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UNIFORM CONVERGENCE OF NONPARAMETRIC CONDITIONAL HAZARD FUNCTION IN THE SINGLE FUNCTIONAL INDEX MODELING FOR DEPENDENT DATA

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Abstract. *We study the nonparametric local linear estimation of the conditional hazard function of a scalar response variable given a functional explanatory variable, when the functional data are α-mixing dependency and we give the uniform almost complete convergence with rates of this function.*

Keywords: *dimension reduction; dependent data; single index model.*
Mathematics subject classification: *62G05; 62G20; 62G07.*

1. INTRODUCTION

The single-index models have been considered for the nonparametric kernel estimation in the multivariate case by [1-3]. In [4] the authors considered the case of functional data for the single index model, they obtained the almost complete convergence in the independent and identically distributed (i.i.d) case for regression function. Then, for the same estimator [5] established the pointwise and uniform almost complete convergence of the conditional density, moreover, the dependent case were achieved by [6, 7], where the authors proposed an estimator for the single functional index model. The contribution of this work is to study the conditional hazard in the single functional index model, for its excellence in many characteristics and due to the flexibility of the model in dimension reduction and it is widely used in econometrics fields with relation to nonparametric and parametric models.

In this paper, we estimate the conditional hazard in the single index model for dependent data of a real variable Y given a functional variable X in the local linear method (see [8]). We point out that the single functional index in this method is intimately limited until now. Concerning this model of conditional density in the kernel method for dependent observation, one we cite the work of [9-11]. Particularly, in the quasi-associated, [12] studied the pointwise almost complete convergence and the uniform almost complete convergence (with the rate) of the kernel estimate of the hazard function of a real random variable conditioned by a functional predictor, and gave a simulation to illustrate their methodology.
Our work relies on studying the conditional hazard function of a scalar response variable Y given a hilbertian random variable in a functional single-index model for the dependence case in a locally linear approach, so under certain conditions we show the uniform almost complete convergence of our estimator.

The structure of this article is as follows: in Section 2, we introduce our model and estimator; hypotheses and comments can be found in Section 3; in Section 4, we present our

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main results; section 5 gives the conclusion. Finally, we complete the proof of the results in the appendix section.

2. SINGLE INDEX MODEL

Let $\{X_i, Y_i\}_{i\in\mathbb{N}}$ be a random processes identically distributed as (X, Y) where Y_i's are valued in $\mathbb{R}$ and X_i takes values in separable Hilbert space with the norm $||.||$ generated by an inner product $<.,.>$. We assume that the regular version of the conditional probability of Y given X exists and bounded. Moreover, we denote the conditional density by $f_\theta^x(y)$ respect to Lebesgue's measure over $\mathbb{R}$. So, denote the conditional hazard function of Y given X by

$$h_\theta^x(y) = \frac{f_\theta^x(y)}{1 - F_\theta^x(y)}, \forall y \in \mathbb{R}$$

where, $F_\theta^x(y) < 1$.

The local linear estimator (see [13]) of the conditional distribution $F_\theta^x(y)$ (resp. the conditional density $f_\theta^x(y)$) is defined as follows:

$$\widehat{F}_\theta^x(y) = \frac{\sum_{1\le i,j\le n} W_{\theta,ij}(x) H({h_H}^{-1}(y - Y_j))}{\sum_{1\le i,j\le n} W_{\theta,ij}(x)}$$

$$\left(\text{resp.}\, \hat{f}_\theta^x(y) = \frac{\sum_{1\le i,j\le n} W_{\theta,ij}(x) H'({h_H}^{-1}(y - Y_j))}{h_H \sum_{1\le i,j\le n} W_{\theta,ij}(x)}\right)$$

with

$$W_{\theta,ij}(x) = \beta_\theta(X_i, x)(\beta_\theta(X_i, x) - \beta_\theta(X_j, x)) K(h_K^{-1} d_\theta(x, X_i)) K(h_K^{-1} d_\theta(x, X_j))$$

and $\beta_\theta(X_i, x) =< x - X_i, \theta >$ is a known bi-functional operator from $\mathcal{H}^2$ into $\mathbb{R}$, such that $\forall x_1, x_2 \in \mathcal{H}$, $\forall \theta \in \mathcal{H}$, d_θ is a semi-metric associated to the single index $\theta \in \mathcal{H}$ defined by $d_\theta(x_1, x_2) := | < x_1 - x_2, \theta > |$, with the kernel K.

H is a distribution function (respectively, H' is the derivative of H), and $h_K = h_{K,n}$(respectively, $h_H = h_{H,n}$) is a sequence of positive real numbers. Finally, the local linear estimator of the hazard function is given by

$$\hat{h}_\theta^x(y) = \frac{\hat{f}_\theta^x(y)}{1 - \widehat{F}_\theta^x(y)}.$$

Now, we define the definition of α mixing sequence. The sequence is said to be α-mixing (strong mixing), if the mixing coefficient $\alpha(n) \xrightarrow{n\to\infty} 0$ such that

$$\alpha(n) = \sup\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| : A \in \mathfrak{I}_1^k \text{ and } B \in \mathfrak{I}_{k+n}^\infty, k \in \mathbb{N}^*\}$$

and $\mathfrak{I}_i^k$ denote the σ-algebra generated by the random variables $\{(X_i, Y_i), j \le i \le k\}$.

3. ASSUMPTIONS

In the rest of the paper, we denote by C, C', C'' and $C_{\theta,x}$ some strictly positive constants. $\forall x \in \mathcal{H}$ and $i, j = 1, ., , n$,

$$K_{\theta,i}(x) = K(h_K^{-1} d_\theta(x, X_i)) \text{ and } \forall y \in \mathbb{R}, H_j(y) = H({h_H}^{-1}(y - Y_j)).$$

On the other hand, we denote x a fixed point in $\mathcal{H}$, $\mathcal{N}_x$ is a fixed neighborhood of x and $S_\mathbb{R}$ is a fixed compact of $\mathbb{R}$. We consider the following cover of the compacts $S_\mathcal{H}$ and $\Theta_\mathcal{H}$:

$$S_\mathcal{H} \in \bigcup_{j=1}^{N^{S_\mathcal{H}}} B(x_j, r_n) \text{ and } \Theta_\mathcal{H} \in \bigcup_{j''=1}^{N^{\Theta_\mathcal{H}}} B(t_{j''}, r_n)$$

and $\forall x \in \mathcal{H}, \forall \theta \in \Theta_\mathcal{H}$, we set

$$\boldsymbol{j}(\boldsymbol{x}) = \mathbf{arg} \min_{\boldsymbol{j} \in \{\mathbf{1},\ldots,\boldsymbol{N}^{\boldsymbol{S}_\mathcal{H}}\}} ||\boldsymbol{x} - \boldsymbol{x}_{\boldsymbol{j}}|| \ \mathbf{and} \boldsymbol{j}''(\boldsymbol{\theta}) = \mathbf{arg} \min_{\boldsymbol{j}'' \in \{\mathbf{1},\ldots,\boldsymbol{N}^{\boldsymbol{\Theta}_\mathcal{H}}\}} ||\boldsymbol{\theta} - \boldsymbol{t}_{\boldsymbol{j}''}||$$

where $x_j, t_{j''} \in \mathcal{H}$ and r_n is a sequence of positif numbers.

