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NEW ARITHMETIC FUNCTION RELATED TO THE LEAST COMMON MULTIPLE

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Abstract.*The author's last two papers concerned the study of arithmetic functions related to the greatest common divisor. In the present paper, similarly to the two previous papers we will define new multiplicative arithmetic functionsrelated to the least common multiple and we will study several interesting properties of them. Also, we will discuss the values of two special functionsof them at perfect numbers.*

Keywords*: arithmetic function; greatest common divisor; least common multiple; perfect number.*

1. INTRODUCTION

Throughout this paper, we let $\mathbb{N}^*$denote the set $\mathbb{N} \setminus \{0\}$ of positive integers and we let (m, n) and $[m, n]$denote the greatest common divisor and the least common multiple of any two integers $m$and n, respectively. Let the prime factorization of the positive integer $n > 1$ be

$$n = \prod_{i=1}^{r} p_i^{e_i}$$

where$r, e_1, e_2, \dots, e_r$ are positive integers and $p_1, p_2, \dots, p_r$ are different primes.

In number theory, a complex valued function defined on $\mathbb{N}^*$ is called an arithmetic function. Some of them have very rich properties, which allows - using them - to solve problems in arithmetic and number theory. Arithmetic functions are very important in many fields of theoretical and applied sciences, and they appear almost in all major domains of mathematics or its applications. For example, the famous Riemann hypothesis, which is one of the most difficult unsolved problems, can be stated in terms of the sum-of-divisors function (see, e.g. [1, 2]).

The author and Derbal [3], for a positive integer α presented and studied some elementary properties of the following arithmetic function:

$$f_\alpha(n) = \prod_{i=1}^{r} p_i^{(e_i, \alpha)}, f_\alpha(1) = 1, \tag{1}$$

which can be considered as a generalization of the radical function, since

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$$f_1(n) = \prod_{i=1}^{r} p_i = \text{rad}(n) \text{ for all } n.$$

Also, the author [4] discussed other properties of the functions f_α and defined new integer sequences related to them. In the present paper, similarly to [3-4] we will study new arithmetic functions g_α related the least common multiple, which will define by substituting the (e_i, α) for the $[e_i, \alpha]$ in (1), and we will discuss the values of g_2 and g_4 at perfect numbers.

2. MAIN RESULTS

Let α be a positive integer. Then we define g_α to be the arithmetic function such that:

$$\begin{cases} g_\alpha(1) = 1, \\ g_\alpha(n) = \prod_{i=1}^{r} p_i^{[e_i,\alpha]}. \end{cases}$$

It can be easily seen that g_α is a multiplicative function for all α, since

$$g_\alpha(mn) = g_\alpha(m) g_\alpha(n) \text{ whenever } (m, n) = 1. \tag{2}$$

It is not completely multiplicative, since for a prime number p:

$$g_\alpha(p^2) = \begin{cases} p^{2\alpha}, \text{ if } \alpha \text{ is odd;} \\ p^\alpha, \text{if } \alpha \text{ is even.} \end{cases}$$

while

$$g_\alpha(p) g_\alpha(p) = p^{2\alpha} \text{ for all } \alpha.$$

The next theorem gives, for all odd positive integers α, a condition for m and n (which are not necessarily co-prime) to be satisfied the equation (2).

Theorem 2.1. Let α be an odd positive integer. Then

$$g_\alpha(mn) = g_\alpha(m) g_\alpha(n)$$

for all square-free positive integers m and n.

Proof: Let m and n be square-free positive integers. Then it follows that:

$$g_\alpha(mn) = \prod_{p|m, p\nmid n} p^\alpha \prod_{q\nmid m, q|n} q^\alpha \prod_{s|m, s|n} r^{[2,\alpha]}, \tag{3}$$

where p, q, and s are prime numbers. Also, we have

$$g_\alpha(m) g_\alpha(n) = mn = \prod_{p|m, p\nmid n} p^\alpha \prod_{q\nmid m, q|n} q^\alpha \prod_{s|m, s|n} r^{2\alpha}. \tag{4}$$

The right-hand sides of (3) and (4) are equal only if $[2, \alpha] = 2\alpha$, i.e., only if α is odd, as claimed. This ends the proof.

