ORIGINAL PAPER

INTERSECTIONS OF RULED SURFACES CORRESPONDING TO CURVES ON PSEUDO SPHERES IN DUAL SPACE

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Manuscript received: 22.05.2022; Accepted paper: 03.02.2023;
Published online: 30.03.2023.

Abstract. *In this article, firstly, the intersection of two ruled surfaces corresponding to two different curves on $\mathbb{S}_1^2$ is investigated. The conditions for the intersection of these ruled surfaces in $\mathbb{R}_1^3$ are expressed by theorems with bivariate functions. Then, the intersection of two ruled surfaces corresponding to two different curves on $\mathbb{H}^2$ is examined. Similarly, the conditions for the intersection of these ruled surfaces in $\mathbb{R}_1^3$ are shown by some illustrative theorems with bivariate functions. Finally, some examples are given to support the main results.*

Keywords: *dual space; pseudo spheres; ruled surfaces; surface intersection; bivariate functions.*

1. INTRODUCTION

The geometry of surface is one of the main research topics of differential geometry, see [1, 2]. Moreover, it has many application areas such as physics, engineering, surface modelling, etc. Especially, the ruled surface, which is obtained by moving a line along the curve, has many application areas and its geometric interpretation has been explored by a lot of authors such as [3-6]. Although the theory of surface is widely studied, there are not many studies on the intersection of surfaces. Therefore, the surface/surface intersection problem is extensively explored by a lot of authors in geometry. In [7], algorithms are presented to calculate differential geometric properties of intersection curves of two surfaces, such as parametric-parametric, parametric-implicit, and implicit-implicit. In [8], the unit tangent vector of the tangential intersection curve of two surfaces is obtained for all three types of SSI problems. Then, the geodesic torsion of the intersection curve is calculated. In [9], the intersection of ruled surfaces is shown in detail. The intersection curve of these ruled surfaces corresponds to solutions in the zero set, except for some singular points, redundant solutions, and degenerate cases. In [10], the transverse intersection curve of two spacelike parametric surfaces is studied. In [11], the spacelike transverse intersection curve of the spacelike surface and the timelike surface is investigated.

Dual numbers, which are hypercomplex numbers, were introduced by W.K. Clifford in 1873. Then, E. Study established the relation between the geometry of lines and the unit dual sphere with the mapping, which is called E. Study mapping. In E. Study mapping, there exists one-to-one correspondence between the points on the unit dual sphere DS^2 and the directed lines of space of lines $\mathbb{R}^3$. For detailed information about the properties of dual numbers, see [12]. Using E. Study mapping, the curve on the unit dual sphere corresponds to the ruled surfaces in Euclidean space, see [13] and [14]. In [13], surfaces are defined by using

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https://doi.org/10.46939/J.Sci.Arts-23.1-a05

dual vectors and line transformations. Also, a new perspective is given for the transformation of parametric surfaces. In [14], some properties about the ruled surface generated by the natural lift curve, which is the curve obtained by the endpoints of the unit tangent vectors of the main curve, is examined. We know that E. Study mapping has also been defined in $\mathbb{R}_1^3$. This mapping says that there exists one-to-one correspondence between the directed timelike (resp. spacelike) lines in $\mathbb{R}_1^3$ and ordered pairs of vectors on $\mathbb{S}_1^2$ and $\mathbb{H}^2$, see for detail in [15]. Utilizing E. Study mapping, the curve on the dual Lorentz unit sphere (or dual hyperbolic unit sphere) corresponds to the ruled surfaces in Lorentzian space, see [16]. Additionally, some geometric properties of ruled surfaces produced by natural lift curves in $\mathbb{R}_1^3$ are investigated in the same study. Also, for detailed information about surfaces in Lorentz space and its properties, see [17].

