldsymbol{T}) = F \times \boldsymbol{T},$$

appears and the charged particle follows a trajectory, called $\boldsymbol{T}$-magnetic curve. It is clear that $\boldsymbol{T}$ is parallel to F, then $\phi(\boldsymbol{T}) = 0$, so the particle moves parallel to F. If $\boldsymbol{T}$ is orthogonal to F, then the Lorentz force is maximum and the particle has a circular trajectory. If $\boldsymbol{T}$ makes a constant angle with F, then the particle has a helical trajectory with the effect of the Lorentz force, [3]. For further information on these curves and applications, see [4-6].

This paper is organized as follows. Section 2 is devoted to fundamental informations on $\mathbb{R}_1^3$, pseudo null curves and magnetic curves. Our results are presented in Section 3. The conditions for the Killing magnetic field of a pseudo null curve to be pseudo null are found. Then, in this case Frenet formulas are derived. Finally, the conditions for the transformation of the pseudo null curve to its pseudo null Killing magnetic field with equal torsions are obtained. Section 4 consists of concluding remarks.

2. PRELIMINARIES

The space $\mathbb{R}^3$ endowed with the Lorentzian product,

$$\langle m, n\rangle_L = m_1 n_1 + m_2 n_2 - m_3 n_3,$$

where $m = (m_1, m_2 m_3), n = (n_1, n_2, n_3)\epsilon\mathbb{R}^3$, is said to be Minkowski 3-space and denoted by $\mathbb{R}_1^3$. A vector m in $\mathbb{R}_1^3$ is said to be spacelike if $\langle m, m\rangle_L > 0$ or $m = 0$, timelike if $\langle m, m\rangle_L < 0$ and null if $\langle m, m\rangle_L = 0$ and $m \neq 0$. These are said to be the causal characters of m. The Lorentzian norm of m is defined as the nonnegative real number,

$$\|m\|_L = \sqrt{|\langle m, m\rangle_L|}.$$

The causal character of the tangent vector of a regular curve in $\mathbb{R}_1^3$ determines the causal character of the curve [7]. Let α be a unit speed spacelike curve, whose the tangent vector is $\boldsymbol{T}$. If $\boldsymbol{T}'(s)$ is null for any s, then α is said to be a pseudo null curve in $\mathbb{R}_1^3$. The principal normal vector of α is defined as $\boldsymbol{N}(s) = \boldsymbol{T}'(s)$. The vector $\boldsymbol{B}$ is said to be the binormal vector of α, defined as the unique vector orthogonal to $\boldsymbol{T}$ such that $\langle \boldsymbol{N}, \boldsymbol{B}\rangle_L = -1$. In this case, the Frenet formulas are

$$\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}' \\ \boldsymbol{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & -\tau & 0 \\ \kappa & 0 & \tau \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{bmatrix} \tag{2.1}$$

where

$$\langle \boldsymbol{T}, \boldsymbol{T}\rangle_L = 1, \langle \boldsymbol{N}, \boldsymbol{N}\rangle_L = \langle \boldsymbol{B}, \boldsymbol{B}\rangle_L = 0,$$
$$\langle \boldsymbol{N}, \boldsymbol{B}\rangle_L = -1, \langle \boldsymbol{T}, \boldsymbol{N}\rangle_L = \langle \boldsymbol{T}, \boldsymbol{B}\rangle_L = 0, \tag{2.2}$$

with the following properties of cross product $\times$,

$$\boldsymbol{B} \times \boldsymbol{N} = \boldsymbol{T}, \boldsymbol{T} \times \boldsymbol{N} = \boldsymbol{N}, \boldsymbol{B} \times \boldsymbol{T} = \boldsymbol{B}. \tag{2.3}$$

Here, τ is said to be the pseudo torsion of α. The curvature κ can be either 0 or 1. Since α is a line when $\kappa = 0$, we assume that $\kappa = 1$. Hence, the Frenet formulas become

$$\begin{aligned} \boldsymbol{T}' &= \boldsymbol{N}, \\ \boldsymbol{N}' &= -\tau \boldsymbol{N}, \\ \boldsymbol{B}' &= \boldsymbol{T} + \tau \boldsymbol{B}. \end{aligned} \tag{2.4}$$

On the other hand, there are different approaches in framing a pseudo null curve, see [8-11]. Let (K^n, g) be a n-dimensional Riemannian manifold and ∇ be an affine connection defined on K^n. If for every $P, Q, R \epsilon \chi(K^n)$ the conditions:

i: $\nabla_P Q - \nabla_Q P = [P, Q]$,
ii: $Pg(Q, R) = g(\nabla_P Q, R) + g(Q, \nabla_P R)$,

hold, then ∇ is said to be Riemannian connection or Levi-Civita connection of K^n [12].