Suppose that $N^{S_\mathcal{H}}$ and $N^{\Theta_\mathcal{H}}$ are the minimal numbers of open balls with radius r_n in $\mathcal{H}$ which are required to cover $S_\mathcal{H}$ and $\Theta_\mathcal{H}$. We introduce the following assumptions to determine our main result.

$$\forall \boldsymbol{h_K} > \mathbf{0}, \mathbb{P}(| < \boldsymbol{X} - \boldsymbol{x}, \boldsymbol{\theta} > | < \boldsymbol{h_K}) = \boldsymbol{\phi_{\theta,x}}(\boldsymbol{h_K}) > 0.$$

(U1) There exists a differentiable function $\phi(\cdot)$ such that $\forall x \in S_\mathcal{H}$, and $\forall \theta \in \Theta_\mathcal{H}$

$$0 < C\phi(h_K) \leq \phi_{\theta,x}(h_K) \leq C'\phi(h_K) < \infty \text{ and } \exists \eta_0 > 0, \forall \eta < \eta_0, \phi'(\eta) < 0,$$

where ϕ' is the first derivative function of ϕ and $\phi(0) = 0$.

(U2) The functions F_θ^x and f_θ^x satisfy:

$$\begin{cases} \exists a_1, a_2 > 0, \forall (y, y') \in S_\mathbb{R}^2, \forall (x, x') \in \mathcal{N}_x \times \mathcal{N}_x, \forall \theta \in \Theta, \\ (i)\ |f_\theta^x(y) - f_\theta^{x'}(y')| \leq C'(||x - x'||^{a_1} + |y - y'|^{a_2}), \\ (ii)\ |F_\theta^x(y) - F_\theta^{x'}(y')| \leq C''(||x - x'||^{a_1} + |y - y'|^{a_2}). \end{cases}$$

(U3) The pairs $(X_i, Y_j)_{i,j \in \mathbb{N}}$ satisfies:

(a) $\exists a > (5 + \sqrt{17})/2, \exists c > 0: \forall n \in \mathbb{N}, \alpha(n) \leq cn^{-a}$.
(b) $0 \leq C\phi(h_K)^{1+a} \leq \varphi_{\theta,x}(h_K) \leq C'\phi(h_K)^{\frac{1}{1+a}}$

where $\varphi_{\theta,x}(h_K) = \sup_{i \neq j} \mathbb{P}((X_i, X_j) \in B(x, h_K) \times B(x, h_K))$.

(U4) The bi-functional function $\beta_\theta(\cdot,\cdot)$ is lipschitzian continuous function and satisfying:

$$\forall x' \in S_{\mathcal{H}}, C d_\theta(x',x) \leq |\beta_\theta(x,x')| \leq C' d_\theta(x',x).$$

(U5) (a) The kernel K is a positive, lipschitzian and differentiable function, supported within $(-1,1)$.

(b) The kernel H' is a positive, bounded and lipschitzian continuous function, such that:

$$\int |t|^{a_2} H'(t)\mathrm{d}t < \infty \text{ and } \int H'^2(t)\mathrm{d}t < \infty.$$

(U6) The bandwidth h_K satisfies: $\exists n_0 \in \mathbb{N}, \forall \eta > n_0$,

$$-\frac{1}{\phi_{\theta,x}(h_K)} \int_{-1}^{1} \phi_{\theta,x}(th_K, h_K) \frac{\mathrm{d}}{\mathrm{d}t}(t^2 K(t))\mathrm{d}t > C'' > 0$$

and

$$h_K \int_{B(x,h_K)} \beta_\theta(u,x)\mathrm{d}P(u) = o(\int_{B(x,h_K)} \beta_\theta^2(u,x)\mathrm{d}P(u))$$

where $B(x,h) = \{z \in \mathcal{H} |\, d_\theta(z,x) \leq h\}$ and $\mathrm{d}P(u)$ is the probability measure of X.

(U7) For some $\lambda > 0$, the bandwidth h_H satisfies $\lim_{n\to\infty} n^\lambda h_H = \infty$ and for $r_n = O\left(\frac{\log n}{n}\right)$ the sequences $N^{S_{\mathcal{H}}}$ and $N^{\Theta_{\mathcal{H}}}$ satisfy

$$\frac{\log^2 n}{n h_H \phi(h_K)} \leq \log N^{S_{\mathcal{H}}} + \log N^{\Theta_{\mathcal{H}}} \leq \frac{n h_H \phi(h_K)}{\log n}$$

and

$$\sum_{n=1}^{\infty} n^{(3\lambda+1)/2} (N^{S_{\mathcal{H}}} N^{\Theta_{\mathcal{H}}})^{1-\eta} < \infty \text{ for} \eta > 1.$$

(U8) $\exists \beta, \eta_0 > 0, C n^{\frac{4-a+3\beta}{a+1}+\eta_0} \leq h_H \phi(h_K)$ and $\phi(h_K) \leq C n^{\frac{1}{1-a}}$.

Comments on assumptions: As usually in functional statistics and in the independent case, the conditions (U1) and (U4) are standard hypotheses (see [11]). (U2) is about regularity and boundary conditions. Hypotheses (U5) and (U7) are a technical conditions (see [8]). Particulary, for the dependence frame, (U3) indicate that the observations are $\boldsymbol{\alpha}$-mixing dependency. Likewise, we find the condition (U6) in [8] and we put (U8) that needed for our asymptotic results.

4. MAIN RESULT: UNIFORM ALMOST COMPLETE CONVERGENCE

Theorem 1. Under assumptions (U1)-(U8), we have:

$$\sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sup_{y\in S_{\mathbb{R}}} |\hat{h}_\theta^x(y) - h_\theta^x(y)| = O(h_K^{a_1} + h_H^{a_2}) + O_{\text{a.co.}}(\sqrt{\frac{\log(N^{S_{\mathcal{H}}} N^{\Theta_{\mathcal{H}}})}{n h_H \phi(h_K)}}).$$

Proof: The proof is based on the decomposition in Theorem 3.1 of [15] and the results below.

