ORIGINAL PAPER

ROTATION MINIMIZING SPHERICAL MOTIONS AND HELICES

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Abstract. *In this study we give the structure of the motion RMM with the spherical curve orbit by using rotation minimizing frames (RMF). Further, quaternionic helices and their characterizations of being CCR curve are given with the help of unit quaternions which are related spherical frame motion.*

Keywords: *spherical motions; rotation minimizing motions; angular velocity; quaternionic helices*

1. INTRODUCTION

For the spherical motions, the orbit of a point can be found as a product of the vector with point coordinates and the motion matrix. Let rotation matrix be $A(t)$ and the point be P, then $\alpha(t) = A(t)P$ gives us the orbit curve of the point P. As an inverse problem if $\alpha(t)$ is a spherical curve, then there are many orthogonal matrix $A(t)$ which gives us the curve $\alpha(t)$. The angular velocity matrix of the motion is

$$W(t) = \dot{A}(t)A^{-1}(t) \tag{1}$$

where $\cdot = {}^{d}/_{dt}$.

And also, it is an antisymmetric matrix. We know that each antisymmetric matrix is a vector W(t) corresponding. And ‖W(t)‖ is angular velocity of the motion. If angular speed ‖W(t)‖ is minimum, then the motion is called Rotating Minimizing Motion (RMM). In "Rotation minimizing spherical motions" by B. Jüttler [1], for the determination of these motions, quaternions have been used. In this study, rotating minimizing motion (RMM) has been produced, using rotating minimizing frame (RMF). Also, how to get quaternionic helices has been examined under the condition the special orbit curves [2-3].

1.1. PRELIMINARIES

Let $\alpha(t) \in \mathbb{R}^3$ be a curve with unit tangent vector field $T = \alpha'(t)$ and normal field $X(t)$. If X' and T are propotional, then $X(t)$ is called rotation minimizing vector. Also $\{T(t), X(t), T \times X(t)\}$ is called Rotation Minimizing Frame (RMF) or Bishop Frame. Let $z(t), t \in [0,1]$ is curve on the unit sphere S^2. For spherical motion $U(t)$ and arbitrary point $P \in S^2$.

$$z(t) = U(t)p \tag{2}$$

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where there are many spherical motions $U(t)$ (can be written) to get the orbit $z(t)$. The spherical motion (t) , with minimum angular speed has been called rotating minimizing motions RMM. For more detail see “Rotating Minimizing Spherical Motions, by B. Jüttler.

Definition 1. CCR curve: Let $\alpha(t)$ be a curve, if k_{i+1}/k_i is constant, then the curve α is called constant curvature ratio (CCR) curve. It is well known that, general helices have been characterized by the condition $\tau/_{\kappa}$ is constant (Lancert’s theorem). Hence CCR curves are the generalized helices in $\mathbb{R}^n$, as the general helices in $\mathbb{R}^3$ [4].

Definition 2. Real Quaternion: A real quaternion is defined with

$$q = ae_1 + be_2 + ce_3 + de_4$$

Such that

$$e_i \times e_i = -e_i, (1 \leq i \leq 3),$$

$$e_i \times e_j = -e_j \times e_i = e_k, (1 \leq i, j \leq 3),$$

where (ijk) is even permutation of (123) and $q \in Q$

Let p and q be any two elements of Q. The product of p and q is defined by

$$p \times q = S_P S_q - \langle V_p, V_q \rangle + S_P V_q + S_q V_p + V_P \wedge V_q$$

where, S_r and V_r denote scalar and vector part q ∈ Q, respectively and we have used the inner product and the cross product in Euclidean space. The conjugate of the quaternion

$$q = ae_1 + be_2 + ce_3 + de_4$$

is denoted by cq and given by

$$cq = S_q - V_q = de_4 - ae_1 - be_2 - ce_3$$

From this, we define the symemetric non- degenerate real-valued bilinear from h as follows:

$$h: Q \times Q \to \mathbb{R}$$
$$: (p, q) \to h(p, q) = \frac{1}{2}(p \times cq + q \times cp)$$