A 2-form η on K^n is a function

$$\eta: \chi(K^n) \times \chi(K^n) \to C^\infty(K^n, \mathbb{R}),$$

satisfying the following two conditions [12]:

i: $\eta(P, Q)$ is linear in P and in Q for all $P, Q \epsilon \chi(K^n)$,
ii: η is skew-symmetric, that is, $\eta(P, Q) = \eta(Q, P)$ for all $P, Q \epsilon \chi(K^n)$.

If the exterior derivative of η vanishes, that is $d\eta = 0$, η is said to be a closed 2-form on K^n. A magnetic field F on (K^n, g) is a closed 2-form and the Lorentz force ϕ associated to F is a $C^\infty(K^n, \mathbb{R})$ linear transformation $\phi: \chi(K^n) \to \chi(K^n)$ such that

$$F(P, Q) = g(\phi(P), Q),$$

for all $P, Q \epsilon \chi(K^n)$.

The magnetic trajectories of F are curves γ in K^n that satisfy the Lorentz equation

$$\nabla_{\gamma'} \gamma' = \phi(\gamma').$$

Since ϕ is skew-symmetric, the curve γ must have constant speed [9].

3. NEW PROPERTIES OF KILLING MAGNETIC FIELDS

3.1. T-MAGNETIC CASE

Let α be a unit speed $\boldsymbol{T}$-magnetic pseudo null curve in Minkowski 3-space. Then, the necessary and sufficient condition for α to be a $\boldsymbol{T}$-magnetic trajectory of a Killing magnetic field V is

$$V = -\mu \boldsymbol{T} - \boldsymbol{N},$$

along the curve α. In this case, V is a spacelike vector, [13].

Theorem 3.1.1 The Killing magnetic field V of a $\boldsymbol{T}$ -magnetic curve α is pseudo null if and only if

$$\mu' = -1.$$

Proof: Differentiating the equality $V = -\mu\boldsymbol{T} - \boldsymbol{N}$, one can get

$$\overline{\boldsymbol{T}} = \mu'\boldsymbol{T} + (\tau - \mu)\boldsymbol{N}.$$

Since, $\langle \overline{\boldsymbol{T}}, \overline{\boldsymbol{T}} \rangle_L = (\mu')^2 = 1$, we can take $\mu' = -1$. Then,

$$\overline{\boldsymbol{T}} = \boldsymbol{T} + \gamma\boldsymbol{N}, \tag{3.1}$$

where $\gamma = \tau + s + c$. In this case $\mu = -s - c$.

Theorem 3.1.2 The Frenet vectors of the Killing magnetic field V of a $\boldsymbol{T}$-magnetic curve α are

$$\overline{\boldsymbol{T}} = \boldsymbol{T} + \gamma\boldsymbol{N},$$
$$\overline{\boldsymbol{N}} = (1 + \gamma' - \tau\gamma)\boldsymbol{N},$$
$$\overline{\boldsymbol{B}} = \frac{\gamma}{\eta}\boldsymbol{T} + \frac{\gamma^2}{2\eta}\boldsymbol{N} + \frac{1}{\eta}\boldsymbol{B},$$

and the torsion of V is

$$\bar{\tau} = \tau - \frac{\eta'}{\eta}, \tag{3.2}$$

where $\eta = 1 + \gamma' - \tau\gamma$.