Lemma 1. Under assumptions (U1), (U2) and (U5), we obtain:

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|f_\theta^x(y)-\mathbb{E}[\hat{f}_{\theta,N}^x(y)]| = O(h_K^{a_1})+O(h_H^{a_2})$$

and

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|F_\theta^x(y)-\mathbb{E}[\hat{F}_{\theta,N}^x(y)]| = O(h_K^{a_1})+O(h_H^{a_2}).$$

Lemma 2. Under assumptions (U1)and (U3)-(U8), we get:

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}|1-\hat{g}_{\theta,D}^x| = O_{\text{a.co.}}\left(\sqrt{\frac{\log(N^{S_{\mathcal{H}}}N^{\Theta_{\mathcal{H}}})}{n\phi(h_K)}}\right) \text{ and } \sum_{i=1}^{\infty}\mathbb{P}\left(\inf_{\theta\in\Theta_{\mathcal{H}}}\inf_{x\in S_{\mathcal{H}}}\hat{g}_{\theta,D}^x<1/2\right)<\infty.$$

Lemma 3. Under assumptions (U1), (U2)-(U8), we obtain:

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|\hat{f}_{\theta,N}^x(y)-\mathbb{E}[\hat{f}_{\theta,N}^x(y)]| = O_{\text{a.co.}}\left(\sqrt{\frac{\log(N^{S_{\mathcal{H}}}N^{\Theta_{\mathcal{H}}})}{nh_H\phi(h_K)}}\right)$$

and

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|\hat{F}_{\theta,N}^x(y)-\mathbb{E}[\hat{F}_{\theta,N}^x(y)]| = O_{\text{a.co.}}\left(\sqrt{\frac{\log(N^{S_{\mathcal{H}}}N^{\Theta_{\mathcal{H}}})}{n\phi(h_K)}}\right).$$

Corollary 1. Under the conditions of Theorem 1, we get:

$$\exists\mu>0,\sum_{n=1}^{\infty}\mathbb{P}\left(\inf_{\theta\in\Theta_{\mathcal{H}}}\inf_{x\in S_{\mathcal{H}}}\inf_{y\in S_{\mathbb{R}}}|1-\hat{F}_\theta^x(y)|<\mu\right)<\infty.$$

Through these results, we observe that's the same rate of convergence as [15], where the data are independent.

5. CONCLUSION

This contribution concerns the nonparametric estimation of the conditional hazard function in the presence of a functional explanatory variable when the data are generated from a model of regression to a single index, of which we have considered an estimator by the local linear approach. As asymptotic results in the case of dependent data, we have established the uniform almost complete convergence of our constructed estimator. Moreover, given the importance of the point of high-risk in the fields of survival analysis and reliability theory, our model makes it possible to construct this estimator based on the estimator of the conditional hazard function.

6. APPENDIX

We use the procedure of Masry (1986) for proofing the Lemma 4 and Lemma 5 that we need for proved our lemmas.

Lemma 4. Under assumptions (U1)-(U8), we have

$$S_{\theta,n}^2 = O(n\phi(h_K)),$$

where, for $k = 0,1,2$, we define $S_{\theta,n}^2$ as follows,

$$S_{\theta,n}^2 = \sum_{i,j=1}^{n} |Cov(\Lambda_\theta^{i^k}(x), \Lambda_\theta^{j^k}(x))| \text{and } \Lambda_\theta^{i^k}(x) = \frac{1}{h_K^k}\left(K_{\theta,i}(x)\beta_{\theta,i}^k(x) - \mathbb{E}[K_{\theta,i}(x)\beta_{\theta,i}^k(x)]\right).$$

Proof: For determine $S_{\theta,n}^2$, we have for $k = 0,1,2$ that

$$S_{\theta,n}^2 = \sum_{i\neq j=1}^{n} Cov(\Lambda_\theta^{i^k}(x), \Lambda_\theta^{j^k}(x)) + n\text{Var}(\Lambda_\theta^{1^k}(x)).$$

First, we calculate the quantity $\sum_{i\neq j=1}^{n} Cov(\Lambda_\theta^{i^k}(x), \Lambda_\theta^{j^k}(x))$. So, collecting this latter over the sets S'_1 and S'_2, where, for the sequence v_n, we define the sets as follow

$$S'_1 = \{(i,j), 1 \le i-j \le v_n\} \text{ and } S_{2'} = \{(i,j), v_n + 1 \le i-j \le n-1\}.$$

Then, the sum over S_1, and under (U1), (U3)(b), and (U5)(b), permit us to write

$$\sum_{(i,j)\in S'_1} Cov(\Lambda_\theta^{i^k}(x), \Lambda_\theta^{j^k}(x)) \le Cnv_n(\mathbb{P}(X_i, X_j) \in B(x,h_K) \times B(x,h_K)$$
$$+\mathbb{P}(X_i \in B(x,h_K))\mathbb{P}(X_j \in B(x,h_K)))$$
$$\le Cnv_n(\phi(h_K)^{1+a} + \phi(h_K)^2)$$
$$\le Cnv_n\phi(h_K)^{1+a}.$$

Second, regarding the sum over the set S'_2, we use Davydov-Rio's inequality, for all $i \neq j$

$$|Cov(\Lambda_\theta^{i^k}(x), \Lambda_\theta^{j^k}(x))| \le C\alpha(|i-j|),$$

and under (U3) (a)

$$|\sum_{(i,j)\in S'_2} Cov(\Lambda_\theta^{i^k}(x), \Lambda_\theta^{j^k}(x))| \le C\frac{nv_n^{-a+1}}{a-1}, \tag{1}$$

then, under (U8) and by choosing $v_n = (\phi(h_K))^{-1/a}$ in (1), we get

$$\sum_{(i,j)\in S'_1} Cov(\Lambda_\theta^{i^k}(x), \Lambda_\theta^{j^k}(x)) \leq Cn\phi(h_K)n^{-t(\frac{ad-1}{a})}, \tag{2}$$

thus,

$$\sum_{(i,j)\in S'_1} Cov(\Lambda_\theta^{i^k}(x), \Lambda_\theta^{j^k}(x)) = O(n\phi(h_K)). \tag{2}$$

Also, for (1) we use the same v_n, for $a > 2$ and for $k = 0,1,2$, we obtain

$$\sum_{(i,j)\in S'_2} Cov(\Lambda_\theta^{i^k}(x), \Lambda_\theta^{j^k}(x)) = O(n\phi^{(a-1)/a}(h_K)). \tag{3}$$

About the variance term, and under (U1), we have

$$\begin{aligned} \mathrm{Var}(\Lambda_\theta^{1^k}(x)) &\leq C(\phi(h_K) + \phi^2(h_K)) \\ &\leq C\phi(h_K). \end{aligned} \tag{4}$$

Finally, from (2), (3), and (4) we get

$$S_{\theta,n}^2 = O(n\phi(h_K)).$$

Lemma 5. Under assumptions (U1)-(U8), we have

$$\tilde{S}_{\theta,n}^2 = O(nh_H\phi(h_K)),$$

where, for $k = 0,1$, we define $\tilde{S}_{\theta,n}^2$ as follows,

$$\boldsymbol{\tilde{S}_{\theta,n}^2 = \sum_{i,j=1}^{n} |Cov(\Upsilon_{\theta,i}^k(x), \Upsilon_{\theta,j}^k(x))| \text{and } \Upsilon_{\theta,i}^k = \frac{1}{h_K^k}[K_{\theta,i}(x)H_i(y)\beta_{\theta,i}^k(x) - \mathbb{E}[K_{\theta,i}(x)H_i(y)\beta_{\theta,i}^k(x)]}.$$