It is called the quaternionic inner product. The norm of the q quaternion is

$$\|q\|^2 = q \times \alpha p = \alpha q \times q = a^2 + b^2 + c^2 + d^2 \text{ [5].}$$

Definition 3. Quaternionic curve: A 3-dimensional Euclidean space 3 is defined with the space of spatial quaernion $\{\beta \in Q, \beta + \alpha\beta = 0\}$ in an obvious manner. Let $I = [0,1]$ be an invertal in the real line and $s \in I$ be the arc-length parameter along the smooth curve

$$\beta: I \in \mathbb{R} \to Q$$

$$: s \to \beta(s) = \sum \beta_i\,(s) e_{i,} (1 \le i \le 3).$$

The tangent vector $\beta'(t) = t(s)$ has unit length $\|t(s)\| = 1$ for all s. it follows

$$t' \times \alpha t + t \times \alpha t' = 0$$

which implies t' is orthogonal to t and $t' \times \alpha t$ is a special quaternion. Let $\{t, n_1, n_2\}$ be the Frenet frame of $\beta(s)$. Then Frenet formula is given by

$$t' = k n_1$$

$$n_1' = -kt + r n_2$$

$$n_2' = -r n_1$$

where t, n_1, n_2 are the unit tangent, the unit principal normal and the unit binormal vector of a quaternionic curve β, respectively. The functions k, r are called the principal curvature and the torsion of β, respectively [5].

Definition 4. (Quaternionic helices): Let $\gamma(s)$ be a quaternionic curve in Q and axis of the curve $\gamma(s)$ be the unit vector U. Then $\gamma(s)$ is quaternionic helices if and only if $\langle T, U \rangle$ is constant. Therefore, characterization of quaternionic helices can be given as follows:

Theorem 1. Let $\gamma = \gamma(s)$ be a quaternionic curve in Q, with nonzero curvatures $K(s), k(s), r(s)$, then $\gamma = \gamma(s)$ is a general helix if and only if the function

$$\left(\frac{K}{k}\right)^2 + \frac{1}{(r-K)^2}\left(\left(\frac{K}{k}\right)'\right)^2 \tag{3}$$

is constant [5].

2. STRUCTURE OF RMM

In the following theorem we will give a method to structure the motion RMM. Let $\alpha(t) \subset S^2$ be a spherical curve, we will give the spherical motion with angular speed vector

$$w(t) = \alpha(t) \times \alpha'(t) \tag{4}$$

where $()' = {}^{d}/_{dt}$. The spherical motion $U(t)$ of orbit $\alpha(t)$ is rotating minimizing motion if and only if angular speed of spherical motion is $(t) = \alpha(t) \times \alpha'(t)$, see in B. Jüttler. For the spherical curve $\alpha(t)$, Sabban Frame can be given as in the following way $\{\alpha(t), T(t), S(t)\}$, where $T = \frac{\alpha'(t)}{\|\alpha'(t)\|}$ and

$$\begin{bmatrix} \alpha'(t) \\ T'(t) \\ S'(t) \end{bmatrix} = \begin{bmatrix} 0 & m & 0 \\ -m & 0 & n \\ 0 & -n & 0 \end{bmatrix} \begin{bmatrix} \alpha(t) \\ T(t) \\ S(t) \end{bmatrix} \tag{5}$$

is spherical frame with geodesic curvature $n/_m = k_g$, where $\|\alpha'(t)\| = m$. The Darboux vector field of the spherical motion with the frame $\{\alpha(t), T(t), S(t)\}$ is $W = n\alpha(t) + mS(t)$ and the angular speed of this motion is

$$\|W(t)\| = \sqrt{n^2 + m^2} \tag{6}$$

If the vector fields T and S are rotationed around $\alpha(t)$, as much as $\theta = -\int n(t)dt$, we get frame $\{\alpha, M_1, M_2\}$. Since

$$\begin{aligned} M_{\mathbf{1}} &= cos\theta T + sin\theta S \\ M_2 &= -sin\theta T + cos\theta S \end{aligned} \tag{7}$$

