ORIGINAL PAPER

INTERSECTIONS OF TWO RULED SURFACES CORRESPONDING TO CURVES ON THE UNIT DUAL SPHERE

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Abstract. *In this study, considering two different curves on the unit dual sphere, $D\mathbb{S}^2$, we investigate the intersection of two different ruled surfaces in $\mathbb{R}^3$ by using E. Study mapping. The conditions for the intersection of these ruled surfaces in $\mathbb{R}^3$ are expressed by theorems with bivariate functions. Finally, some examples are given to support the main results.*

Keywords: *dual space; ruled surface; bivariate functions; the intersection of ruled surfaces*

1. INTRODUCTION

The theory of surfaces has an important area in differential geometry, see [1, 2]. Moreover, it has several applications in engineering, architecture, physics, surface modelling, etc. Especially, the ruled surface, which is obtained by moving a line along the curve, has many applications and its geometric interpretation has been studied by a lot of authors such as [3-7]. However, there are many researches about the theory of surfaces while there are a little researches about the intersection of surfaces in literature. Hence, the surface/surface intersection problem has attracted significiant research attention in geometry. In [8], algorithms for computing the differential geometric properties of intersection curves of two surfaces, where the combination of two surfaces can be parametric–parametric, implicit–implicit and parametric–implicit, are represented in detail. In [9], the unit tangent vector of the tangential intersection curve of two surfaces in all three types of SSI problems have been found. Then the geodesic torsion of the intersection curve have been calculated. In [10], the intersection of ruled surfaces have been denoted. Furthermore, to each connected component of the surface intersection curve corresponds a connected component in the zero-set except for some singular points, redundant solutions and degenerate cases.

Dual numbers, which are hypercomplex numbers, were introduced by W.K. Clifford in 1873. Later, E. Study established the relation between the geometry of lines and the unit dual sphere with mapping, which is called E. Study mapping. In E. Study mapping, there exists a one-to-one correspondence between the oriented lines in Euclidean space and the points on the unit dual sphere, $D\mathbb{S}^2$. For detailed information about the properties of dual numbers, see [11]. Using the E. Study mapping, the curve on the unit dual sphere corresponds to the ruled surfaces in Euclidean space, see [12, 13]. In [12], surfaces are defined by using dual vectors and line transformations. Also, a new perspective is given for the transformation of parametrically surfaces. In [13], some properties about the ruled surface generated by the

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natural lift curve, which is the curve obtained by the end points of the unit tangent vectors of the main curve, have been examined.

In this study, firstly, considering E. Study mapping, for any given two separate dual curves on the unit dual sphere, $D\mathbb{S}^2$, two corresponding ruled surfaces in $\mathbb{R}^3$ are given. In addition, the parameter curves of these ruled surfaces are presented. Then, some basic theorems are proved for the intersection of parameter curves with bivariate functions. Then, some examples are given to support the main results.

This study is organized as follows: In Section 2, some basic definitions and theorems about dual vectors, ruled surfaces and the intersection of two ruled surface are denoted. In Section 3, the intersections of the ruled surfaces corresponding to two different curves on the unit dual sphere, $D\mathbb{S}^2$, is represented. Then, some examples are given. In Section 4, obtained results are discussed.

2. PRELIMINARIES

In this section, basic definitions and concepts about dual vectors, ruled surfaces, and the intersection of two ruled surfaces are represented, respectively.

The set of dual numbers is defined as

$$ID = \{P = p + \varepsilon p^* : (p, p^*) \in \mathbb{R} \times \mathbb{R},\ \varepsilon^2 = 0\}. \tag{1}$$

Here, p and p^* are real and dual components of P , respectively. If $\vec{p}$ and $\vec{p}^*$ are vectors in $\mathbb{R}^3$, then $\vec{P} = \vec{p} + \varepsilon \vec{p}^*$ is called dual vectors. Let $\vec{P} = \vec{p} + \varepsilon \vec{p}^*$ and $\vec{R} = \vec{r} + \varepsilon \vec{r}^*$ be dual vectors. The addition, the inner product and the vector product are shown as follows.