In the first part of this study, according to the E. Study mapping, any two different curves on the Lorentz unit dual sphere correspond to two separate timelike ruled surfaces, two separate spacelike ruled surfaces, or one spacelike and one timelike ruled surfaces. Firstly, consider the corresponding ruled surfaces as two different timelike ruled surfaces. Here, the intersection of the parameter curves of two timelike ruled surfaces is given by some theorems with the help of bivariate functions. Secondly, take the corresponding ruled surfaces as two different spacelike ruled surfaces. Similarly, the intersection of the parameter curves of two spacelike ruled surfaces is expressed with some theorems with the help of bivariate functions. Thirdly, regard the corresponding ruled surfaces as the spacelike and timelike ruled surfaces. Similarly, the intersection of parameter curves of spacelike and timelike ruled surfaces are specified with some theorems with the help of bivariate functions. In the second part of the study, again according to E. Study mapping, any two separate curves on the dual hyperbolic sphere correspond to two different timelike ruled surfaces. Here, the intersection of the parameter curves of two timelike ruled surfaces is presented with some theorems with the help of bivariate functions. All the theorems mentioned above are verified by examples.

This study is organized as follows: In Section 2, some basic definitions and theorems about the topic are denoted. In Section 3, firstly, the intersections of two separate timelike ruled surfaces corresponding to two different curves on the dual Lorentz unit sphere, $\mathbb{S}_1^2$, are represented. Secondly, the intersections of the two separate spacelike ruled surfaces corresponding to two different curves on the dual Lorentz unit sphere, $\mathbb{S}_1^2$, are given. Thirdly, the intersections of spacelike and timelike ruled surfaces corresponding to two different curves on the dual Lorentz unit sphere, $\mathbb{S}_1^2$, are shown. In Section 4, the intersections of the two separate timelike ruled surfaces corresponding to two different curves on the dual hyperbolic unit sphere, $\mathbb{H}^2$, are presented. In Section 5, obtained results are discussed.

2. PRELIMINARIES

This section explains some basic definitions and concepts of Lorentz space, dual space, dual Lorentz space, ruled surfaces, and the intersection of two ruled surfaces. Note that in the whole paper, the 3-dimensional Lorentzian space is shown with $\mathbb{R}_1^3$. The Lorentzian scalar product is given as

$$\langle , \rangle_L = a_1^2 + a_2^2 - a_3^2, \tag{1}$$

where $\vec{a} = (a_1, a_2, a_3)$ is a vector at $\mathbb{R}_1^3$. Any vector $\vec{a}$ is said to be spacelike if $\langle \vec{a}, \vec{a} \rangle_L > 0$ or $\vec{a} = 0$, timelike if $\langle \vec{a}, \vec{a} \rangle_L < 0$ and lightlike if $\langle \vec{a}, \vec{a} \rangle_L = 0$ for $\vec{a} \neq 0$. Also, the Lorentzian vector product of vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ is defined as

$$\vec{a} \times_L \vec{b} = (a_2b_3 - a_3b_2, a_1b_3 - a_3b_1, a_2b_1 - a_1b_2) \text{ [17]}.$$

The norm of vector a is denoted by

$$\|\vec{a}\|_L = \sqrt{|\langle \vec{a}, \vec{a} \rangle|_L} \text{ [17]}.$$

The set of dual numbers is represented as

$$ID = \{A = a + \varepsilon a^* : (a, a^*) \in \mathbb{R} \times \mathbb{R},\ \varepsilon^2 = 0\},$$

here, a and a^* are real and dual parts of A, respectively. Here, $A = a + \varepsilon a^*$ is called a dual number.