we have,

$$\begin{aligned} M_{\mathbf{1}}' &= -mcos\theta\, \alpha(t) \\ M_2{}' &= msin\theta\, \alpha(t) \end{aligned} \tag{8}$$

$$\begin{bmatrix} \alpha'(t) \\ M_{\mathbf{1}}'(t) \\ M_2{}'(t) \end{bmatrix} = \begin{bmatrix} 0 & mcos\theta & -msin\theta \\ -mcos\theta & 0 & n \\ msin\theta & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha(t) \\ M_{\mathbf{1}}(t) \\ M_2(t) \end{bmatrix}$$

So the frame $\{\alpha, M_1, M_2\}$ is a RMF along the curve $\int \alpha(t)dt$. The following theorem can be given for the structure of the spherical motion RMM of the curve with the orbit $\alpha(t)$.

Theorem 2. Let $\alpha(t)$ be spherical curve and $\{\alpha, M_1, M_2\}$ be a RMF along the curve $\int \alpha(t)dt$. For the curve with orbit $\alpha(t) = U(t)e_1$, $U(t)$ is RMM (motion) , where $e_1 = (1,0,0)$ and $U(t) = [\alpha(t), M_{\mathbf{1}}(t), M_2(t)\ ]$.

Proof: From (6) for the frame $\{\alpha(t), T(t), S(t)\}$ angular speed of the spherical motion is $\|W(t)\| = \sqrt{m^2 + n^2}$. Let's find the motion which make the angular speed is minimum. First derivation equations of spherical motion $U(t) = [\alpha(t), M_{\mathbf{1}}(t), M_2(t)\ ]$ which are produced by RMF frame $\{\alpha(t), M_{\mathbf{1}}(t), M_2(t)\}$ has been calculated as in the following way

$$\begin{bmatrix} \alpha'(t) \\ M_{\mathbf{1}}'(t) \\ M_2{}'(t) \end{bmatrix} = \begin{bmatrix} 0 & mcos\theta & -msin\theta \\ -mcos\theta & 0 & n \\ msin\theta & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha(t) \\ M_{\mathbf{1}}(t) \\ M_2(t) \end{bmatrix} \tag{9}$$

The Darboux vector field of this motion is

$$w(t) = msin\theta M_1 + mcos\theta M_2 \tag{10}$$

If

$$M_{\mathbf{1}} = cos\theta T + sin\theta S$$
$$M_2 = -sin\theta T + cos\theta S \tag{11}$$

replace in (9) we get $w(t) = mS(t)$. Where angular speed of the motion $U(t) = [\alpha(t), M_{\mathbf{1}}(t), M_2(t)\ ]$ is $\|w(t)\| = m$ and $\|w(t)\| \leq \|W(t)\|$. Hence angular speed is minimum.

Proposition 1. Spherical motion $U(t) = [M_1 \quad M_2 \quad \alpha]$ is RMM motion with orbit $\alpha(t)$ of the point $e_3 = (0,0,1)$

Example 1. For the small circle

$$\alpha = \alpha(t) = (sin\psi\cos(t\phi)\,, sin\psi\sin(t\phi)\,, cos\psi)$$

Lets find RMM motion with orbit $\alpha(t)$ of the point $e_1 = (1,0,0)$. Since

$$\alpha'(t) = (-\phi sin\psi\sin(t\phi)\,, sin\psi\cos(t\phi)\,, 0)$$

$$\|\alpha'(t)\| = \phi sin\psi = m$$

$$T = (-\sin(t\phi)\,, \cos(t\phi)\,, 0)$$

$$S = (-cos\psi\cos(t\phi)\,, -cos\psi\sin(t\phi)\,, sin\psi)$$

and $S' = -nT$, we get $n = \phi cos\psi$. Also the geodesic curvature is $\frac{n}{m} = k_g = cot\psi$. Darboux vector field of the frame $\{\alpha, M_1, M_2\}$ is

$$W = mS = \phi sin\psi S$$

$$= (-\cos(t\phi)\,\phi sin\psi cos\psi, -sin(t\phi)\phi sin\psi cos\psi, \phi sin^2\psi + \phi)$$