Proof of Lemma 5: In the same manner of Lemma 4 where $v_{n'} \to \infty$, we calculate the summation of $\sum_{i\neq j}^{n} |Cov(\Upsilon_i^k, \Upsilon_j^k)|$ over $\tilde{S}_1$ and $\tilde{S}_2$

$$\tilde{S}_1 = \{(i,j) \text{ such that } 0 < |i-j| < v_{n'}\} \text{ and } \tilde{S}_2 = \{(i,j) \text{ such that } v_{n'} + 1 < |i-j| < n-1\},$$

So, under (U1)and (U3)-(U6) and some calculate, we get

$$\sum_{(i,j)\in\tilde{S}_1}^{n} Cov(\Upsilon_i^k, \Upsilon_j^k) \leq Cnh_H^2\phi(h_K), \tag{5}$$

and for $\tilde{S}_2$, we use again Davydov-Rio's inequality for bounded mixing processes,

$$\sum_{(i,j)\in\tilde{S}_2}^{n} Cov(\mathrm{Y}_i^k, \mathrm{Y}_j^k) \leq Cnv_n'^{1-a}, \tag{6}$$

then, taking $v_{n'} = (\frac{1}{h_H^2\phi(h_K)})^{1/a}$ in (5) and (6), we obtain

$$\sum_{i\neq j}^{n} |Cov(\mathrm{Y}_i^k, \mathrm{Y}_j^k)| = O(n(h_H^2\phi(h_K))^{(a-1)/a}), \tag{7}$$

now, for the variance term, we use the same concepts in Lemma 5

$$\mathrm{Var}(\mathrm{Y}_1^k) \leq h_H\phi(h_K). \tag{8}$$

Finally, from (7) and (8), we obtain

$$\tilde{S}_{\theta,n}^2 = O(nh_H\phi(h_K)).$$

Proof of Lemma 1: By the Lemma 4.3 in [8] in the regression function, the proofs of Lemma 1 can be compleed, due to the bias term is not constrained by the dependence shape.

Proof of Lemma 2: We have by adoption from the proof of Lemma 4.4 in [8]

$$\begin{aligned}\hat{g}_{\theta,D}^x &= \underbrace{\frac{n^2h_K^2\phi_{\theta,x}^2(h_K)}{n(n-1)\mathbb{E}[W_{\theta,12}(x)]}}_{S_1}[\underbrace{(\frac{1}{n}\sum_{j=1}^{n}\frac{K_{\theta,j}(x)}{\phi_{\theta,x}(h_K)})}_{S_{\theta,2}^x}\underbrace{(\frac{1}{n}\sum_{i=1}^{n}\frac{K_{\theta,i}(x)\beta_{\theta,i}^2(x)}{h_K^2\phi_{\theta,x}(h_K)})}_{S_{\theta,4}^x} \\ &- \underbrace{(\frac{1}{n}\sum_{j=1}^{n}\frac{K_{\theta,j}(x)\beta_{\theta,j}(x)}{h_K\phi_{\theta,x}(h_K)})}_{S_{\theta,3}^x}\underbrace{(\frac{1}{n}\sum_{i=1}^{n}\frac{K_{\theta,i}(x)\beta_{\theta,i}(x)}{h_K\phi_{\theta,x}(h_K)})}_{S_{\theta,3}^x}].\end{aligned}$$

We have that $\mathbb{E}[\hat{g}_{\theta,D}^x] = 1$. So,

$$\boldsymbol{\hat{g}_{\theta,D}^x - \mathbb{E}[\hat{g}_{\theta,D}^x] = S_1((S_{\theta,2}^x S_{\theta,4}^x - \mathbb{E}[S_{\theta,2}^x S_{\theta,4}^x]) - ((S_{\theta,3}^x)^2 - \mathbb{E}[(S_{\theta,3}^x)^2]))},$$

we put

$$\begin{aligned}\boldsymbol{S_{\theta,2}^x S_{\theta,4}^x - \mathbb{E}[S_{\theta,2}^x S_{\theta,4}^x]} &= \boldsymbol{(S_{\theta,2}^x - \mathbb{E}[S_{\theta,2}^x])(S_{\theta,4}^x - \mathbb{E}[S_{\theta,4}^x]) + (S_{\theta,4}^x - \mathbb{E}[S_{\theta,4}^x])\mathbb{E}[S_{\theta,2}^x] +} \\ &\quad \boldsymbol{(S_{\theta,2}^x - \mathbb{E}[S_{\theta,2}^x])\mathbb{E}[S_{\theta,4}^x] - Cov(S_{\theta,2}^x S_{\theta,4}^x)}. \\ \boldsymbol{(S_{\theta,3}^x)^2 - \mathbb{E}[(S_{\theta,3}^x)^2]} &= \boldsymbol{(S_{\theta,3}^x - \mathbb{E}[S_{\theta,3}^x])^2 + 2(S_{\theta,3}^x - \mathbb{E}[S_{\theta,3}^x])\mathbb{E}[S_{\theta,3}^x] - \mathrm{Var}[S_{\theta,3}^x]}.\end{aligned}$$

Thus, these decompositions allows us to proved the following equations

$$S_1 = O(1) \text{ and } \mathbb{E}[S_{\theta,z}^x] = O(1) \text{ for} z = 2,3,4. \tag{9}$$

$$\mathrm{Var}[S_{\theta,3}^x] = \mathrm{o}(\sqrt{\frac{\log(N^{S_{\mathcal{H}}}N^{\Theta_{\mathcal{H}}})}{n\phi(h_K)}}), \tag{10}$$

$$Cov(S^x_{\theta,2}, S^x_{\theta,4}) = \mathrm{o}(\sqrt{\frac{\log(N^{S_\mathcal{H}} N^{\Theta_\mathcal{H}})}{n\phi(h_K)}}), \tag{11}$$

$$\sum_{n=1}^{\infty} \mathbb{P}\{|S^x_{\theta,z} - \mathbb{E}[S^x_{\theta,z}]| > \eta \sqrt{\frac{\log(N^{S_\mathcal{H}} N^{\Theta_\mathcal{H}})}{n\phi(h_K)}}\} < \infty, \text{forz} = 2,3,4. \tag{12}$$