The addition is

$$\vec{P} + \vec{R} = (\vec{p} + \vec{r}) + \varepsilon(\vec{p}^* + \vec{r}^*),$$

and the inner product is

$$\vec{P} \cdot \vec{R} = \langle \vec{p}, \vec{r} \rangle + \varepsilon(\langle \vec{p}^*, \vec{r} \rangle + \langle \vec{p}, \vec{r}^* \rangle).$$

The vector product is also presented as

$$\vec{P} \times_D \vec{R} = \vec{p} \times \vec{r} + \varepsilon(\vec{p}^* \times \vec{r} + \vec{p} \times \vec{r}^*).$$

Moreover, the norm of $\vec{P} = \vec{p} + \varepsilon \vec{p}^*$ is given as

$$\|\vec{P}\| = \sqrt{\vec{p} \cdot \vec{p}} + \frac{\vec{p} \cdot \vec{p}^*}{\sqrt{\vec{p} \cdot \vec{p}}}.$$

The norm of $\vec{P} = \vec{p} + \varepsilon \vec{p}^*$ is valid only for $\vec{p} \neq 0$. If the norm of $\vec{P}$ is equal to 1, then $\vec{P}$ is called a unit dual vector. The unit dual sphere which consists of all unit dual vectors are shown as

$$D\mathbb{S}^2 = \{\vec{P} = \vec{p} + \varepsilon \vec{p}^* : \|\vec{P}\| = 1\}. \tag{2}$$

For more information on dual vectors, see [11]. The ruled surface is defined as a surface, which has been swept out by a straight line L moving along a curve β. The various positions of the generating line L are called the rulings of the surface. Such surfaces always have a parametrization given below:

$$\phi(w,s) = \beta(w) + se(w), \tag{3}$$

where β is called the base curve and e is called the director curve, see [1].

Theorem 1. (E. Study mapping) There exists a one-to-one correspondence between the oriented lines in $\mathbb{R}^3$ and the points on $D\mathbb{S}^2$, see [11].

Theorem 2. Let $\bar{\alpha}(w) = \alpha(w) + \varepsilon\alpha^*(w)$ be the curve on the unit dual sphere $D\mathbb{S}^2$. In $\mathbb{R}^3$, the ruled surface generated by the curve on the unit dual sphere is shown as

$$\phi(w,s) = \alpha(w) \times \alpha^*(w) + s\alpha(w), \tag{4}$$

where $C(w) = \alpha(w) \times \alpha^*(w)$ is the base curve of the ruled surface $\phi(w,s)$, see [12,13].

Now, we will present some expressions for the intersection curve of two ruled surfaces in $\mathbb{R}^3$. Let $\phi^A(w_1,s)$ and $\phi^B(w_2,t)$ be two ruled surfaces defined by

$$\phi^A(w_1,s) = \beta_1(w_1) + se_1(w_1), \tag{5}$$

$$\phi^B(w_2,t) = \beta_2(w_2) + te_2(w_2). \tag{6}$$

where, $\beta_1(w_1)$ and $\beta_2(w_2)$ are the base curves of surfaces $\phi^A(w_1,s)$ and $\phi^B(w_2,t)$, respectively. Also, $e_1(w_1)$ and $e_2(w_2)$ are the directions of the surfaces $\phi^A(w_1,s)$ and $\phi^B(w_2,t)$, respectively.

The w_1-parameter curve of $\phi^A(w_1,s)$ with a constant s_0-parameter and the w_2-parameter curve of $\phi^B(w_1,t)$ with a constant t_0 -parameter are indicated by $L_A(w_1) = \phi^A(w_1,s_0)$ and $L_B(w_2) = \phi^B(w_1,t_0)$, respectively, see [10]. As $\phi^A(w_1,s)$ and $\phi^B(w_2,t)$ ruled surfaces intersect,

$$\phi^A(w_1,s) = \phi^B(w_2,t),$$

and we obtain

$$\beta_1(w_1) - \beta_2(w_2) = -se_1(w_1) + te_2(w_2).$$

Since these three vectors are linearly dependent, the following equation can be presented:

$$\mu(w_1,w_2) = det\{e_1(w_1), e_2(w_1), [\beta_1(w_1) - \beta_2(w_2)]\},$$

see [10].

Theorem 3. Let $\phi^A(w_1,s)$ and $\phi^B(w_2,t)$ be two ruled surfaces in $\mathbb{R}^3$. If

$$\mu(w_1,w_2) = det\{e_1(w_1), e_2(w_2), [\beta_1(w_1) - \beta_2(w_2)]\} = 0,$$

Then the parameter curves $L_A(w_1)$ and $L_B(w_2)$ intersect, see [10].