Theorem 2.2. For each natural number $n \geq 2$, there are finitely many positive integers α such that $g_\alpha(n) = n$.

Proof: Let α be a positive integer and we let e denote the greatest common divisor of $e_1, e_2, \dots, e_r$. Then we wish to prove that $g_\alpha(n) = n$ if and only if $\alpha | e$, i.e., $\alpha | e_i (1 \leq i \leq r)$, which means that there are finitely many positive integers α such that $g_\alpha(n) = n$. Indeed

$$\begin{aligned} g_\alpha(n) = n &\Leftrightarrow \prod_{i=1}^{r} p_i^{[e_i,\alpha]} = \prod_{i=1}^{r} p_i^{e_i} \\ &\Leftrightarrow p_i^{[e_i,\alpha]} = p_i^{e_i} (1 \leq i \leq r) \\ &\Leftrightarrow [e_i, \alpha] = e_i (1 \leq i \leq r) \\ &\Leftrightarrow \alpha | e_i (1 \leq i \leq r), \end{aligned}$$

as required. The theorem is proved.

Theorem 2.3. Let α and β be positive integers. Then for every $n \in \mathbb{N}^*$ we have

$$g_\alpha\left(g_\beta(n)\right) = g_{[\alpha,\beta]}(n).$$

In particular, if $(\alpha, \beta) = 1$ then $g_\alpha\left(g_\beta(n)\right) = g_{\alpha\beta}(n)$.

*Proof:*The theorem is true for$n = 1$. If $n > 1$, then

$$\begin{aligned} g_\alpha\left(g_\beta(n)\right) &= g_\alpha\left(\prod_{i=1}^{r} p_i^{[e_i,\beta]}\right) \\ &= \prod_{i=1}^{r} p_i^{[\alpha,[e_i,\beta]]} \\ &= \prod_{i=1}^{r} p_i^{[e_i,[\alpha,\beta]]} \\ &= g_{[\alpha,\beta]}(n), \end{aligned}$$

as required. The proof is complete.

Clearly if $\alpha = 1$, then for all positive integers m and n:

$$[g_\alpha(m), g_\alpha(n)] = g_\alpha([m, n]), \tag{5}$$

since $[g_1(m), g_1(n)] = [m, n] = g_1([m, n])$. Also, this equation holds for all positive integers α when $m = n$. The next theorem gives three sufficient conditions for m and n to be satisfied the equation (5) for all α.

Theorem 2.4. The positive integers $m < n$ satisfy the equation (5) for all α if at least one of the following three conditions holds:

1. $(m, n) = 1$.
2. $\mathrm{rad}(m) = 1$or $\mathrm{rad}(n) = 1$.
3. m is a unitary divisor of n, i.e., $(m, {}^{n}/_{m}) = 1$.

Proof:
1. Assume that $(m,n)=1$. Then $[m,n]=mn$, $\left(g_\alpha(m),g_\alpha(n)\right)=1$, and $[g_\alpha(m),g_\alpha(n)]=g_\alpha(m)g_\alpha(n)$. Hence

2.
$$g_\alpha([m,n])=g_\alpha(mn)=g_\alpha(m)g_\alpha(n) \text{ since } (m,n)=1$$
$$=[g_\alpha(m),g_\alpha(n)],$$

which proves the equation (5) when $(m,n)=1$.