We have that (9) is proved in [8]. So, it requires to prove (10), (11) and (12). Concerning (12), we consider the following decomposition

$$\begin{aligned}
\sup_{\theta\in\Theta_\mathcal{H}} \sup_{x\in S_\mathcal{H}} |S^x_{\theta,z} - \mathbb{E}[S^x_{\theta,z}]| \le\ & \underbrace{\sup_{\theta\in\Theta_\mathcal{H}} \sup_{x\in S_\mathcal{H}} |S^x_{\theta,z} - S^{x_{j(x)}}_{\theta,z}|}_{\mathrm{E}_1} \\
& + \underbrace{\sup_{\theta\in\Theta_\mathcal{H}} \sup_{x\in S_\mathcal{H}} |S^{x_{j(x)}}_{\theta,z} - S^{x_{j(x)}}_{t_{j''(\theta)},z}|}_{\mathrm{E}_2} \\
& + \underbrace{\sup_{\theta\in\Theta_\mathcal{H}} \sup_{x\in S_\mathcal{H}} |S^{x_{j(x)}}_{t_{j''(\theta)},z} - \mathbb{E}[S^{x_{j(x)}}_{t_{j''(\theta)},z}]|}_{\mathrm{E}_3} \\
& + \underbrace{\sup_{\theta\in\Theta_\mathcal{H}} \sup_{x\in S_\mathcal{H}} |\mathbb{E}[S^{x_{j(x)}}_{t_{j''(\theta)},z}] - \mathbb{E}[S^{x_{j(x)}}_{\theta,z}]|}_{\mathrm{E}_4} \\
& + \underbrace{\sup_{\theta\in\Theta_\mathcal{H}} \sup_{x\in S_\mathcal{H}} |\mathbb{E}[S^{x_{j(x)}}_{\theta,z}] - \mathbb{E}[S^x_{\theta,z}]|}_{\mathrm{E}_5}
\end{aligned}$$

Applying the Fuck-Nagaev exponential inequality (Proposition A.11(ii) in [14]), for all $\epsilon > 0, r > 0$, we have

$$\begin{aligned}
\mathbb{P}(\mathrm{E}_3 > \epsilon) &= \mathbb{P}\left(\max_{j''\in\{1,\dots,N^{\Theta_\mathcal{H}}\}} \max_{j'\in\{1,\dots,N^{S_\mathcal{H}}\}} |S^{x_{j(x)}}_{t_{j''(\theta)},z} - \mathbb{E}[S^{x_{j(x)}}_{t_{j''(\theta)},z}]| > \epsilon\right) \\
&\le \boldsymbol{N^{\Theta_\mathcal{H}} N^{S_\mathcal{H}}} \max_{j''\in\{1,\dots,N^{\Theta_\mathcal{H}}\}} \max_{j'\in\{1,\dots,N^{S_\mathcal{H}}\}} \mathbb{P}\left(\left|\boldsymbol{S^{x_{j(x)}}_{t_{j''(\theta)},z}} - \mathbb{E}\left[\boldsymbol{S^{x_{j(x)}}_{t_{j''(\theta)},z}}\right]\right| > \boldsymbol{\epsilon}\right) \\
&\le \boldsymbol{N^{\Theta_\mathcal{H}} N^{S_\mathcal{H}}} \max_{j''\in\{1,\dots,N^{\Theta_\mathcal{H}}\}} \max_{j'\in\{1,\dots,N^{S_\mathcal{H}}\}} \mathbb{P}\left(|\sum_{i=1}^{n} \boldsymbol{\Lambda^i_{t_{j''(\theta)},z}(x_{j(x)})}| > \boldsymbol{n\phi(h_K)\epsilon}\right)
\end{aligned} \tag{13}$$

where, for $k = 0,1,2,$ and for $z = 2,3,4$,

$$\Lambda^{i^k}_{t_{j''(\theta)},z}(x_{j(x)}) = \frac{1}{h_K^k}\left(K_{t_{j''(\theta)},i}(x_{j(x)})\beta^k_{t_{j''(\theta)},i}(x_{j(x)}) - \mathbb{E}[K_{t_{j''(\theta)},i}(x_{j(x)})\beta^k_{t_{j''(\theta)},i}(x_{j(x)})]\right),$$

and since $\mathbb{E}[|\Lambda^i_{t_{j''(\theta)},z}(x_{j(x)})|^p] \le C\phi(h_K)$, for $p > 2$, $0 < C < \infty$, we get

$$\mathbb{P}(\mathrm{E}_3 > \epsilon) \le N^{\Theta_\mathcal{H}} N^{S_\mathcal{H}} (1 + \frac{\epsilon^2 n^2 \phi(h_K)^2}{r S_n^2})^{-r/2} + N^{\Theta_\mathcal{H}} N^{S_\mathcal{H}} n r^{-1} (\frac{r}{\epsilon n \phi(h_K)})^{a+1}$$

we put

$$A_1 = N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}} (1 + \frac{\epsilon^2 n^2 \phi(h_K)^2}{r S_n^2})^{-r/2}, A_2 = N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}} n r^{-1} (\frac{r}{\epsilon n \phi(h_K)})^{a+1}.$$

Now, taking $r = C(\log n)^2$, $\epsilon = \frac{\kappa\sqrt{S_n^2 \log N^{S_{\mathcal{H}}} N^{\Theta_{\mathcal{H}}}}}{n\phi(h_K)}$ in A_1 and by Lemma 5, we get

$$A_1 \leq N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}} (1 + \frac{\kappa^2 \log n}{(\log n)^2})^{-(\log n)^2/2} = N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}} \exp(-\kappa^2 \frac{\log n}{2}) = C N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}} n^{-\kappa^2/2},$$

therefore, for favorable κ, $A_1 \leq C n^{-\kappa^2/2}$, about A_2, the same r and ϵ in pervious calculation and Lemma 5, we obtain

$$\boldsymbol{A_2 \leq C N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}} n^{1-(a+1)/2} \phi(h_K)^{-(a+1)/4} (\log n)^{(3a-1)/2}}$$

then, by (U8), there exist $\nu > 0$ such that

$$A_2 \leq C n^{-1-\nu},$$

So,

$$\mathbb{P}\left(\mathrm{E}_3 > \kappa \sqrt{\frac{\log N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}}}{n\phi(h_K)}} \right) < \infty.$$

We can write the term E_1 by

$$\begin{aligned}
\mathbf{E_1} \quad & \leq \frac{C}{n h_K^{k-2} \phi(h_K)} \sup_{\theta \in \Theta_{\mathcal{H}}} \sup_{x \in S_{\mathcal{H}}} \sum_{i=1}^n |K_{\theta,i}(x) \beta_{\theta,i}^{k-2}(x) \mathbf{1}_{B(x,h)}(X_i) \\
& - K_{\theta,i}(x_{j(x)}) \beta_{\theta,i}^{k-2}(x_{j(x)}) \mathbf{1}_{B(x_{j(x)},h)}(X_i)| \\
& \leq \frac{C}{n h_K^{k-2} \phi(h_K)} \sup_{\theta \in \Theta_{\mathcal{H}}} \sup_{x \in S_{\mathcal{H}}} \sum_{i=1}^n K_{\theta,i}(x) \mathbf{1}_{B(x,h)(X_i)} |\beta_{\theta,i}^{k-2}(x) - \beta_{\theta,i}^{k-2}(x_{j(x)}) \mathbf{1}_{B(x_{j(x)},h)}(X_i)| \\
& + \frac{C}{n h_K^{k-2} \phi(h_K)} \sup_{\theta \in \Theta_{\mathcal{H}}} \sup_{x \in S_{\mathcal{H}}} \sum_{i=1}^n \beta_{\theta,i}^{k-2}(x_{j(x)}) \mathbf{1}_{B(x_{j(x)},h)(X_i)} |K_{\theta,i}(x) \mathbf{1}_{B(x,h)(X_i)} - K_{\theta,i}(x_{j(x)})| \\
& := \mathbf{E_{1,1}} + \mathbf{E_{1,2}}.
\end{aligned}$$