For $\mu(w_1, w_2) = 0$, the parameter curves $L_A(w_1)$ and $L_B(w_2)$ intersect. However, there are some points in the solution set of the equation $\mu(w_1, w_2) = 0$ that is not at the intersection of these two ruled surfaces. The intersection includes these points when the directors of the ruled surfaces $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ are parallel. Thus, in the case of parallelism of the direction vectors of two ruled surfaces, there may be some points that are not at the intersection of two ruled surfaces, see [10].

The parallelism of the direction vectors of the ruled surfaces $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ is represented by bivariate functions as follows:

$$\Theta(w_1, w_2) = \|e_1(w_1) \times e_2(w_2)\|^2,$$

see [10].

The parallelism of $\beta_1(w_1) - \beta_2(w_2)$ and the direction vector of $\phi^A(w_1, s)$ are shown with bivariate functions as follows:

$$\theta_1(w_1, w_2) = \|e_1(w_1) \times [\beta_1(w_1) - \beta_2(w_2)]\|^2,$$

see [10].

Similarly, the parallelism of $\beta_1(w_1) - \beta_2(w_2)$ and the direction vector of the $\phi^B(w_2, t)$ are shown with bivariate functions as follows:

$$\theta_2(w_1, w_2) = \|e_2(w_2) \times [\beta_1(w_1) - \beta_2(w_2)]\|^2,$$

see [10].

Theorem 4. The parameter curves $L_A(w_1)$ and $L_B(w_2)$ intersect if and only if

$$\Theta(w_1, w_2) = \theta_1(w_1, w_2) = \theta_2(w_1, w_2) = 0,$$

see [10].

For more detailed information on the intersection of two ruled surfaces, see [10].

3. INTERSECTIONS OF TWO RULED SURFACES CORRESPONDING TO CURVES ON THE UNIT DUAL SPHERE

In this section, the intersections of the ruled surfaces corresponding to two different curves on the unit dual sphere $D\mathbb{S}^2$ are examined and some examples are given to support the main results. Let $\bar{\alpha}_1 = \alpha_1 + \varepsilon\alpha_1^*$ and $\bar{\alpha}_2 = \alpha_2 + \varepsilon\alpha_2^*$ be curves on the unit dual sphere $D\mathbb{S}^2$. With the help of E. Study mapping, let $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ be the ruled surfaces corresponding to these curves given above in $\mathbb{R}^3$. These ruled surfaces are denoted as follows:

$$\phi^A(w_1, s) = \alpha_1(w_1) \times \alpha_1^*(w_1) + s\alpha_1(w_1) \tag{7}$$

and

$$\phi^B(w_2, t) = \alpha_2(w_2) \times \alpha_2^*(w_2) + t\alpha_2(w_2). \tag{8}$$

Here, the base curves of the ruled surfaces $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ are

$$C_A = \alpha_1(w_1) \times \alpha_1^*(w_1)$$

and

$$C_B = \alpha_2(w_2) \times \alpha_2^*(w_2).$$

The w_1-parameter curve of $\phi^A(w_1, s)$with a constant s_0-parameter is presented by the following equation:

$$K_A(w_1) = \phi^A(w_1, s_0).$$

Similarly, the w_2-parameter curve of the surface $\phi^B(w_2, t)$ with a constant t_0-parameter is presented by the following equation:

$$K_B(v) = \phi^B(w_2, t_0).$$

As $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ intersect,

$$\phi^A(w_1, s) = \phi^B(w_2, t),$$

and we write

$$C_A(w_1) - C_B(w_2) = -s\alpha_1(w_1) + t\alpha_2(w_2).$$

Since these three vectors are linearly dependent, the following equation can be presented

$$\xi(w_1, w_2) = \det\{\alpha_1(w_1), \alpha_2(w_2), [\mathrm{C}_A(w_1) - \mathrm{C}_B(w_2)]\ \} = 0.$$

Theorem 5. Let $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ be the two ruled surfaces corresponding to two different curves on the unit dual sphere in $\mathbb{R}^3$. If

$$\xi(u, v) = \det\{\alpha_1(w_1), \alpha_2(w_2), [\mathrm{C}_A(w_1) - \mathrm{C}_B(w_2)]\ \} = 0,$$

then the parameter curves $K_A(w_1)$ and $K_B(w_2)$ intersect.