3. Next, suppose that $\mathrm{rad}(m)=1$, i.e., m is a square-free integer. Write $m=\prod_{i=1}^{r}p_i$and $n=\prod_{j=1}^{s}q_j^{e_j}$. It follows that
$$[m,n]=\prod_{j=1}^{s}q_j^{e_j}\prod_{p_i\nmid n}p_i \text{ and } g_\alpha([m,n])=\prod_{j=1}^{s}q_j^{[e_j,\alpha]}\prod_{p_i\nmid n}p_i{}^{\alpha}.$$

On the other hand, we have
$$[g_\alpha(m),g_\alpha(n)]=\left[\prod_{i=1}^{r}p_i{}^{\alpha},\prod_{j=1}^{s}q_j^{[e_j,\alpha]}\right]=\prod_{j=1}^{s}q_j^{[e_j,\alpha]}\prod_{p_i\nmid n}p_i{}^{\alpha},$$

which confirms the equation (5) when $\mathrm{rad}(m)=1$ or $\mathrm{rad}(n)=1$.

4. Now let m be a unitary divisor of n. Then $[m,n]=n$ and $g_\alpha([m,n])=g_\alpha(n)$. Also, we have
$$[g_\alpha(m),g_\alpha(n)]=[g_\alpha(m),g_\alpha({}^{mn}/_{m})]$$
$$=[g_\alpha(m),g_\alpha(m)g_\alpha({}^{n}/_{m})](\text{since}(m,{}^{n}/_{m})=1)$$
$$=g_\alpha(m)g_\alpha({}^{n}/_{m})=g_\alpha(n),$$

from which the validity of the equation (5), when m is a unitary divisor of n, follows. The proof of the theorem is complete.

Remark 2.5. Same reasoning as above allows us to show that:

$$\left(g_\alpha(m),g_\alpha(n)\right)=g_\alpha\left((m,n)\right)$$

when at least one of the conditions of Theorem 2.4 holds.

Theorem 2.6 Let $m<n$ be positive integers such that $p^r \parallel m$ and $p^s \parallel n$ for a prime p and positive integers r and s with $2\le r<s$and $r\nmid s$. Then

$$[g_s(m),g_s(n)]\neq g_s([m,n]).$$

*Proof:*On one hand
$$p^s\parallel n\Rightarrow p^s\parallel[m,n] \text{ (since } r<s)$$
$$\Rightarrow p^{[s,s]}=p^s\parallel g_s([m,n]).$$

On the other hand,
$$p^r\parallel m\Rightarrow p^{[s,r]}=p^{sr/(s,r)}\parallel g_s(m)$$
$$\Rightarrow p^{sr/(s,r)}|[g_s(m),g_s(n)].$$

Since $1\le(s,r)<r$, it follows that $s<sr/(s,r)$. Thus

$$p^s < p^{sr/(s,r)} || [g_s(m), g_s(n)], \text{i.e.}, [g_s(m), g_s(n)] \neq g_s([m,n]),$$

which concludes this proof.

Theorem 2.7. Let $\alpha > 1$ be a positive integer. Then the following two systems of inequalities:

$$\begin{cases} m < n \\ g_\alpha(m) < g_\alpha(n) \end{cases} \text{and} \begin{cases} m < n \\ g_\alpha(m) > g_\alpha(n) \end{cases}$$

hold for infinitely many positive integers m and n.

*Proof:*Let $\alpha > 1$ be a positive integer. For the first system, we put $m = p$ and $n = q$ where p and q are prime numbers with $p < q$. Then it follows that

$$g_\alpha(m) = g_\alpha(p) = p^\alpha < q^\alpha = g_\alpha(p) = g_\alpha(n),$$

which confirms the first system. For the second one, let p be an odd prime number and let M be the set of the positive multiples of α. Let us choose aninteger k such that$\alpha \neq k \in M$. So $[k, \alpha] = k,\ k - 1 \notin M$ since $\alpha > 1$, and$[k-1, \alpha] > k$. Now we put $m = \varphi(p^k) = (p - 1)p^{k-1}$, where φ is the totient's Euler function (see e.g.,[5, Chapter 2]) and $n = p^k$. Clearly, $m < n$ and we have

$$\begin{aligned} g_\alpha\left(\varphi(p^k)\right) &= g_\alpha((p-1)p^{k-1}) \\ &= g_\alpha(p-1) g_\alpha(p^{k-1}) \text{since} (p-1, p^{k-1}) = 1 \\ &= g_\alpha(p-1) p^{[k-1,\alpha]} \\ &> p^{[k-1,\alpha]} \text{ (since } p - 1 > 1) \\ &> p^k \text{ (since} [k-1, \alpha] > k) \\ &= p^{[k,\alpha]} = g_\alpha(p^k) \\ &\Rightarrow g_\alpha(m) > g_\alpha(n), \end{aligned}$$

which confirms the second system and completes this proof.