Concerning the term $\mathrm{E}_{1,1}$

$$1_{B(x,h)}(X_i) |\beta_{\theta,i}(x) - \beta_{\theta,i}(x_{j(x)}) 1_{B(x_{j(x)},h)}(X_i)| \leq$$

$$C r_n 1_{B(x,h) \cap B(x_{j(x)},h)}(X_i) + C h_K 1_{B(x,h) \cap \overline{B(x_{j(x)},h)}}(X_i)$$

and

$$1_{B(x,h)}(X_i) |\beta_{\theta,i}^2(x) - \beta_{\theta,i}^2(x_{j(x)}) 1_{B(x_{j(x)},h)}(X_i)| \leq$$

$$C r_n h_K 1_{B(x,h) \cap B(x_{j(x)},h)}(X_i) + C h_K^2 1_{B(x,h) \cap \overline{B(x_{j(x)},h)}}(X_i),$$

for $k = 3,4$

$$1_{B(x,h)}(X_i) |\beta_{\theta,i}^{k-2}(x) - \beta_{\theta,i}^{k-2}(x_{j(x)}) 1_{B(x_{j(x)},h)}(X_i)| \leq$$

$$C r_n h_K^{k-2} 1_{B(x,h) \cap B(x_{j(x)},h)}(X_i) + C h_K^{k-2} 1_{B(x,h) \cap \overline{B(x_{j(x)},h)}}(X_i).$$

So,

$$\mathrm{E}_{1,1} \leq \frac{Cr_n}{nh_K\phi(h_K)} \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sum_{i=1}^{n} K_{\theta,i}(x) 1_{B(x,h)\cap B(x_{j(x)},h)}(X_i) +$$

$$\frac{C}{n\phi(h)} \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sum_{i=1}^{n} K_{\theta,i}(x) 1_{B(x,h)\cap \overline{B(x_{j(x)},h)}}(X_i).$$

Concerning the term $\mathrm{E}_{1,2}$

$$\mathbf{1}_{B(x_{j(x)},h)(X_i)} |K_{\theta,i}(x)\mathbf{1}_{B(x,h)}(X_i) - K_{\theta,i}(x_{j(x)})\mathbf{1}_{B(x,h)\cup\overline{B(x,h)}}(X_i)|$$
$$\leq \mathbf{1}_{B(x_{j(x)},h)\cap B(x,h)}(X_i)|K_{\theta,i}(x) - K_{\theta,i}(x_{j(x)})| + K_{\theta,i}(x_{j(x)})\mathbf{1}_{B(x_{j(x)},h)\cap\overline{B(x,h)}}(X_i)$$

and by

$$|\beta_{\theta,i}^{k-2}(x_{j(x)})|\mathbf{1}_{B(x_{j(x)},h)}(X_i)\mathbf{1}_{B(x_{j(x)},h)(X_i)}|K_{\theta,i}(x)\mathbf{1}_{B(x,h)}(X_i) - K_{\theta,i}(x_{j(x)})|$$
$$\leq Ch_K^{k-2}\frac{r_n}{h_K}\mathbf{1}_{B(x_{j(x)},h)\cap B(x,h)}(X_i) + K_{\theta,i}(x_{j(x)})\mathbf{1}_{B(x_{j(x)},h)\cap\overline{B(x,h)}}(X_i)$$

we get

$$\begin{aligned}\mathrm{E}_{1,2} \quad &\leq \frac{Cr_n}{nh_K\phi(h_K)} \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sum_{i=1}^{n} 1_{B(x,h)\cap B(x_{j(x)},h)}(X_i) \\ &+ \frac{C}{n\phi(h_K)} \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sum_{i=1}^{n} K_{\theta,i}(x_{j(x)}) 1_{B(x_{j(x)},h_K)\cap\overline{B(x,h_K)}}(X_i)\end{aligned}$$

finally, by $\mathrm{E}_{1,1}$ and $\mathrm{E}_{1,2}$, we get

$$\mathrm{E}_1 \leq \frac{Cr_n}{nh_K\phi(h_K)} \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sum_{i=1}^{n} 1_{B(x_{j(x)},h_K)\cup B(x,h_K)}(X_i),$$

we put

$$T_i = \frac{Cr_n}{h_K} \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} 1_{B(x_{j(x)},h_K)\cup B(x,h_K)}(X_i)$$

and under hypotheses (U1), (U3)(b), (U8), and Lemma 4, we have that

$$S_n^2 = \sum_{i,j=1}^{n} |Cov(T_i,T_j)| = O(n\phi(h_K)).$$

So, from E_3, we obtain $\mathrm{E}_1 = \mathrm{O}_{a.co.}(\sqrt{\frac{\log(N^{\Theta_{\mathcal{H}}}N^{S_{\mathcal{H}}})}{n\phi(h_K)}})$, and we deduce

$$\mathrm{E}_5 \leq \mathrm{E}_1 = \mathrm{O}_{a.co.}(\sqrt{\frac{\log(N^{\Theta_{\mathcal{H}}}N^{S_{\mathcal{H}}})}{n\phi(h_K)}}).$$

We using the same ideas those used in E_1 and E_5, we get

$$\mathrm{E}_4 \leq \mathrm{E}_2 = \mathrm{O}_{a.co.}(\sqrt{\frac{\log(N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}})}{n\phi(h_K)}}).$$

Now, for (10) and (11), we will use results that obtained in Lemma 5, we obtain

$$\boldsymbol{Var[S^x_{\theta,3}] = o(\frac{1}{n\phi(h_K)}) = o(\sqrt{\frac{log(N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}})}{n\phi(h_K)}})}$$

and

$$\boldsymbol{Cov(S^x_{\theta,2}, S^x_{\theta,4}) = \mathrm{O}(\frac{1}{n\phi(h_K)}) = \mathrm{o}(\sqrt{\frac{\log(N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}})}{n\phi(h_K)}})}.$$

Finally, from this later results, we deduce that $\sum_{i=1}^{\infty} \mathbb{P}(\inf_{\theta\in\Theta_{\mathcal{H}}} \inf_{x\in S_{\mathcal{H}}} \hat{g}^x_{\theta,D} < 1/2) < \infty$, because we have $\inf_{\theta\in\Theta_{\mathcal{H}}} \inf_{x\in S_{\mathcal{H}}} \hat{g}^x_{\theta,D} < 1/2$ and this latter, involved that

$$1 - \hat{g}^x_{\theta,D} < 1/2 \Rightarrow \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} |1 - \hat{g}^x_{\theta,D}| < 1/2,$$

so, after the step of insertion the probability on both sides, we finished the proof of lemma.