For $\xi(w_1, w_2) = 0$, the parameter curves $K_A(w_1)$ and $K_B(w_2)$ intersect. However, there are some points in the solution set of the equation $\xi(w_1, w_2) = 0$ that is not at the intersection of these two ruled surfaces. The intersection includes these points as the base curves of the ruled surfaces $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ are parallel. Thus, in the case of parallelism of the direction vectors of two ruled surfaces, there may be some points that are not at the intersection of two ruled surfaces, see [10].

According to the parallelism of these three vectors with each other, the bivariate functions are represented as follows, respectively:

$$\Gamma(w_1, w_2) = \|\alpha_1(w_1) \times \alpha_2(w_2)\|^2,$$

$$\gamma_1(w_1, w_2) = \|\alpha_1(w_1) \times [C_A(w_1) - C_B(w_2)]\|,^2$$

$$\gamma_2(w_1, w_2) = \|\alpha_2(w_2) \times [C_A(w_1) - C_B(w_2)]\|.^2$$

Theorem 6. Let $K_A(w_1)$ and $K_B(w_2)$ be the parameter curves of the ruled surfaces $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$, which correspond to two different curves on the unit dual sphere. In this case, the parameter curves $K_A(w_1)$ and $K_B(w_2)$ intersect if and only if

$$\Gamma(w_1, w_2) = \gamma_1(w_1, w_2) = \gamma_2(w_1, w_2) = 0.$$

Example 1: Let $\alpha_1(w_1)=\left(-\frac{1}{\sqrt{2}}\cos w_1,\ \frac{1}{\sqrt{2}}\sin w_1, \frac{1}{\sqrt{2}}\right)$, $\alpha_1^*(w_1)=\left(\frac{1}{\sqrt{2}}\ \sin w_1\ , \frac{1}{\sqrt{2}}\cos w_1\ , 0\ \right)$, $\alpha_2(w_2)=\left(\frac{1}{\sqrt{2}}\ \sin w_2,\ \frac{1}{\sqrt{2}}\cos w_2\ , \frac{1}{\sqrt{2}}\right)$ and $\alpha_2^*(w_2)=\left(\frac{1}{\sqrt{2}}\cos w_2\ , -\frac{1}{\sqrt{2}}\sin w_2\ , 0\right)$ be vectors in $\mathbb{R}^3$. Since $|\alpha_1(w_1)|=1$ and $\langle \alpha_1(w_1), \alpha_1^*(w_1)\rangle=0$, the $\bar{\alpha}_1(w_1)=\alpha_1(w_1)+\varepsilon\alpha_1^*(w_1)$ is on the unit dual sphere $D\mathbb{S}^2$. Similarly, the $\bar{\alpha}_2(w_2)=\alpha_2(w_2)+\varepsilon\alpha_2^*(w_2)$ is on the unit dual sphere $D\mathbb{S}^2$. With the help of E. Study mapping, the ruled surface corresponding to the $\bar{\alpha}_1(w_1)=\alpha_1(w_1)+\varepsilon\alpha_1^*(w_1)$ is

$$\phi^A(w_1,s)=\left(-\frac{1}{2}\cos w_1,\ \frac{1}{2}\sin w_1, -\frac{1}{2}\right)+s\left(-\frac{1}{\sqrt{2}}\cos w_1, \frac{1}{\sqrt{2}}\sin w_1, \frac{1}{\sqrt{2}}\right).$$

Here, the base curve and director of this ruled surface are

$$C_A(w_1)=\left(-\frac{1}{2}\cos w_1\ , \frac{1}{2}\sin w_1, -\frac{1}{2}\right)$$

and

$$\alpha_1(w_1)=\left(-\frac{1}{\sqrt{2}}\cos w_1,\ \frac{1}{\sqrt{2}}\sin w_1, \frac{1}{\sqrt{2}}\right).$$