The set of ID-module is given by

$$ID^3 = \{(A_1, A_2, A_3) : A_i = a_i + a_i^* \in ID, 1 \leq i \leq 3\}. \tag{2}$$

The dual vector $\vec{A} = (A_1, A_2, A_3)$ can be represented as $\vec{A} = \vec{a} + \varepsilon \vec{a}^*$, where $\vec{a} = (a_1, a_2, a_3)$ and $\vec{a}^* = (a_1^*, a_2^*, a_3^*)$ are real vectors in $\mathbb{R}^3$. Suppose that $\vec{A} = \vec{a} + \varepsilon \vec{a}^*$ and $\vec{B} = \vec{b} + \varepsilon \vec{b}^*$ are dual vectors. The Lorentzian inner product and the Lorentzian vector product are defined as follows: The Lorentzian inner product is shown as

$$\langle \vec{A}, \vec{B} \rangle_L = \langle \vec{a}, \vec{b} \rangle_L + \varepsilon \big(\langle \vec{a}^*, \vec{b} \rangle_L + \langle \vec{a}, \vec{b}^* \rangle_L \big).$$

Lorentzian cross product is given as

$$\vec{A} \times_L \vec{B} = \vec{a} \times_L \vec{b} + \varepsilon \big(\vec{a}^* \times_L \vec{b} + \vec{a} \times_L \vec{b}^* \big).$$

The space ID^3 defined by the Lorentzian inner product is called the dual Lorentzian space, denoted by ID_1^3. In ID_1^3, a dual Lorentzian vector $\vec{A}$ is said to be spacelike if $\vec{a}$ is spacelike, timelike if $\vec{a}$ is timelike, and lightlike if $\vec{a}$ is lightlike, respectively. The Lorentzian norm of dual Lorentzian vector $\vec{A} = \vec{a} + \varepsilon \vec{a}^*$ is presented as

$$\|\vec{A}\|_L = \sqrt{|\langle \vec{a}, \vec{a} \rangle|_L} + \frac{\langle \vec{a}, \vec{a} \rangle_L}{\sqrt{|\langle \vec{a}, \vec{a} \rangle|_L}}. \tag{3}$$

Let $\mathbb{S}_1^2$ and $\mathbb{H}^2$ be the dual Lorentzian unit sphere and the dual hyperbolic unit sphere in ID_1^3, respectively. $\mathbb{S}_1^2$ and $\mathbb{H}^2$ are shown in the following equations [5]:

$$\mathbb{S}_1^2 = \big\{ \vec{A} = \vec{a} + \varepsilon \vec{a}^* \in ID_1^3 : \langle \vec{a}, \vec{a} \rangle_L = 1, \langle \vec{a}, \vec{a}^* \rangle_L = 0 \big\},$$

$$\mathbb{H}^2 = \big\{ \vec{A} = \vec{a} + \varepsilon \vec{a}^* \in ID_1^3 : \langle \vec{a}, \vec{a} \rangle_L = -1, \langle \vec{a}, \vec{a}^* \rangle_L = 0 \big\}.$$

In $\mathbb{R}^3$, the ruled surface is defined as a surface created by the movement of a straight line along a curve β. Such surfaces always have a parametrization denoted below:

$$\phi(w,s)=\beta(w)+se(w), \tag{4}$$

where β is called the base curve and e is called the director curve, see [1].

In $\mathbb{R}_1^3$, the ruled surface is said to be spacelike if the normal vector of the ruled surface at every point is a timelike vector, timelike if the normal vector of the ruled surface at every point is a spacelike vector.

Theorem 1. (E. Study mapping) There exists one-to-one correspondence between the directed timelike (resp. spacelike) lines in $\mathbb{R}_1^3$ and ordered pairs of vectors on $\mathbb{S}_1^2$ and $\mathbb{H}^2$, see [15].

Theorem 2. Let $\bar{\alpha}(w)=\alpha(w)+\varepsilon\alpha^*(w)$ be the curve on the dual Lorentz unit sphere $\mathbb{S}_1^2$ (or the dual hyperbolic unit sphere $\mathbb{H}^2$). In $\mathbb{R}_1^3$, the ruled surface generated by the curve on the dual Lorentz unit sphere (or the dual hyperbolic unit sphere) is shown as

$$\phi(w,s)=\alpha(w)\times_L\alpha^*(w)+s\alpha(w), \tag{5}$$

where $D(w)=\alpha(w)\times_L\alpha^*(w)$ is the base curve of the ruled surface $\phi(w,s)$, see [16].