Now we will examine the matrix of the motion RMM. Since

$$U(t) = [M_1 \quad M_2 \quad \alpha]$$

$$M_1 = cos\theta T + sin\theta S$$

$$M_2 = -sin\theta T + cos\theta S$$

and

$$\theta = -\int \phi cos\psi dt$$

$$= -\phi cos\psi \int dt$$

$$= -\,\phi cos\psi t$$

We have obtained that

$$U(\mathrm{t}) = \begin{bmatrix} \sin\psi\cos(t\phi) & -\cos\theta\sin(t\phi) - \sin\theta\cos\psi\cos(t\phi) & \sin\theta\sin(t\phi) - \cos\theta\cos\psi\cos(t\phi) \\ \sin\psi\sin(t\phi) & \cos\theta\cos(t\phi) - \sin\theta\cos\psi\sin(t\phi) & -\sin\theta\cos(t\phi) - \cos\theta\cos\psi\sin(t\phi) \\ \cos\psi & \sin\theta\sin\psi & \cos\theta\sin\psi \end{bmatrix} \quad (12)$$

Hence matrix $U(t)$ gives us RMM and $U(t)e_1 = \alpha(t)$. In addition $\|w(t)\| = m$ is the minimum angular speed.

3. SPHERICAL MOTION AND QUATERNIONIC HELICES

Let $A(t) = [U_1 \quad U_2 \quad U_3]$ is orthogonal matrix. $\dot{A}A^T = W(t)$ is angular velocity matrix. And vector $W(t)$, according to the matrix $W_{3\times3}$, is the darboux vector. We can write,

$$W \times U_1 = U_1{}'$$

$$W \times U_2 = U_2{}'$$

$$W \times U_3 = U_3{}'$$

There is unit quaternion $Q(t)$ as if $AX = Q(t) \times X \times \bar{Q}(t)$, (quaternion product), hence $\frac{\dot{Q}}{\|\dot{Q}\|} = N_1(t)$ and

$$2\dot{Q}(t)\,\bar{\mathrm{Q}}(\mathrm{t}) = W(\mathrm{t}),$$

$$W_0(t) = \frac{W(t)}{\|W(t)\|},$$

$$\frac{W_0{}'(\mathrm{t})}{\left\|W_0{}'(\mathrm{t})\right\|} = n$$

Under the conditions

$$\begin{aligned} \boldsymbol{W_0} \times \boldsymbol{Q} &= \boldsymbol{N_1} \\ \boldsymbol{n_1} \times \boldsymbol{Q} &= \boldsymbol{N_2} \\ \boldsymbol{n_2} \times \boldsymbol{Q} &= \boldsymbol{N_3} \end{aligned} \quad (13)$$

We get the following quaternionic frame

$$\begin{bmatrix} Q \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}' = \begin{bmatrix} 0 & K & 0 & 0 \\ -K & 0 & k & 0 \\ 0 & -k & 0 & r-K \\ 0 & 0 & K-r & 0 \end{bmatrix} \begin{bmatrix} Q \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}$$

Theorem 3. Let $\gamma = \gamma(t)$ be quaternionic curve in $\mathbb{R}^4$ with unit speed and non zero curvatures $K(t), k(t)$ and $r(t) - K(t)$. The curve

$$\gamma(t) = \int Q(t)dt$$

is a helix with the axis

$$U = \cos\theta\left[Q + \frac{K}{k}N_2 + \frac{1}{r-K}\left(\frac{K}{k}\right)' N_3\right]$$

in $\mathbb{R}^4$ if and only if

$$\left(\frac{K}{k}\right)^2 + \frac{1}{(r-K)^2}\left(\left(\frac{K}{k}\right)'\right)^2 = c \tag{14}$$

($c \in \mathbb{R}$) (D.W. Yoon) [5].

