ORIGINAL PAPER

THE ADJOINT MAP AND DUAL MAP OF MULTILINEAR MAP

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Abstract. *Let V_i be the vector spaces and V_i^* be their dual spaces for $1 \leq i \leq r$. In this paper, $(V_1 \times V_2 \times \ldots \times V_r)^* \cong V_1^* \times V_2^* \times \ldots \times V_r^*$ is examined. Furthermore, the dual map F^* and the adjoint map F' of the r-linear map $F: V_1 \times V_2 \times \ldots \times V_r \to W$ are defined and for their matrices, the equality $[F^*] = [F'] = [\bar{F}]^T$ is found. Analog, the dual map F^* of the r-variable vector valued linear map $F: V_1 \times V_2 \times \ldots \times V_r \to W_1 \times W_2 \times \ldots \times W_r$ is defined and for their matrices, the equality $[F^*] = [\bar{F}]^T$ is true, for the vector spaces V_i and W_i .*

Keywords: *adjoint maps; dual space; multilinear maps; dual maps.*

1. INTRODUCTION

Let V be a vector space over a field $\mathfrak{F}$ and V^* be a dual space of V. Recall that a symmetric bilinear form $B: V \times V \to \mathfrak{F}$ is corresponding to a linear map $\varphi: V \to V^*$. Backward, a linear map $\varphi: V \to V^*$ is corresponding to a bilinear form $B: V \times V \to \mathfrak{F}$ such that

$$\varphi(x): V \to \mathfrak{F}$$
$$y \mapsto B(x, y)$$

If two vector spaces V and W have non-degenerate bilinear forms, then $\varphi: V \to V^*$ and $\psi: W \to W^*$are linear isomorphism, [3-4]. The dual map L^* of the linear map $L: V \to W$ is defined such that

$$L^*: W^* \to V^*$$
$$\beta^* \mapsto L^*(\beta^*): V \to \mathfrak{F}$$
$$\alpha \mapsto [L^*(\beta^*)](\alpha) = \beta^*\overline{(L(\alpha))}.$$

Also, we can write $L^* \circ \beta^* = \beta^* \circ \bar{L}$.

If $\{\alpha_j\}$ is a basis in finite-dimensional inner product space V and $\{\alpha_j^*\}$ is the dual basis in dual space V^* such that

$$\langle \alpha_i^*, \alpha_j \rangle = \delta_{ij}, \qquad 1 \leq i, j \leq n,$$

and $\{\beta_k\}$ is a basis in finite-dimensional inner product space W and $\{\beta_k^*\}$ is the dual basis in dual space W^* such that

$$\langle \beta_l^*, \beta_k \rangle = \delta_{lk}, \qquad 1 \leq l, k \leq m,$$

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then, we have

$$L(\alpha_j) = \sum_{k=1}^{m} a_{kj}\beta_k \quad \text{and} \quad L^*(\beta_k^*) = \sum_{j=1}^{n} a_{jk}^*\alpha_j^*. \tag{1}$$

Hence, we obtain two matrices $A^* = [a_{jk}^*]_{n\times m}$ and $A = [a_{kj}]_{m\times n}$ such that $A^* = \bar{A}^T$.

Moreover, the linear map $L^{ad} = L' = \varphi^{-1} \circ L^* \circ \psi: W \to V$ is called ***the adjoint map*** of L. The adjoint map L' has $A' = [a_{jk}]_{n\times m}$ such that $A' = A^* = \bar{A}^T$.

If V and W are two inner product spaces, then it provides $\langle L(\alpha), \beta\rangle_W = \langle \alpha, L'(\beta)\rangle_V$. Furthermore, there is only $x \in V$ such that linear functional $F: V \to \mathfrak{F}$, $F(v) = \langle v, x\rangle$. Then, the vector $x \in V$ is called ***the Reisz vector***. Thus, there is a linear isomorphism

$$\begin{aligned} \varphi: V &\to V^* \\ x &\mapsto \varphi(x): V \to \mathfrak{F} \\ &\qquad v \mapsto (\varphi(x))_{(v)} = \langle v, x\rangle, \end{aligned} \tag{2}$$

and it is called ***the isomorphism of inner product spaces***. Thus, we can write the inner product such that

$$\langle \alpha^*, \beta^*\rangle_{V^*} = \langle \varphi^{-1}(\alpha^*), \varphi^{-1}(\beta^*)\rangle_V \tag{3}$$

over dual vector space V^*, [3-4].