Proof of Lemma 3: Firstly, about $\mathbb{E}[\hat{f}^x_{\theta,N}(y)]$, so, by the fact that $S_{\mathbb{R}}$ is a compact set, we can write that $S_{\mathbb{R}} \in \bigcup_{J=1}^{N^{\mathbb{R}}} (y_J - l_n, y_J + l_n)$ and putting $l_n = n^{-\lambda-1/2}$and $z_y = \arg\min_{J\in\{1,\dots,N^{\mathbb{R}}\}} |y - y_J|$, we obtain

$$\begin{aligned}
\sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sup_{y\in S_{\mathbb{R}}} |f^x_{\theta,N}(y) - \mathbb{E}[f^x_{\theta,N}(y)]| \leq\ & \underbrace{\sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sup_{y\in S_{\mathbb{R}}} |f^x_{\theta,N}(y) - f^{x_{j(x)}}_{\theta,N}(y)|}_{\mathrm{E}'_1} \\
& + \underbrace{\sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sup_{y\in S_{\mathbb{R}}} |f^{x_{j(x)}}_{\theta,N}(y) - f^{x_{j(x)}}_{t_{j''(\theta)},N}(y)|}_{\mathrm{E}'_2} \\
& + \underbrace{\sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sup_{y\in S_{\mathbb{R}}} |f^{x_{j(x)}}_{t_{j''(\theta)},N}(y) - f^{x_{j(x)}}_{t_{j''(\theta)},N}(z_y)|}_{\mathrm{E}'_3} \\
& + \underbrace{\sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sup_{y\in S_{\mathbb{R}}} |f^{x_{j(x)}}_{t_{j''(\theta)},N}(z_y) - \mathbb{E}[f^{x_{j(x)}}_{t_{j''(\theta)},N}(z_y)]|}_{\mathrm{E}'_4} \\
& + \underbrace{\sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sup_{y\in S_{\mathbb{R}}} |\mathbb{E}[f^{x_{j(x)}}_{t_{j''(\theta)},N}(z_y)]| - \mathbb{E}[f^{x_{j(x)}}_{t_{j''(\theta)},N}(y)]|}_{\mathrm{E}'_5} \\
& + \underbrace{sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sup_{y\in S_{\mathbb{R}}} |\mathbb{E}[f^{x_{j(x)}}_{t_{j''(\theta)},N}(y)]| - \mathbb{E}[f^{x_{j(x)}}_{\theta,N}(y)]|}_{\mathrm{E}'_6} \\
& + \underbrace{\sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sup_{y\in S_{\mathbb{R}}} |\mathbb{E}[f^{x_{j(x)}}_{\theta,N}(y)]| - \mathbb{E}[f^x_{\theta,N}(y)]|}_{\mathrm{E}'_7}
\end{aligned}$$

the same steps that using in E_1 and E_5, we get

$$\mathrm{E'}_7 \leq \mathrm{E'}_1 = \mathrm{O}_{a.co.}\left(\sqrt{\frac{\log(N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}})}{n\phi(h_K)}}\right).$$

we have for all $\eta > 0$

$$\mathbb{P}\left(\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|f_{t_{j''(\theta)},N}^{x_{j(x)}}(z_y) - \mathbb{E}[f_{t_{j''(\theta)},N}^{x_{j(x)}}(z_y)]| > \eta\sqrt{\frac{\log(N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}})}{nh_H\phi(h_K)}}\right)$$
$$= \mathbb{P}\left(\max_{j''\in\{1,\dots,N^{\Theta_{\mathcal{H}}}\}}\max_{j'\in\{1,\dots,N^{S_{\mathcal{H}}}\}}\max_{z_y\in\{1,\dots,N^{\mathbb{R}}\}}|f_{t_{j''(\theta)},N}^{x_{j(x)}}(z_y) - \mathbb{E}[f_{t_{j''(\theta)},N}^{x_{j(x)}}(z_y)] > \eta\sqrt{\frac{\log(N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}})}{nh_H\phi(h_K)}}\right)$$
$$\leq N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}} N^{\mathbb{R}} \max_{j''\in\{1,\dots,N^{\Theta_{\mathcal{H}}}\}}\max_{j'\in\{1,\dots,N^{S_{\mathcal{H}}}\}}\max_{z_y\in\{1,\dots,N^{\mathbb{R}}\}}$$
$$\mathbb{P}\left(|f_{t_{j''(\theta)},N}^{x_{j(x)}}(z_y) - \mathbb{E}[f_{t_{j''(\theta)},N}^{x_{j(x)}}(z_y)]| > \eta\sqrt{\frac{\log(N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}})}{nh_H\phi(h_K)}}\right),$$

then, by adaption of the proof of Lemma 2, we have

$$f_{\theta,N}^x(y) = S_1[T_{\theta,2}(y)S_{\theta,4}^x - T_{\theta,5}(y)S_{\theta,3}^x]$$

where $S_1, S_{\theta,3}^x$ and $S_{\theta,4}^x$ are the same terms in Lemma 2, and

$$T_{\theta,2}(y) = \frac{1}{n}\sum_{j=1}^{n}\frac{K_{\theta,j}(x)H_j(y)}{h_H\phi_{\theta,x}(h_K)},\; T_{\theta,5}(y) = \frac{1}{n}\sum_{j=1}^{n}\frac{K_{\theta,j}(x)\beta_j(x)H_j(y)}{h_H h_K\phi_{\theta,x}(h_K)}.$$

then, we have the decomposition

$$f_{\theta,N}^x(y) - \mathbb{E}[f_{\theta,N}^x(y)] = S_1((T_{\theta,2}(y)S_{\theta,4}^x - \mathbb{E}[T_{\theta,2}(y)S_{\theta,4}^x]) - (T_{\theta,5}(y)S_{\theta,3}^x - \mathbb{E}[T_{\theta,5}(y)S_{\theta,3}^x])).$$

which implies that

$$\boldsymbol{T_{\theta,2}(y)S_{\theta,4}^x} - \mathbb{E}[\boldsymbol{T_{\theta,2}(y)S_{\theta,4}^x}] = (\boldsymbol{T_{\theta,2}^x(y)} - \mathbb{E}[\boldsymbol{T_{\theta,2}^x(y)}])(\boldsymbol{S_{\theta,4}^x} - \mathbb{E}[\boldsymbol{S_{\theta,4}^x}]) + (\boldsymbol{S_{\theta,4}^x}$$
$$-\mathbb{E}[\boldsymbol{S_{\theta,4}^x}])\mathbb{E}[\boldsymbol{T_{\theta,2}^x(y)}] + (\boldsymbol{T_{\theta,2}^x(y)} - \mathbb{E}[\boldsymbol{T_{\theta,2}^x(y)}])\mathbb{E}[\boldsymbol{S_{\theta,4}^x}] - \boldsymbol{Cov}(\boldsymbol{T_{\theta,2}^x(y)}, \boldsymbol{S_{\theta,4}^x});$$

$$T_{\theta,5}(y)S_{\theta,3}^x - \mathbb{E}[T_{\theta,5}(z_y)S_{\theta,3}^x] = (T_{\theta,5}^x(y) - \mathbb{E}[T_{\theta,5}^x(y)])(S_{\theta,3}^x - \mathbb{E}[S_{\theta,3}^x])$$
$$+(S_{\theta,3}^x - \mathbb{E}[S_{\theta,3}^x])\mathbb{E}[T_{\theta,5}^x(y)] + \left(T_{\theta,5}^x(y) - \mathbb{E}\left[T_{\theta,5}^x(y)\right]\right)\mathbb{E}\left[S_{\theta,3}^x\right] - Cov\left(T_{\theta,5}^x(y), S_{\theta,3}^x\right).$$