Example 2. Let $\alpha: I \longrightarrow E^3$, $t \longrightarrow \alpha(t)$ be a curve with Frenet frame $\{T, N, B\}$. $A(t) = [T \quad N \quad B]$ is orthogonal matrix. For

$$W = \tau T + \kappa B$$
$$W_0 = \frac{\tau}{\sqrt{\kappa^2+\tau^2}}T + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}B$$

and $C = W_0 \times N$, we get

$$\begin{bmatrix} W_0 \\ -C \\ N \end{bmatrix}' = \begin{bmatrix} 0 & g & 0 \\ -g & 0 & f \\ 0 & -f & 0 \end{bmatrix}\begin{bmatrix} W_0 \\ -C \\ N \end{bmatrix}$$

where $f = \sqrt{\tau^2+\kappa^2}$, $g = \tau f$. Also the curvatures are

$$K = \frac{\|W_0\|}{2} = \frac{f}{2}$$

$$k = g$$

$$r = f$$

Hence we can say that $\int Q(s)\,ds$ is helix $\Leftrightarrow \frac{1}{4}\left(\frac{f}{g}\right)^2 + \frac{1}{f^2}\left(\left(\frac{f}{g}\right)'\right)^2 = c$, ($c \in \mathbb{R}$).

Proposition 2. Let $\alpha: I \longrightarrow E^3$

$$\kappa = r\cos t$$

$$\tau = r\sin t$$

and constant precession slant helix curve.

For the constants f and g , $\int Q(\mathrm{s})\,\mathrm{ds}$ is a CCR curve, because the curvatures K, k, r are constants and their rations are constants too [6].

Proposition 3. Let $\frac{f}{g} = \lambda$, and α be a slant helix. Then $\int Q(\mathrm{s})\,\mathrm{ds}$ is a CCR curve in $\mathbb{R}^4$ [7]. Actually we have

$$K = k_1 = \frac{f}{2}$$

$$k_2 = g$$

$$k_3 = r - K = f - \frac{f}{2}$$

$$= K$$

Since

$$\frac{k_1}{k_2} = \frac{f}{2g} = c\,,\ (c \in R)$$

$$\frac{k_2}{k_3} = \frac{2g}{f} = c\,,\ (c \in \mathbb{R})$$

where k_1, k_2, k_3 are the curvatures of the curve $\int Q(s)ds$. We can say that $\int Q(\mathrm{s})\,\mathrm{ds}$ is a CCR curve in $\mathbb{R}^4$.

4. RMM MOTION AND QUATERNIONIC HELICES

In this section we will produce RMM motion with the help of a given curve in $\mathbb{R}^3$. Further, we will give the characterization of the being a helix according to this RMM motion. Let $\beta(t) \subset \mathbb{R}^3$ be a curve and $\{T, M_1, M_2\}$ be a RMF frame on the curve β. Frame $\{T, M_1, M_2\}$ is the RMF frame on the $\int T ds$. Since $U e_1 = T = \beta(t)$, and $U = [T \quad M_1 \quad M_2]$, U give us the motion RMM which is the orbit T of the point e_1. Hence Darboux vector of the motion RMM is $w = \kappa B$. Where $w_0 = B$ and w_0 is the unit of w. Now we can give the following theorem.

Theorem 4. Let $\beta(t) \subset E^3$ be a curve which is not helix. On the curve β, let Q be a quaternionic curve based on the frame RMF, then $\int Q(t)dt$ is a helix if and only if

$$\frac{1}{4}\left(\frac{\kappa}{\tau}\right)^2 + \frac{1}{\kappa^2}\left(\left(\frac{\tau}{\kappa}\right)'\right)^2 = c^2,\ \left(\frac{\tau}{\kappa} \neq c\right)$$