Proof: Differentiating both sides of the equality (3.1), we get

$$\overline{\boldsymbol{N}} = (1 + \gamma' - \tau\gamma)\boldsymbol{N}. \tag{3.3}$$

Assume that

$$\overline{\boldsymbol{B}} = a\boldsymbol{T} + b\boldsymbol{N} + c\boldsymbol{B}. \tag{3.4}$$

From the equalities (3.1), (3.3) and (3.4), we compute

$$\overline{\boldsymbol{B}} \times \overline{\boldsymbol{N}} = c(1 + \gamma' - \tau\gamma)\boldsymbol{T} + \boldsymbol{a}(1 + \gamma' - \tau\gamma)\boldsymbol{N}.$$

Since we must have $\overline{\boldsymbol{B}} \times \overline{\boldsymbol{N}} = \overline{\boldsymbol{T}}$, we obtain

$$c = \frac{1}{1+\gamma'-\tau\gamma}, a = \frac{\gamma}{1+\gamma'-\tau\gamma}.$$

Since $\overline{\boldsymbol{B}} \times \overline{\boldsymbol{T}} = \overline{\boldsymbol{B}}$, we get

$$c\gamma = a, \frac{a\gamma}{2} = b.$$

Hence, we obtain

$$\overline{\boldsymbol{B}} = \frac{\gamma}{1+\gamma'-\tau\gamma}\boldsymbol{T} + \frac{\gamma^2}{2(1+\gamma'-\tau\gamma)}\boldsymbol{N} + \frac{1}{1+\gamma'-\tau\gamma}\boldsymbol{B}. \tag{3.5}$$

Let us denote $1+\gamma'-\tau\gamma = \eta$. Differentiating both sides of the equality (3.5), we derive

$$\overline{\boldsymbol{B}}' = \left(\left(\frac{\gamma}{\eta}\right)' + \frac{1}{\eta}\right)\boldsymbol{T} + \left(\frac{\gamma}{\eta} + \left(\frac{\gamma^2}{2\eta}\right)' - \frac{\tau\gamma^2}{2\eta}\right)\boldsymbol{N} + (\left(\frac{1}{\eta}\right)' + \frac{\tau}{\eta})\boldsymbol{B}. \tag{3.6}$$

On the otherhand,

$$\begin{aligned}\overline{\boldsymbol{B}}' &= \overline{\boldsymbol{T}} + \bar{\tau}\overline{\boldsymbol{B}} \\ &= \left(1 + \frac{\bar{\tau}\gamma'}{\eta}\right)\boldsymbol{T} + \left(\frac{\bar{\tau}\gamma^2}{2\eta} + \gamma\right)\boldsymbol{N} + \frac{\bar{\tau}}{\eta}\boldsymbol{B}.\end{aligned} \tag{3.7}$$

Considering the equalities (3.6) and (3.7), we obtain

$$\bar{\tau} = \tau - \frac{\eta'}{\eta}.$$

Let α and $\bar{\alpha}$ be unit speed pseudo null curves in $\mathbb{R}_1^3$, the transformation $\varphi: \alpha \to \bar{\alpha}$ satisfy the conditions stated in [10]. In this case, for α and $\bar{\alpha}$, one of the following statements are true. They are:

i: pseudo null helices having equal torsions $\bar{\tau} = \tau$,

ii: pseudo null circles.

Then, the following theorem can be given.

Theorem 3.1.3 Let α be a pseudo null $\boldsymbol{T}$-magnetic curve in $\mathbb{R}_1^3$ parameterized by arc-length parameter s with torsion τ and Frenet frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$. Then, φ tansforms α to a pseudo null curve given by

$$V = (s+c)\boldsymbol{T} - \boldsymbol{N},$$

with $\bar{\tau} = \tau$, if the following differential equation holds,

$$1 + \tau' - \tau s - \tau^2 + c\tau = 0.$$

Proof: Considering the equality (3.2), $\bar{\tau} = \tau$ if and only if $\eta' = 0$. Since $\eta = 1 + \gamma' - \tau\gamma$ and $\gamma = \tau + s + c$, we get

$$\eta = 2 + \tau' - \tau^2 - \tau(s-c) = constant,$$

so

$$\tau' - \tau^2 - \tau(s-c) = constant.$$

Differentiating both sides of the equality $V=(s+c)\boldsymbol{T}-\boldsymbol{N}$, we get $V'=\boldsymbol{T}+(\tau+s+c)\boldsymbol{N}$. Differentiating both sides of the last equality, we derive

$$V''=\boldsymbol{T}'+(\tau+s+c)'\boldsymbol{N}+(\tau+s+c)\boldsymbol{N}'$$

$$=\left(2+\tau'-\tau^2-\tau(s-c)\right)\boldsymbol{N}.$$

Since $\overline{\boldsymbol{N}}=\boldsymbol{N}$, we must have $1+\tau'-\tau s-\tau^2+c\tau=0$.