1.1. MULTILINEAR MAP

Let V_i, $1 \le i \le r$ and W be vector spaces and $\{e_{i1}, e_{i2}, \dots, e_{ik_i}\}$ and $\{\beta_1, \beta_2, \dots, \beta_m\}$ be their bases, respectively. Recall that, $\dim W = m, \dim V_i = k_i$.

Then, the matrix A of linear map $L: V_1 \times V_2 \times \dots \times V_r \to W$ corresponding to the basis $\{(e_{1j_1}, e_{2j_2}, \dots, e_{rj_r})\}, 1 \le j_1 \le k_1, 1 \le j_2 \le k_2, \dots, 1 \le j_r \le k_r$ is

$$A = [a_{s(j_1 j_2 \dots j_r)}]_{m\times(k_1 k_2 \dots k_r)}$$

such that

$$L(e_{1j_1}, e_{2j_2}, \dots, e_{rj_r}) = \sum_{s=1}^{m} a_{s(j_1 j_2 \dots j_r)}\beta_s.$$

Also, for the r-linear maps $F: V \times V \times \dots \times V \to W$ and $G: V' \times V' \times \dots \times V' \to W'$, there exists a set of linear bijections $\varphi_1, \varphi_2, \dots, \varphi_r: V \to V'$ and $\psi: W \to W'$ and

$$G(\varphi_1(v_1), \varphi_2(v_2), \dots, \varphi_r(v_r)) = \psi(L(v_1, v_2, \dots, v_r))$$

is provided. Then, the r-linear maps F and G are called ***equivalent*** and can be written $F = (\varphi_1, \varphi_2, \dots, \varphi_r)$ and $G = (\psi_1, \psi_2, \dots, \psi_t)$, [1-2].

2. DUAL SPACE OF CARTESIAN PRODUCT SPACES

Theorem 1. Let $V_i, 1 \leq i \leq r$ be vector spaces. Then,

$$(V_1 \times V_2 \times \ldots \times V_r)^* \cong V_1^* \times V_2^* \times \ldots \times V_r^*. \tag{4}$$

Proof. We can write

$$\begin{aligned} \varphi_i &: V_i \to V_i^* \\ \alpha_i &\mapsto \varphi(\alpha_i): V_i \to \mathfrak{F} \\ &\qquad \beta_i \mapsto B_{V_i}(\alpha_i \beta_i), \quad 1 \leq i \leq r. \end{aligned}$$

By the above linear isomorphism, we can define linear isomorphisms

$$\begin{aligned} M &: V_1 \times V_2 \times \ldots \times V_r \to V_1^* \times V_2^* \times \ldots \times V_r^* \\ &(\alpha_1, \alpha_2, \ldots, \alpha_r) \mapsto M(\alpha_1, \alpha_2, \ldots, \alpha_r) = \big(\varphi_1(\alpha_1), \varphi_2(\alpha_2), \ldots, \varphi_r(\alpha_r)\big) \end{aligned} \tag{5}$$

and

$$\begin{aligned} \Phi &: V_1 \times V_2 \times \ldots \times V_r \to (V_1 \times V_2 \times \ldots \times V_r)^* \\ &(\alpha_1, \alpha_2, \ldots, \alpha_r) \mapsto \Phi(\alpha_1, \alpha_2, \ldots, \alpha_r): V_1 \times V_2 \times \ldots \times V_r \to \mathfrak{F} \\ &\qquad (\beta_1, \beta_2, \ldots, \beta_r) \mapsto [\Phi(\alpha_1, \alpha_2, \ldots, \alpha_r)](\beta_1, \beta_2, \ldots, \beta_r) = \textstyle\prod_{i=1}^{r} B_{V_i}(\alpha_i, \beta_i). \end{aligned} \tag{6}$$

Hence, we get a diagram as follows:

$$\begin{array}{ccc} V_1^* \times V_2^* \times \ldots \times V_r^* & \xrightarrow{L} & (V_1 \times V_2 \times \ldots \times V_r)^* \\ \uparrow \varphi_i & \Phi & \nearrow \\ V_1 \times V_2 \times \ldots \times V_r & & \end{array}$$