Proposition 1. Let $\bar{\alpha}(w)=\alpha(w)+\varepsilon\alpha^*(w)$ be the curve on the dual Lorentz unit sphere $\mathbb{S}_1^2$. If the base curve $D(w)=\alpha(w)\times_L\alpha^*(w)$ is a spacelike curve and director curve $\alpha(w)$ is a spacelike curve, then $\alpha^*(w)$ is a timelike curve. Therefore, $\phi(w,s)$ is a spacelike ruled surface on $\mathbb{R}_1^3$, see [16].

Proposition 2. Let $\bar{\alpha}(w)=\alpha(w)+\varepsilon\alpha^*(w)$ be the curve on the dual Lorentz unit sphere $\mathbb{S}_1^2$. If the base curve $D(w)=\alpha(w)\times_L\alpha^*(w)$ is a timelike curve and director curve $\alpha(w)$ is a spacelike curve, then $\alpha^*(w)$ is a spacelike curve. Therefore, $\phi(w,s)$ is a timelike ruled surface on $\mathbb{R}_1^3$, see [16].

Proposition 3. Let $\bar{\alpha}(w)=\alpha(w)+\varepsilon\alpha^*(w)$ be the curve on the dual hyperbolic unit sphere $\mathbb{H}^2$. If the base curve $D(w)=\alpha(w)\times_L\alpha^*(w)$ is a spacelike curve and director curve $\alpha(w)$ is a timelike curve, then $\alpha^*(w)$ is a spacelike curve. Therefore, $\phi(w,s)$ is a timelike ruled surface on $\mathbb{R}_1^3$, see [16].

Proposition 4. Let $\bar{\alpha}(w)=\alpha(w)+\varepsilon\alpha^*(w)$ be the curve on the dual hyperbolic unit sphere $\mathbb{H}^2$. If the base curve $D(w)=\alpha(w)\times_L\alpha^*(w)$ is a spacelike curve and director curve $\alpha(w)$ is a timelike curve, then $\alpha^*(w)$ is a timelike curve. Therefore, $\phi(w,s)$ is a timelike ruled surface on $\mathbb{R}_1^3$, see [16].

Now, we will present some expressions for the intersection curve of two ruled surfaces in $\mathbb{R}^3$. Let $\phi^A(w_1,s)$ and $\phi^B(w_2,t)$ be two ruled surfaces defined by

$$\phi^A(w_1,s)=\beta_1(w_1)+se_1(w_1), \tag{6}$$

$$\phi^B(w_2,t)=\beta_2(w_2)+te_2(w_2). \tag{7}$$

Here, $\beta_1(w_1)$ and $\beta_2(w_2)$ are the base curves of surfaces $\phi^A(w_1,s)$ and $\phi^B(w_2,t)$, respectively. Also, $e_1(w_1)$ and $e_2(w_2)$ are the directions of the surfaces $\phi^A(w_1,s)$ and $\phi^B(w_2,t)$, respectively. The w_1-parameter curve of $\phi^A(w_1,s)$ with a constant s_0-parameter and the w_2-parameter curve of $\phi^B(w_1,t)$ with a constant t_0 -parameter are indicated by

$L_A(w_1) = \phi^A(w_1, s_0)$ and $L_B(w_2) = \phi^B(w_1, t_0)$, respectively, see [9]. As $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ ruled surfaces intersect,

$$\phi^A(w_1, s) = \phi^B(w_2, t),$$

and we obtain

$$\beta_1(w_1) - \beta_2(w_2) = -\, se_1(w_1) + te_2(w_2).$$

Since these three vectors are linearly dependent, the following equation can be presented [9]:

$$\mu(w_1, w_2) = \boldsymbol{det}\{e_1(w_1), e_2(w_2), [\beta_1(w_1) - \beta_2(w_2)]\}.$$

Theorem 3. Let $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ be two ruled surfaces in $\mathbb{R}^3$. If

$$\mu(w_1, w_2) = \boldsymbol{det}\{e_1(w_1), e_2(w_2), [\beta_1(w_1) - \beta_2(w_2)]\} = 0,$$

then the parameter curves $L_A(w_1)$ and $L_B(w_2)$ intersect, see [9].