3.2. N-MAGNETIC CASE

Let α be a unit speed $\boldsymbol{N}$-magnetic pseudo null curve in Minkowski 3-space. Then, the necessary and sufficient condition for α to be a $\boldsymbol{N}$-magnetic trajectory of a Killing magnetic field V is

$$V=-\tau\boldsymbol{T}+\mu\boldsymbol{N}$$

along the curve α. In this case, V is a spacelike vector [13].

Theorem 3.2.1 The Killing magnetic field V of a $\boldsymbol{N}$-magnetic curve α is pseudo null if and only if

$$\tau'=-1.$$

Proof: Differentiating the equality $V=-\tau\boldsymbol{T}+\mu\boldsymbol{N}$, one can get

$$\overline{\boldsymbol{T}}=-\tau'\boldsymbol{T}+(-\tau+\mu'-\tau\mu)\boldsymbol{N}.$$

Since, $\langle\overline{\boldsymbol{T}},\overline{\boldsymbol{T}}\rangle_L=(-\tau')^2=1$, we can take $\tau'=-1$. Then,

$$\overline{\boldsymbol{T}}=\boldsymbol{T}+\gamma\boldsymbol{N}, \tag{3.8}$$

where $\gamma=-\tau+\mu'-\tau\mu$.

Theorem 3.2.2 The Frenet equations of the Killing magnetic field V of a $\boldsymbol{N}$-magnetic curve α are

$$\begin{aligned}\overline{\boldsymbol{T}}&=\boldsymbol{T}+\gamma\boldsymbol{N},\\ \overline{\boldsymbol{N}}&=(1+\gamma'-\tau\gamma)\boldsymbol{N},\\ \overline{\boldsymbol{B}}&=\frac{\gamma}{\eta}\boldsymbol{T}+\frac{\gamma^2}{2\eta}\boldsymbol{N}+\frac{1}{\eta}\boldsymbol{B},\end{aligned}$$

and the torsion of V is

$$\bar{\tau}=\tau-\frac{\eta'}{\eta}, \tag{3.9}$$

where $\eta=1+\gamma'-\tau\gamma$.

Proof: Differentiating both sides of the equality (3.8), we get

$$\overline{\boldsymbol{N}} = (1+\gamma' - \tau\gamma)\boldsymbol{N}. \tag{3.10}$$

Assume that

$$\overline{\boldsymbol{B}} = a\boldsymbol{T} + b\boldsymbol{N} + c\boldsymbol{B}. \tag{3.11}$$

From the equalities (3.8), (3.10) and (3.11), we compute

$$\overline{\boldsymbol{B}} \times \overline{\boldsymbol{N}} = c(1+\gamma'-\tau\gamma)\boldsymbol{T} + a(1+\gamma'-\tau\gamma)\boldsymbol{N}.$$

Since we must $\overline{\boldsymbol{B}} \times \overline{\boldsymbol{N}} = \overline{\boldsymbol{T}}$, have we obtain

$$c = \frac{1}{1+\gamma'-\tau\gamma}, a = \frac{\gamma}{1+\gamma'-\tau\gamma}.$$

Since $\overline{\boldsymbol{B}} \times \overline{\boldsymbol{T}} = \overline{\boldsymbol{B}}$ we get

$$c\gamma = a, \frac{a\gamma}{2} = b.$$

Hence, we obtain

$$\overline{\boldsymbol{B}} = \frac{\gamma}{1+\gamma'-\tau\gamma}\boldsymbol{T} + \frac{\gamma^2}{2(1+\gamma'-\tau\gamma)}\boldsymbol{N} + \frac{1}{1+\gamma'-\tau\gamma}\boldsymbol{B}. \tag{3.12}$$

Let us denote $1+\gamma'-\tau\gamma = \eta$. Differentiating both sides of the equality (3.12), we derive

$$\overline{\boldsymbol{B}}' = \left(\left(\frac{\gamma}{\eta}\right)' + \frac{1}{\eta}\right)\boldsymbol{T} + \left(\frac{\gamma}{\eta} + \left(\frac{\gamma^2}{2\eta}\right)' - \frac{\tau\gamma^2}{2\eta}\right)\boldsymbol{N} + \left(\left(\frac{1}{\eta}\right)' + \frac{\tau}{\eta}\right)\boldsymbol{B}. \tag{3.13}$$