Thus, we can write

$$L\big(\varphi_1(\alpha_1), \varphi_2(\alpha_2), \ldots, \varphi_r(\alpha_r)\big) = \Phi(\alpha_1, \alpha_2, \ldots, \alpha_r)$$

and so $L \circ M = \Phi$. Therefore, $L = M^{-1} \circ \Phi$ is a linear isomorphism. This gives us

$$(V_1 \times V_2 \times \ldots \times V_r)^* \cong V_1^* \times V_2^* \times \ldots \times V_r^*.$$

Note 1. Since $L\big(\varphi_1(\alpha_1), \varphi_2(\alpha_2), \ldots, \varphi_r(\alpha_r)\big) \in (V_1 \times V_2 \times \ldots \times V_r)^*$ we have

$$\begin{aligned} &\big[L\big(\varphi_1(\alpha_1), \varphi_2(\alpha_2), \ldots, \varphi_r(\alpha_r)\big)\big](\beta_1, \beta_2, \ldots, \beta_r) \\ &\qquad = \Big(\big(\varphi_1(\alpha_1)\big)(\beta_1), \big(\varphi_2(\alpha_2)\big)(\beta_2), \ldots, \big(\varphi_r(\alpha_r)\big)(\beta_r)\Big). \end{aligned}$$

For $\forall(\alpha_1^*, \alpha_2^*, \ldots, \alpha_r^*) \in V_1^* \times V_2^* \times \ldots \times V_r^*$, we can write

$$\begin{aligned} (\alpha_1^*, \alpha_2^*, \ldots, \alpha_r^*) &= \big(\varphi_1(\alpha_1), \varphi_2(\alpha_2), \ldots, \varphi_r(\alpha_r)\big) \\ &= (L^{-1} \circ \Phi)(\alpha_1, \alpha_2, \ldots, \alpha_r). \end{aligned} \tag{7}$$

3. THE DUAL MAP AND ADJOINT MAP OF MULTILINEAR MAP

Let $V_1, V_2, \dots, V_r, W$ be vector spaces over $\mathfrak{F}$ and $F: V_1 \times V_2 \times \dots \times V_r \to W$ be a r-linear map. Then with the following diagram

$$\begin{array}{ccc} V_1 \times V_2 \times \dots \times V_r & \xrightarrow{\boldsymbol{F}} & \boldsymbol{W} \\ \downarrow \boldsymbol{\varphi_i} & & \downarrow \boldsymbol{\psi} \\ V_1^* \times V_2^* \times \dots \times V_r^* & \xrightarrow{\boldsymbol{G}} & \boldsymbol{W}^* \end{array}$$

we can write

$$G\big(\varphi_1(\alpha_1), \varphi_2(\alpha_2), \dots, \varphi_r(\alpha_r)\big) = \psi\big(F(\alpha_1, \alpha_2, \dots, \alpha_r)\big). \tag{8}$$

3.1. THE DUAL MAP AND ITS MATRIX

The map

$$\begin{aligned} &F^*: W^* \to V_1^* \times V_2^* \times \dots \times V_r^* \\ &\quad \beta^* \mapsto F^*(\beta^*): V_1 \times V_2 \times \dots \times V_r \to \mathfrak{F} \\ &\qquad (\alpha_1, \alpha_2, \dots, \alpha_r) \mapsto [F^*(\beta^*)](\alpha_1, \alpha_2, \dots, \alpha_r) = \beta^*\left(\overline{F(\alpha_1, \alpha_2, \dots, \alpha_r)}\right) \end{aligned} \tag{9}$$

is called ***the dual map*** of the map F .

Theorem 2. Let V and W be vector spaces over $\mathfrak{F}$ and $L: V \times V \times \dots \times V \to W$ be a r-linear map. If the matrix of the map L is A, and the matrix of the dual map L^* is A^*, then

$$A^* = \bar{A}^T \tag{10}$$

is obtained.