So, our results are from the following assertions

$$S_1 = O(1) \text{ and } \mathbb{E}[S_{\theta,s}^x] = O(1), \text{for } s = 3,4 \tag{14}$$

$$\mathbb{E}[T_{\theta,s}(y)] = O(1), \text{for } s = 2,5 \tag{15}$$

$$Cov(T_{\theta,2}(y), S_{\theta,4}^x) = \mathrm{o}\left(\sqrt{\frac{\log N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}}}{n\phi(h_K)}}\right), \tag{16}$$

$$Cov(T_{\theta,5}(y), S^x_{\theta,3}) = \mathrm{o}(\sqrt{\frac{\log(N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}})}{n\phi(h_K)}}), \tag{17}$$

$$\sum_{n=1}^{\infty} \mathbb{P}\left\{|T_{\theta,s}(y) - \mathbb{E}[T_{\theta,s}(y)]| > \eta\sqrt{\frac{\log(N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}})}{nh_H\phi(h_K)}}\right\} < \infty, \text{for } s = 2{,}5. \tag{18}$$

Again, by the lemma in [8], we obtain (14) and (15). In other hand for (18), by the same manner in Lemma 2 for $\epsilon, r > 1$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}\{|T_{\theta,s}(y) - \mathbb{E}[T_{\theta,s}(y)]| > \epsilon\} \leq N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}} N^{\mathbb{R}} \mathbb{P}\{|\mathrm{Y}_{\theta,i}(y) - \mathbb{E}[\mathrm{Y}_{\theta,i}(y)]| > nh_H\phi(h_K)\epsilon\}$$

$$\leq N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}} N^{\mathbb{R}} (A'_1 + A'_2), \tag{19}$$

where

$$A'_1 = N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}} N^{\mathbb{R}} (1 + \frac{n^2 h_H^2 \phi(h_K)^2 \epsilon^2}{r\tilde{S}_n^2})^{-r/2}$$

and

$$A'_2 = N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}} N^{\mathbb{R}} n r^{-1} (\frac{r}{nh_H\phi(h_K)\epsilon})^{a+1}.$$

The choice of $\epsilon = C\eta \frac{\sqrt{\tilde{S}_n^2 \log n}}{nh_H\phi(h_K)}$ and $r = C(\log n)^2$, and from Lemma 5 we have that $\tilde{S}_n^2 = O(nh_H\phi(h_K))$, and by taking $N^{\mathbb{R}} = \frac{1}{l_n} = n^{\lambda+1/2}$, for $\nu > 0$, we obtain

$$CN^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}} N^{\mathbb{R}} (A'_1 + A'_2) \leq Cn^{-1-\nu},$$

finally

$$\sum_{n=1}^{\infty} \mathbb{P}\left\{|T_{\theta,s(y)} - \mathbb{E}[T_{\theta,s}(y)]| > \eta\sqrt{\frac{\log(N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}})}{nh_H\phi(h_K)}}\right\} < \infty.$$

Secondly, for the covariance terms, by the same arguments used in (6) for (16) and (17) to get

$$Cov(T_{\theta,2}(y), S_{\theta,4}) = o(\sqrt{\frac{(\log N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}})}{n\phi(h_K)}}),$$

$$Cov(T_{\theta,5}(y), S_{\theta,3}) = o(\sqrt{\frac{\log(N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}})}{n\phi(h_K)}}).$$

By using the same ideas of E'_1 and E'_7, we get

$$\mathrm{E}'_6 \leq \mathrm{E}'_2 = O_{a.co.}(\sqrt{\frac{\log(N^{\Theta_{\mathcal{H}}} N^{S_{\mathcal{H}}})}{nh_H\phi(h_K)}}).$$

The Lipschitz's condition on H permit us to write

$$|\boldsymbol{f_{\theta,N}(y) - f^{x}_{\theta,N}(z_y)}| \leq \frac{\boldsymbol{C}}{\boldsymbol{nh_H\phi_{\theta,x}(h_K)}} \sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}} \sum_{i\neq j=1}^{n} \boldsymbol{W_{\theta,ij}(x)|H'(y) - H'(z_y)|}$$

$$\leq C\frac{l_n}{h_H^2}\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}} S^x_{\theta,s}$$

$$\leq C\frac{l_n}{h_H^2}.$$

We have that $S^x_{\theta,s}$ is proved in Lemma 2, and taking the same l_n in pervious calculs , we get

$$\mathrm{E'}_5 \leq \mathrm{E'}_3 = o_{a.co.}\left(\sqrt{\frac{\log(N^{\Theta_{\mathcal{H}}}N^{S_{\mathcal{H}}})}{nh_H\phi(h_K)}}\right).$$

Secondly, we use the same strides for $\mathbb{E}[\hat{F}^x_{\theta,N}(y)]$ to get the required.

Proof of Corollary 1: We have that

$$\inf_{\theta\in\Theta_{\mathcal{H}}}\inf_{x\in S_{\mathcal{H}}}\inf_{y\in S_{\mathbb{R}}}|1-\hat{F}^x(y)| \leq (1-\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}\hat{F}^x(y))/2$$

$$\Longrightarrow \sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|\hat{F}^x(y)-F^x(y)| \geq (1-\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}\hat{F}^x(y))/2$$

then, by the probability on this terms, we get then,

$$\mathbb{P}\left\{\inf_{\theta\in\Theta_{\mathcal{H}}}\inf_{x\in S_{\mathcal{H}}}\inf_{y\in S_{\mathbb{R}}}|1-\hat{F}^x(y) \leq (1-\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}\hat{F}^x(y))/2\right\}$$
$$\leq \mathbb{P}\left\{\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}|\hat{F}^x(y)-F^x(y)| \geq (1-\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}\hat{F}^x(y))/2\right\} < \infty.$$

So, taking $\mu = (1-\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{y\in S_{\mathbb{R}}}\hat{F}^x(y))/2$ and under the Lemmas 2 and Lemma 3, the proof of corollary is complete.

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