Example 3. For the helix $\alpha(\mathrm{t}) = \left(\frac{\sqrt{3}}{2}\sin t, \frac{-\sqrt{3}}{2}\cos t, \frac{1}{2}t\right)$, let's find

i. the RMM motion with the orbit tangential imagine T
ii. Q quaternion
iii. Quaternionic frame and quaternionic curvatures
iv. the CCR curve in $\mathbb{R}^4$ for $W = \frac{\sqrt{3}}{2}B$,

i. Let's find the RMM motion with the orbit tangential imagine T, then it is trivial that Frenet frame $\{T, N, B\}$ is

$$T(t)=\left(\frac{\sqrt{3}}{2}\cos t,\frac{\sqrt{3}}{2}\sin t,\frac{1}{2}\right)$$

$$\mathrm{N}(t)=\left(-\sin t,\cos t,0\right)$$

$$\mathrm{B}(t)=\left(-\frac{1}{2}\cos t,-\frac{1}{2}\sin t,\frac{\sqrt{3}}{2}\right)$$

$$\kappa(t)=\frac{\sqrt{3}}{2}$$

$$\tau(t)=\frac{1}{2}$$

$$\theta=\int\frac{1}{2}dt=\frac{1}{2}t$$

Hence Frenet frame $\{T, N, B\}$ can be written as RMF frame $\{T, M_1, M_2\}$ and

$$M_1=\cos\theta N-\sin\theta B$$

$$M_2=\sin\theta N+\cos\theta B$$

For RMF frame, $A=\left[-M_2 \quad M_1 \quad T\right]$, the orbit of the point $e_3=(0,0,1)$ is T. The rotation matrix of frame RMF $\{T, M_1, M_2\}$ is

$$A(t)=\begin{bmatrix} \sin(t)\sin\left(\frac{t}{2}\right)+\frac{1}{2}\cos(t)\cos\left(\frac{t}{2}\right) & -\cos\left(\frac{t}{2}\right)\sin(t)-\frac{1}{2}\cos(t)\sin\left(\frac{t}{2}\right) & \frac{\sqrt{3}}{2}\cos(t) \\ -\cos t\sin\left(\frac{t}{2}\right)+\frac{1}{2}\sin(t)\cos\left(\frac{t}{2}\right) & \cos(t)\cos\left(\frac{t}{2}\right)-\frac{1}{2}\sin(t)\sin\left(\frac{t}{2}\right) & \frac{\sqrt{3}}{2}\sin t \\ -\frac{\sqrt{3}}{2}\cos\left(\frac{t}{2}\right) & \frac{\sqrt{3}}{2}\sin\left(\frac{t}{2}\right) & \frac{1}{2} \end{bmatrix} \tag{15}$$

ii. Let's find unit quaternion Q which give us spherical motion $A(t)$ in (15) using the equality $AX=Q(t)\times X\times\overline{Q}(t)$ and $X\in\mathbb{R}^3$ we can find quaternion Q as in the following way

$$Q=\left(\frac{\sqrt{3}}{2}\cos\left(\frac{t}{4}\right),-\frac{1}{2}\sin\left(\frac{3t}{4}\right),\frac{1}{2}\cos\left(\frac{3t}{4}\right),\frac{\sqrt{3}}{2}\sin\left(\frac{t}{4}\right)\right)$$

iii. If we use the method is given (13), we can give the Quaternionic frame

$$Q=\left(\frac{\sqrt{3}}{2}\cos\left(\frac{t}{4}\right),-\frac{1}{2}\sin\left(\frac{3t}{4}\right),\frac{1}{2}\cos\left(\frac{3t}{4}\right),\frac{\sqrt{3}}{2}\sin\left(\frac{t}{4}\right)\right)$$

$$N_1=\left(-\frac{1}{2}\sin\left(\frac{t}{4}\right),-\frac{\sqrt{3}}{2}\cos\left(\frac{3t}{4}\right),-\frac{\sqrt{3}}{2}\sin\left(\frac{3t}{4}\right),\frac{1}{2}\cos\left(\frac{t}{4}\right)\right)$$
$$N_2=\left(\frac{1}{2}\cos\left(\frac{t}{4}\right),\frac{\sqrt{3}}{2}\sin\left(\frac{3t}{4}\right),-\frac{\sqrt{3}}{2}\cos\left(\frac{3t}{4}\right),\frac{1}{2}\sin\left(\frac{t}{4}\right)\right)$$
$$N_3=\left(-\frac{\sqrt{3}}{2}\sin\left(\frac{t}{4}\right),-\frac{1}{2}\cos\left(\frac{3t}{4}\right),\frac{1}{2}\sin\left(\frac{3t}{4}\right),\frac{\sqrt{3}}{2}\cos\left(\frac{t}{4}\right)\right)$$

and easily quaternionic curvatures.