Proof. Assume that the basis of the space V is $\{e_1, e_2, \dots, e_r\}$, the basis of the space $V \times V \dots \times V$ is $\{(e_{j_1}, e_{j_2}, \dots, e_{j_r})\}$, and the basis of the space W is $\{\beta_1, \beta_2, \dots, \beta_r\}$. Then, we have

$$L\big(e_{j_1}, e_{j_2}, \dots, e_{j_r}\big) = \sum_{t=1}^{m} a_{t(j_1 j_2 \dots j_r)} \beta_t, \qquad 1 \le j_1, j_2, \dots, j_r \le n.$$

Also, let the basis of the $V^* \times V^* \times \dots \times V^*$ space be $\{(e_{j_1}^*, e_{j_2}^*, \dots, e_{j_r}^*)\}$. Thus, we get

$$L^*(\beta_s^*) = \sum_{i_1, i_2, \dots, i_r = 1}^{n} a^*_{(i_1 i_2 \dots i_r)s} \big(e_{i_1}^*, e_{i_2}^*, \dots, e_{i_r}^*\big), \qquad 1 \le s \le m.$$

These give us

$$A = \big[a_{t(j_1 j_2 \dots j_r)}\big]_{m \times n^r} \quad \text{and} \quad A^* = \big[a^*_{(i_1 i_2 \dots i_r)s}\big]_{n^r \times m}.$$

Since,

$$\left(L^*(\beta_s^*)\right)\left(e_{k_1}, e_{k_2}, \dots, e_{k_r}\right) = \sum_{i_1, i_2, \dots, i_r = 1}^{n} a^*_{(i_1 i_2 \dots i_r)s}\left(e^*_{i_1}, e^*_{i_2}, \dots, e^*_{i_r}\right)\left(e_{k_1}, e_{k_2}, \dots, e_{k_r}\right)$$

we can write

$$\beta_s^*\left(\overline{L\left(e_{k_1}, e_{k_2}, \dots, e_{k_r}\right)}\right) = \sum_{i_1, i_2, \dots, i_r = 1}^{n} a^*_{(i_1 i_2 \dots i_r)s} e^*_{i_1}\left(e_{k_1}\right) e^*_{i_2}\left(e_{k_2}\right) \dots e^*_{i_r}\left(e_{k_r}\right),$$

or

$$\beta_s^*\left(\sum_{t=1}^{m} \bar{a}_{t(k_1 k_2 \dots k_r)} \beta_t\right) = \sum_{i_1, i_2, \dots, i_r = 1}^{n} a^*_{(i_1 i_2 \dots i_r)s} \delta_{i_1 k_1} \delta_{i_2 k_2} \dots \delta_{i_r k_r}.$$

Thus, we see that

$$\sum_{t=1}^{m} \bar{a}_{t(k_1 k_2 \dots k_r)} \delta_{st} = \sum_{i_1, i_2, \dots, i_r = 1}^{n} a^*_{(i_1 i_2 \dots i_r)s} \delta_{i_1 k_1} \delta_{i_2 k_2} \dots \delta_{i_r k_r} .$$

If we take s$= t$ and $i_1 = k_1, i_2 = k_2, \dots i_r = k_r$ then we obtain

$$\bar{a}_{s(k_1 k_2 \dots k_r)} = a^*_{(k_1 k_2 \dots k_r)s}$$

that is

$$A^* = \bar{A}^T.$$

3.2. THE ADJOINT MAP AND ITS MATRIX

The adjoint of the r-linear map $F: V_1 \times V_2 \times \dots \times V_r \to W$, denoted by F', is the r-linear map and defined as

$$F': M^{-1} \circ F^* \circ \psi: W \to V_1 \times V_2 \times \dots \times V_r \quad (11)$$

with the help of the r-linear map

$$M: V_1 \times V_2 \times \dots \times V_r \to V_1^* \times V_2^* \times \dots \times V_r^*$$
$$(\alpha_1, \alpha_2, \dots, \alpha_r) \mapsto M(\alpha_1, \alpha_2, \dots, \alpha_r) = \left(\varphi_1(\alpha_1), \varphi_2(\alpha_2), \dots, \varphi_r(\alpha_r)\right)$$

and the diagram

$$\begin{array}{ccc} \boldsymbol{W}^* & \overset{\boldsymbol{F}^*}{\rightarrow} & V_1^* \times V_2^* \times \dots \times V_r^* \\ \boldsymbol{\psi} \uparrow & & \uparrow \boldsymbol{\varphi_i} \\ \boldsymbol{W} & \overset{\boldsymbol{F'}}{\rightarrow} & V_1 \times V_2 \times \dots \times V_r \end{array}$$

where $M^{-1} = (\varphi_1^{-1}, \varphi_2^{-1}, \dots, \varphi_r^{-1})$ and $M = (\varphi_1, \varphi_2, \dots, \varphi_r)$.