Similarly, the ruled surface corresponding to the $\bar{\alpha}_2(w_2)=\alpha_2(w_2)+\varepsilon\alpha_2^*(w_2)$ is

$$\phi^B(w_2,t)=\left(\frac{1}{2}\sin w_2\ , \frac{1}{2}\cos w_2\ , -\frac{1}{2}\right)+t\left(\frac{1}{\sqrt{2}}\sin w_2\ , \frac{1}{\sqrt{2}}\cos w_2\ , \frac{1}{\sqrt{2}}\right).$$

Here, the base curve and director of this ruled surface are

$$C_B(w_2)=\left(\frac{1}{2}\sin w_2\ , \frac{1}{2}\cos w_2\ , -\frac{1}{2}\right)$$

and

$$\alpha_2(w_2)=\left(\frac{1}{\sqrt{2}}\ \sin w_2\ , \frac{1}{\sqrt{2}}\cos w_2\ \ , \frac{1}{\sqrt{2}}\right).$$

Let's examine the intersection of two ruled surfaces corresponding to two different curves on the unit dual sphere. Let $\phi^A(w_1,s)\ =\ \phi^B(w_2,t)$, the following equation can be written by

$$-\frac{1}{2}(\cos w_1+\sin w_2\ , -\sin w_1+\cos w_2\ , 0)=-s\frac{1}{\sqrt{2}}(-\cos w_1\ , \sin w_1\ , 1)+ t\frac{1}{\sqrt{2}}(\sin w_2, \cos w_2, 1).$$

Here, $C_A(w_1)-C_B(w_2)$ is written as a linear combination of vectors $\alpha_1(w_1)$ and $\alpha_2(w_2)$. Since these three vectors are linearly dependent, $\xi(w_1,w_2)=0$. Since $\xi(w_1,w_2)=0$, the parameter curves $K_A(w_1)$ and $K_B(w_2)$ intersect.

If $\Gamma(w_1,w_2)$, $\gamma_1(w_1,w_2)$, $\gamma_2(w_1,w_2)$ are calculated, we obtain

$$\Gamma(w_1,w_2)=-\frac{1}{4}\{(\sin(w_1-w_2)+3)(\sin(w_1-w_2)-1)\},$$

$$\gamma_1(w_1, w_2) = -\frac{1}{8}\{(\sin(w_1 - w_2) + 3)(\sin(w_1 - w_2) - 1)\},$$

$$\gamma_2(w_1, w_2) = -\frac{1}{8}\{(\sin(w_1 - w_2) + 3)(\sin(w_1 - w_2) - 1)\}.$$

For the $\xi(w_1, w_2) = 0,$ the solution of the $(w_1, w_2) = \left(\frac{\pi}{2}, 0\right)$, $\Gamma(w_1, w_2) = \gamma_1(w_1, w_2) = \gamma_2(w_1, w_2) = 0$. Therefore, the parameter curves $K_A(w_1)$ and $K_B(w_2)$ intersect. As a result, $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ ruled surfaces intersect.

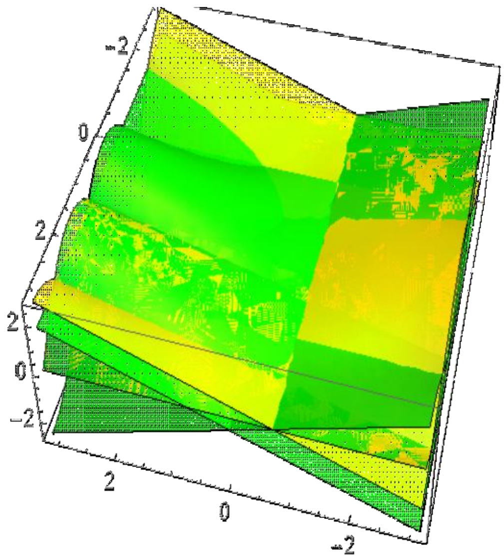

Figure 1. The intersection of two ruled surfaces.

4. CONCLUSIONS

In this paper, considering the E.Study mapping, the curve taken on the unit dual sphere corresponds to the ruled surfaces in Euclidean space. At the same time, the intersection curve of two ruled surfaces in Euclidean space is presented with the help of bivariate functions.Additionally, the intersection curve of two different ruled surfaces in Euclidean space is investigated with the help of E. Study mapping to two different curves taken on the unit dual sphere. Then, the intersection curve for the corresponding ruled surfaces is expressed by theorems. Then, these theorems are supported by an example. These results have an important research field in geometry and solid modeling.

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