$$K=\frac{|w|}{2}=\frac{\sqrt{3}}{4}$$
$$k=\tau=\frac{1}{2}$$
$$r=\kappa=\frac{\sqrt{3}}{2}$$
$$r-K=\frac{\sqrt{3}}{4}=K$$
$$w_0=B$$
$$n_1=\frac{B'}{|B'|}=-N$$
$$n_2=T$$
$$\begin{bmatrix}B\\-N\\T\end{bmatrix}'=\begin{bmatrix}0&\tau&0\\-\tau&0&\kappa\\0&-\kappa&0\end{bmatrix}\begin{bmatrix}B\\-N\\T\end{bmatrix}$$

Hence,

$$K=\frac{\kappa}{2}$$
$$k=\tau$$
$$r=\kappa$$

where $K,k,r-K$ are the curvatures of the quaternionic curve $Q(t)$. Since

$$Q\times B=N_1$$
$$Q\times -N=N_2$$
$$Q\times T=N_3$$

We produced quaternionic frame. Hence we have had the curvatures of the quaternionic curve in 3-dimensional space as

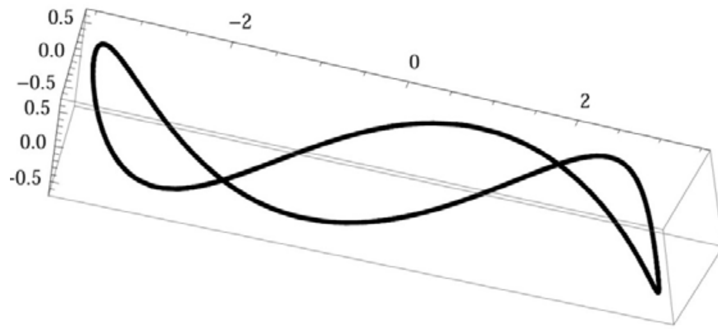

Figure 1. projection of the CCR curve $\gamma(t)$.

$$K = k_1 = \frac{\sqrt{3}}{4}$$
$$k = k_2 = \frac{1}{2}$$
$$r - K = k_3 = \frac{\sqrt{3}}{4}$$

where k_1, k_2, k_3 are the Frenet curvatures of the curve $\int Q(t)dt$.

iv. To find the CCR curve in $\mathbb{R}^4$, since $\frac{k_1}{k_2} = \frac{\sqrt{3}}{2}$ and $\frac{k_2}{k_3} = \frac{2}{\sqrt{3}}$, the curve $\gamma(\mathrm{t}) = \int Q(t)dt$ is a CCR curve as

$$\gamma(\mathrm{t}) = \left(2\sqrt{3}\sin\left(\frac{t}{4}\right), \frac{2}{3}\cos\left(\frac{3t}{4}\right), \frac{2}{3}\sin\left(\frac{3t}{4}\right), -2\sqrt{3}\cos\left(\frac{t}{4}\right)\right)$$

5. CONCLUSION

Motions have an important place in kinematics and physics, Especially, through connecting with the motion of the particle in a given spacetime, physical events are modeled by the geometric equivalences. In robot kinematics, Giving the robot motions with minimum angular speed provides convenience. In the determination of these movements, B. Jüttler used quaternions in his work "Rotating Minimizing Spherical Motions" [1]. In this study, we obtained a rotating minimizing motion (RMM) with the help of a rotating minimizing frame (RMF). We have investigated how to obtain quaternionic helices if orbital curves are specially selected and examined their characterization. In the future, this work for Euclidean space can be generalized to Minkowski space.

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