Theorem 3. Let A, A^* and A' be the matrices of the maps F, F^* and F', respectively. Then

$$A^* = A' = \bar{A}^T. \tag{12}$$

Proof. Since the bases of spaces V and W are $\{e_1, e_2, \dots, e_n\}$ and $\{\beta_1, \beta_2, \dots, \beta_m\}$, respectively; $\{(e_{j_1}, e_{j_2}, \dots, e_{j_r})\}$ is the basis of space $V_1 \times V_2 \times \dots \times V_r$, we can write

$$F(e_{j_1}, e_{j_2}, \dots, e_{j_r}) = \sum_{t=1}^{m} a_{t(j_1 j_2 \dots j_r)} \beta_t, \quad 1 \le j_1, j_2, \dots, j_r \le n\,.$$

Similarly,

$$F^*(\beta_s^*) = \sum_{i_1, i_2, \dots, i_r = 1}^{n} a^*_{(i_1 i_2 \dots i_r)s} (e^*_{i_1}, e^*_{i_2}, \dots, e^*_{i_r})$$

can be written. Thus, we obtain

$$\begin{aligned}
F'(\beta_s^*) &= (M^{-1} \circ F^* \circ \psi)(\beta_s) \\
&= (M^{-1} \circ F^*)(\psi(\beta_s)) \\
&= M^{-1}(F^*(\beta_s^*)) \\
&= M^{-1}\left(\sum_{i_1, i_2, \dots, i_r = 1}^{n} a^*_{(i_1 i_2 \dots i_r)s} (e^*_{i_1}, e^*_{i_2}, \dots, e^*_{i_r}) \right) \\
&= \sum_{i_1, i_2, \dots, i_r = 1}^{n} a^*_{(i_1 i_2 \dots i_r)s} M^{-1}(e^*_{i_1}, e^*_{i_2}, \dots, e^*_{i_r}) \\
&= \sum_{i_1, i_2, \dots, i_r = 1}^{n} a^*_{(i_1 i_2 \dots i_r)s} \left(\varphi_1^{-1}(e^*_{i_1}), \varphi_2^{-1}(e^*_{i_2}), \dots, \varphi_r^{-1}(e^*_{i_r})\right) \\
&= \sum_{i_1, i_2, \dots, i_r = 1}^{n} a^*_{(i_1 i_2 \dots i_r)s} (e_{i_1}, e_{i_2}, \dots, e_{i_r}).
\end{aligned}$$

So this gives us

$$A' = A^* = \bar{A}^T.$$

4. MULTIVARIABLE VECTOR VALUED MAPS

Let V_i and $W_i, 1 \le i \le r$ be vector spaces over $\mathfrak{F}$ and V_i^* and $W_i^*, 1 \le i \le r$ be dual spaces, respectively. We can define linear isomorphisms as

$$\varphi_i : V_i \to V_i^*, \qquad \psi_i : W_i \to W_i^*.$$

Also, we can describe linear isomorphisms as

$$\Phi: V_1 \times V_2 \times \ldots \times V_r \to (V_1 \times V_2 \times \ldots \times V_r)^*$$

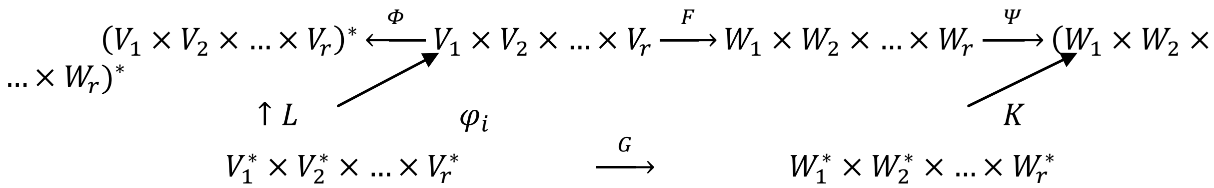
and

$$\Psi: W_1 \times W_2 \times \ldots \times W_r \to (W_1 \times W_2 \times \ldots \times W_r)^*$$