The author and Derbal [3, Theorem 5] discussed the values of f_2 at perfect numbers. Euler has determined all even perfect numbers (EPN), by showing that theyare of the form $2^{p-1}M$, where $M = 2^p - 1$ is a Mersenne prime (see, e.g. [6]). Also, it is well known result of Euler that the form of odd perfect numbers (OPN), if it exists, is $p^a m^2$, where p is a prime with $(p, m) = 1$ and $p \equiv a \equiv 1 \pmod 4$ (see, e.g. [6]). The next theorems give, respectively, the values of g_2 for perfect numbers and the values of g_4 for even perfect numbers.

Theorem 2.8. Let $N > 6$ be a perfect number. Then

$$\frac{g_2(N)}{N} = \begin{cases} M, \text{ if } N = 2^{p-1}M \text{ is(EPN);} \\ p^a, \text{if N} = p^a m^2 \text{ is (OPN).} \end{cases}$$

*Proof:*Let $N = 2^{p-1}M$ be an even perfect number. Then p and M be must prime numbers, from which $[p-1, 2] = p - 1$ since $N > 6$. Thus

$$g_2(N) = 2^{[p-1,2]} M^2 = 2^{p-1} M^2 = NM.$$

Now assume that $N = p^a m^2$ is an odd perfect number. Then $(p, m) = 1$ and $p \equiv a \equiv 1 (\text{mod } 4)$. So $(a, 2) = 1$, i.e., $[a, 2] = 2a$. On the other hand according to Theorem 2.2 we have $g_2(m^2) = m^2$. Hence

$$g_2(N) = p^{[a,2]}m^2 = p^{2a}m^2 = p^a N.$$

The theorem is proved.

Theorem 2.9. Let $N = 2^{p-1}M$ be an even perfect number such that $N > 28$. Then

$$\frac{g_4(N)}{N} = \begin{cases} M^3, \text{ if } p \equiv 1 \text{ (mod 4)}; \\ 2^{p-1}M^3, \text{if } p \equiv -1 \text{ (mod 4)}. \end{cases}$$

Proof: The proof follows at once from $[p-1,4] = p-1$ if $p \equiv 1$ (mod 4) and $[p-1,4] = 2(p-1)$ if $p \equiv -1$ (mod 4).

4. CONCLUSIONS

In this paper, we define new arithmetic functions g_α related the least common multiple and study some interesting properties of them. For example, we show (Theorem 2.2) that the equation $g_\alpha(n) = n$ have a finite solutions as equation on α and we prove (Theorem 2.7) that the sequence $g_\alpha(n)$ ($n \in \mathbb{N}^*$) can not be monotonically increasing or decreasing for all α. Also, we discuss (Theorems 2.8 and 2.9) the values of g_2 and g_4 at perfect numbers. The aim of our next papers will be the study their relationship with $f_\alpha$and other classical arithmetic functions. Also, we will try to find asymptotic formulas for the Dirichlet series associated with g_α and other sums related to them.

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REFERENCES

[1] Robin, G., *Journal de Mathématiques Pures et Appliquées*, **63**, 187, 1984.
[2] Largarias,J. C.,*American Mathematical Monthly*, **109,** 534, 2002.
[3] Mittou, B., Derbal, A., *Notes on Number Theory and Discrete Mathematics*, **27**, 22, 2021.
[4] Mittou, B., *Mathematica Montisnigri***, LIII**, 5, 2022.
[5] Apostol, T. M., *Introduction to Analytic Number Theory*, Springer-Verlage, New York, 1976.
[6] Dickson, L. E., *History of Theory of Numbers Vol. I*, Chelsea Publishing Company, New York, 1952.