On the otherhand,

$$\begin{aligned} \overline{\boldsymbol{B}}' &= \overline{\boldsymbol{T}} + \bar{\tau}\overline{\boldsymbol{B}} \\ &= \left(1 + \tfrac{\gamma\bar{\tau}}{\eta}\right)\boldsymbol{T} + \left(\tfrac{\gamma^2\bar{\tau}}{2\eta} + \gamma\right)\boldsymbol{N} + \tfrac{\bar{\tau}}{\eta}\boldsymbol{B}. \end{aligned} \tag{3.14}$$

Considering the equalities (3.13) and (3.14), we obtain

$$\bar{\tau} = \tau - \frac{\eta'}{\eta}.$$

Theorem 3.2.3 Let α be a pseudo null $\boldsymbol{N}$-magnetic curve in $\mathbb{R}_1^3$ parameterized by arc-length parameter s with torsion τ and Frenet frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$**.** Then, φ tansforms α to a pseudo null curve given by

$$V = -\tau\boldsymbol{T} + \mu\boldsymbol{N},$$

with $\bar{\tau} = \tau$ if the following differential equation holds,

$$\mu'' - 2\tau\mu' + (1+\mu)(\tau^2 - \tau') = 0.$$

Proof: Considering the equality (3.9), $\bar{\tau} = \tau$ if and only if $\eta' = 0$. Since $\eta = 1 + \gamma' - \tau\gamma$ and $\gamma = -\tau + \mu' - \tau\mu$, we get

$$\eta = 1 + \mu'' - 2\tau\mu' + (1 + \mu)(\tau^2 - \tau') = \text{constant},$$

So,

$$\mu'' - 2\tau\mu' + (1 + \mu)(\tau^2 - \tau') = \text{constant}.$$

Differentiating both sides of the equality $V = -\tau\boldsymbol{T} + \mu\boldsymbol{N},$ we get

$$V' = -\tau'\boldsymbol{T} + (-\tau + \mu' - \tau\mu)\boldsymbol{N}.$$

Differentiating both sides of the last equality, we derive

$$V'' = -\tau''\boldsymbol{T'} + (-\tau' + \mu'' - 2\tau\mu' + (1 + \mu)(\tau^2 - \tau')\boldsymbol{N}.$$

Since $\overline{\boldsymbol{N}} = \boldsymbol{N}$, we must have $\tau'' = 0$. As we know that $\tau' = -1$, we have

$$\mu'' - 2\tau\mu' + (1 + \mu)(\tau^2 - \tau') = 0.$$

3.3. B-MAGNETIC CASE

Let α be a unit speed $\boldsymbol{B}$-magnetic pseudo null curve in Minkowski 3-space. Then, the necessary and sufficient condition for α to be a $\boldsymbol{B}$-magnetic trajectory of a Killing magnetic field V is

$$V = -\tau\boldsymbol{T} - \boldsymbol{N} - \mu\boldsymbol{B},$$

along the curve α. In this case, V is a

i: spacelike vector, if $\tau^2 > 2\mu$,
ii: timelike vector, if $\tau^2 < 2\mu$, $\mu > 0$,
iii: null vector, if $\tau^2 = 2\mu$, $\mu \geq 0$, [13].

Theorem 3.3.1 The Killing magnetic field V of a $\boldsymbol{B}$-magnetic curve α is pseudo null if and only if

$$\tau' + \mu = -1.$$

Proof: Differentiating both sides of the equality $V = -\tau\boldsymbol{T} - \boldsymbol{N} - \mu\boldsymbol{B}$, we get

$$\overline{\boldsymbol{T}} = (-\tau' - \mu)\boldsymbol{T} + (-\mu' - \tau\mu)\boldsymbol{B}.$$

Since, $\langle \overline{\boldsymbol{T}}, \overline{\boldsymbol{T}} \rangle_L = (-\tau' - \mu)^2 = 1$, $\tau' + \mu = \pm 1$. Since V is spacelike, we obtain $\tau' + \mu = -1$. Then,

$$\overline{\boldsymbol{T}} = \boldsymbol{T} + \gamma\boldsymbol{B}, \tag{3.15}$$

where $\gamma = -\mu' - \tau\mu$.