Let

$$F: V_1 \times V_2 \times \ldots \times V_r \to W_1 \times W_2 \times \ldots \times W_r$$

be a r-linear map. We thus obtain the diagram as follows:



Considering this, we can write

$$G(\varphi_1(\alpha_1), \varphi_2(\alpha_2), \ldots, \varphi_r(\alpha_r)) = (\psi_1, \psi_2, \ldots, \psi_r)(F(\alpha_1, \alpha_2, \ldots, \alpha_r))$$

$$= \big(\psi_1(F(\alpha_1, \alpha_2, \ldots, \alpha_r)), \ldots, \psi_r(F(\alpha_1, \alpha_2, \ldots, \alpha_r))\big),$$

where

$$M = (\varphi_1, \varphi_2, \ldots, \varphi_r): V_1 \times V_2 \times \ldots \times V_r \to V_1^* \times V_2^* \times \ldots \times V_r^*$$

and

$$N = (\psi_1, \psi_2, \ldots, \psi_r): W_1 \times W_2 \times \ldots \times W_r \to W_1^* \times W_2^* \times \ldots \times W_r^*$$

are linear maps. Therefore, we have

$$(L \circ M)(\alpha_1, \alpha_2, \ldots, \alpha_r) = L(\varphi_1(\alpha_1), \varphi_2(\alpha_2), \ldots, \varphi_r(\alpha_r)) = \Phi(\alpha_1, \alpha_2, \ldots, \alpha_r)$$

and

$$(K \circ N)(\gamma_1, \gamma_2, \ldots, \gamma_r) = K(\psi_1(\gamma_1), \psi_2(\gamma_2), \ldots, \psi_r(\gamma_r)) = \Psi(\gamma_1, \gamma_2, \ldots, \gamma_r).$$

4.1. THE DUAL MAP AND ITS MATRIX

The dual map of the r-linear map $F: V_1 \times V_2 \times \ldots \times V_r \to W_1 \times W_2 \times \ldots \times W_r$ is defined by

$$F^*: W_1^* \times W_2^* \times \dots \times W_r^* \to V_1^* \times V_2^* \times \dots \times V_r^*$$

$$(\gamma_1^*, \gamma_2^*, \dots, \gamma_r^*) \mapsto F^*(\gamma_1^*, \gamma_2^*, \dots, \gamma_r^*): V_1 \times V_2 \times \dots \times V_r \to \mathfrak{F} \tag{13}$$

$$\left(F^*(\gamma_1^*, \gamma_2^*, \dots, \gamma_r^*)\right)(\alpha_1, \alpha_2, \dots, \alpha_r) = (\gamma_1^*, \gamma_2^*, \dots, \gamma_r^*)\overline{\left(F(\alpha_1, \alpha_2, \dots, \alpha_r)\right)}$$

Theorem 4. Let A and A^* be the matrices of the r-variable vector valued r-linear map $F: V \times V \times \dots \times V \to W \times W \times \dots W$ and its dual map F^*, respectively. Then, we have

$$A^* = \overline{A}^T.$$

Proof: Since the bases of the vector spaces V, V^*, W, W^* are $\{v_i\}, \{v_i^*\}, \{\omega_k\}, \{\omega_k^*\}$ where $1 \le i \le n,\ \ 1 \le k \le m$ respectively. We can write

$$F\left(v_{j_1}, v_{j_2}, \dots, v_{j_r}\right)$$

$$= \sum_{k_1, k_2, \dots, k_r = 1}^{m} a_{(k_1 k_2 \dots k_r)(j_1 j_2 \dots j_r)} \left(\omega_{k_1}, \omega_{k_2}, \dots, \omega_{k_r}\right),$$

$$1 \le j_1, j_2, \dots, j_r \le n$$

Thus, we get the following matrix

$$A = \left[a_{(k_1 k_2 \dots k_r)(j_1 j_2 \dots j_r)}\right]_{m^2 \times n^2}.$$

Similarly, considering

$$F^*\left(\omega_{k_1}^*, \omega_{k_2}^*, \dots, \omega_{k_r}^*\right) = \sum_{i_1, i_2, \dots, i_r = 1}^{n} a^*_{(i_1 i_2 \dots i_r)(j_1 j_2 \dots j_r)} \left(v_{i_1}^*, v_{i_2}^*, \dots, v_{i_r}^*\right)$$

we have the matrix

$$A^* = \left[a^*_{(i_1 i_2 \dots i_r)(k_1 k_2 \dots k_r)}\right]_{n^2 \times m^2}.$$

Then, we can write

$$\left[F^*\left(\omega_{k_1}^*, \omega_{k_2}^*, \dots, \omega_{k_r}^*\right)\right]\left(v_{j_1}, v_{j_2}, \dots, v_{j_r}\right)$$

$$= \left(\sum_{i_1, i_2, \dots, i_r = 1}^{n} a^*_{(i_1 i_2 \dots i_r)(k_1 k_2 \dots k_r)} \left(v_{i_1}^*, v_{i_2}^*, \dots, v_{i_r}^*\right)\right)\left(v_{j_1}, v_{j_2}, \dots, v_{j_r}\right)$$

or

$$\left(\omega_{k_1}^*, \omega_{k_2}^*, \dots, \omega_{k_r}^*\right)\left(\overline{F\left(v_{j_1}, v_{j_2}, \dots, v_{j_r}\right)}\right)$$

$$= \sum_{i_1, i_2, \ldots, i_r = 1}^{n} a^*_{(i_1 i_2 \ldots i_r)(k_1 k_2 \ldots k_r)} \left(v^*_{i_1}(v_{j_1}), v^*_{i_2}(v_{j_2}), \ldots, v^*_{i_r}(v_{j_r})\right).$$

Thus, we obtain

$$\left(\omega^*_{k_1}, \omega^*_{k_2}, \ldots, \omega^*_{k_r}\right)\left(\sum_{m_1, m_2, \ldots, m_r = 1}^{m} \bar{a}_{(m_1 m_2 \ldots m_r)(j_1 j_2 \ldots j_r)} \left(\omega_{m_1}, \omega_{m_2}, \ldots, \omega_{m_r}\right)\right)$$

$$= \sum_{i_1, i_2, \ldots, i_r = 1}^{n} a^*_{(i_1 i_2 \ldots i_r)(k_1 k_2 \ldots k_r)} \left(v^*_{i_1}(v_{j_1}), v^*_{i_2}(v_{j_2}), \ldots, v^*_{i_r}(v_{j_r})\right)$$

or

$$\sum_{m_1, m_2, \ldots, m_r = 1}^{m} \bar{a}_{(m_1 m_2 \ldots m_r)(j_1 j_2 \ldots j_r)} \underbrace{\omega^*_{k_1}(\omega_{m_1})}_{\delta_{k_1 m_1}} \underbrace{\omega^*_{k_2}(\omega_{m_2})}_{\delta_{k_2 m_2}} \ldots \underbrace{\omega^*_{k_r}(\omega_{m_r})}_{\delta_{k_r m_r}}$$

$$= \sum_{i_1, i_2, \ldots, i_r = 1}^{n} a^*_{(i_1 i_2 \ldots i_r)(k_1 k_2 \ldots k_r)} \delta_{i_1 j_1} \delta_{i_2 j_2} \ldots \delta_{i_r j_r}.$$

Then, we have

$$k_1 = m_1,$$

$$k_2 = m_2,$$

$$\ldots$$

$$k_r = m_r,$$

$$i_1 = j_1,$$

$$i_2 = j_2,$$

$$\ldots$$

$$i_r = j_r,$$

$$\bar{a}_{(k_1 k_2 \ldots k_r)(j_1 j_2 \ldots j_r)} = a^*_{(j_1 j_2 \ldots j_r)(k_1 k_2 \ldots k_r)}$$

or

$$A^* = \bar{A}^T.$$

5. CONCLUSION

In this study, the dual space of the cartesian product of vector spaces $V_1, V_2, \ldots, V_r$ was defined, and thus the adjiont map and the dual map of the r −linear maps are defined. In addition, the adjoint map and the dual map of the vector valued r −linear maps are considered. In both cases, the corresponding matrices of the dual and the adjoint maps were compared (associated) by using the r −linear maps.

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