Theorem 3.3.2 The Frenet equations of the Killing magnetic field V of a $\boldsymbol{B}$-magnetic curve α are

$$\begin{aligned}\overline{\boldsymbol{T}} &= \boldsymbol{T} + \gamma\boldsymbol{B},\\ \overline{\boldsymbol{N}} &= \gamma\boldsymbol{T} + \boldsymbol{N} + (\gamma' + \gamma\tau)\boldsymbol{B},\\ \overline{\boldsymbol{B}} &= \boldsymbol{B},\end{aligned}$$

and the torsion of V is

$$\bar{\tau} = \tau - \gamma, \tag{3.16}$$

where $\gamma = \tau'' + \tau + \tau\tau'$.

Proof: Differentiating both sides of the equality (3.15), we get

$$\overline{\boldsymbol{N}} = \gamma\boldsymbol{T} + \boldsymbol{N} + (\gamma' + \gamma\tau)\boldsymbol{B}, \tag{3.17}$$

Assume that

$$\overline{\boldsymbol{B}} = a\boldsymbol{T} + b\boldsymbol{N} + c\boldsymbol{B}. \tag{3.18}$$

From the equalities (3.15), (3.17) and (3.18), we compute

$$\overline{\boldsymbol{B}} \times \overline{\boldsymbol{N}} = \big(c - b(\gamma' + \gamma\tau)\big)\boldsymbol{T} + (a - \gamma b)\boldsymbol{N} + \big(\gamma c - a(\gamma' + \gamma\tau)\big)\boldsymbol{B}.$$

Since we must have $\overline{\boldsymbol{B}} \times \overline{\boldsymbol{N}} = \overline{\boldsymbol{T}}$, we obtain

$$a = \gamma b, c - b(\gamma' + \gamma\tau) = 1, \gamma c = a(\gamma' + \gamma\tau) + \gamma.$$

Since $\overline{\boldsymbol{B}} \times \overline{\boldsymbol{T}} = \overline{\boldsymbol{B}}$, we get

$$a = b = 0, c = 1.$$

Hence, we obtain

$$\overline{\boldsymbol{B}} = \boldsymbol{B}. \tag{3.19}$$

Differentiating both sides of the equality (3.19), we derive

$$\overline{\boldsymbol{B}}' = \boldsymbol{B}'. \tag{3.20}$$

On the otherhand,

$$\begin{aligned}\overline{\boldsymbol{B}}' &= \overline{\boldsymbol{T}} + \bar{\tau}\boldsymbol{B}\\ &= \boldsymbol{T} + (\boldsymbol{\gamma} + \bar{\boldsymbol{\tau}})\boldsymbol{B}.\end{aligned} \tag{3.21}$$

Considering the equalities (3.20) and (3.21), we obtain

$$\bar{\tau} = \tau - \gamma.$$

Theorem 3.3.3 Let α be a pseudo null $\boldsymbol{B}$-magnetic curve in $\mathbb{R}_1^3$ parameterized by arc-length parameter s with torsion τ and Frenet frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$. Then, φ tansforms α to a pseudo null curve given by

$$V = -\tau \boldsymbol{T} - \boldsymbol{N} + (1+\tau')\boldsymbol{B},$$

with $\bar{\tau} = \tau$, if the following differential equation holds,

$$\tau'' + \tau\tau' + \tau = 0.$$

Proof: Considering the equality (3.16), $\bar{\tau} = \tau$ if and only if $\gamma = 0$. Since $\gamma = \tau'' + \tau\tau' + \tau$, we obtain

$$\tau'' + \tau\tau' + \tau = 0.$$

4. CONCLUSION

This study is based on investigating the properties of the Killing magnetic field of a pseudo null magnetic curve in $\mathbb{R}_1^3$. Pseudo null condition, Frenet equations and conditions when the curve is transformed to pseudo null Killing magnetic field with equal torsions are obtained. It is aimed to contribute to the theory of magnetic curves in $\mathbb{R}_1^3$. Also, these results can be applied to magnetism theory of physics.

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