For $\mu(w_1, w_2) = 0$, the parameter curves $L_A(w_1)$ and $L_B(w_2)$ intersect. However, there are some points in the solution set of the equation $\mu(w_1, w_2) = 0$ that is not at the intersection of two ruled surfaces. The intersection includes these points when the directors of the ruled surfaces $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ are parallel. Thus, in the case of parallelism of the direction vectors of two ruled surfaces, there may be some points that are not at the intersection of two ruled surfaces, see [9].

The parallelism of the direction vectors of the ruled surfaces $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ is represented by bivariate functions as follows [9]:

$$\Theta(w_1, w_2) = \|e_1(w_1) \times e_2(w_2)\|^2.$$

The parallelism of $\beta_1(w_1) - \beta_2(w_2)$ and the direction vector of $\phi^A(w_1, s)$ are shown with bivariate functions as follows [9]:

$$\theta_1(w_1, w_2) = \|e_1(w_1) \times [\beta_1(w_1) - \beta_2(w_2)]\|^2.$$

Similarly, the parallelism of $\beta_1(w_1) - \beta_2(w_2)$ and the direction vector of the $\phi^B(w_2, t)$ are shown with bivariate functions as follows [9]:

$$\theta_2(w_1, w_2) = \|e_2(w_2) \times [\beta_1(w_1) - \beta_2(w_2)]\|^2.$$

Theorem 4. The parameter curves $L_A(w_1)$ and $L_B(w_2)$ intersect if and only if

$$\Theta(w_1, w_2) = \theta_1(w_1, w_2) = \theta_2(w_1, w_2) = 0.$$

For more detailed information mentioned the intersection of two ruled surfaces above, see [9].

3. INTERSECTION OF TWO RULED SURFACES CORRESPONDING TO CURVES ON THE LORENTZ UNIT SPHERE

In this section, the intersections of two timelike ruled surfaces, two spacelike ruled surfaces and a spacelike and a timelike ruled surfaces corresponding to two different curves on the dual Lorentz unit sphere $\mathbb{S}_1^2$ are examined, respectively. Furthermore, some examples are given to support the main results.

3.1. INTERSECTION OF TWO TIMELIKE RULED SURFACES CORRESPONDING TO CURVES ON THE LORENTZ UNIT SPHERE

Let $\bar{\alpha}_1 = \alpha_1 + \varepsilon\alpha_1^*$ and $\bar{\alpha}_2 = \alpha_2 + \varepsilon\alpha_2^*$ be curves on the dual Lorentz unit sphere $\mathbb{S}_1^2$. With the help of E. Study mapping, let $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ be the timelike ruled surfaces corresponding to these curves given above in $\mathbb{R}_1^3$. These timelike ruled surfaces are denoted as follows:

$$\phi^A(w_1, s) = \alpha_1(w_1) \times_L \alpha_1^*(w_1) + s\alpha_1(w_1) \tag{8}$$

and

$$\phi^B(w_2, t) = \alpha_2(w_2) \times_L \alpha_2^*(w_2) + t\alpha_2(w_2). \tag{9}$$

Here, the timelike base curves of the timelike ruled surfaces $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ are

$$D_A = \alpha_1(w_1) \times_L \alpha_1^*(w_1)$$

and

$$D_B = \alpha_2(w_2) \times_L \alpha_2^*(w_2).$$

The w_1-parameter curve of $\phi^A(w_1, s)$ with a constant s_0-parameter is presented by the following equation:

$$l_A(w_1) = \phi^A(w_1, s_0).$$

Similarly, the w_2-parameter curve of $\phi^B(w_2, t)$ with a constant t_0-parameter is presented by the following equation:

$$l_B(w_2) = \phi^B(w_2, t_0).$$

As $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ intersect,

$$\phi^A(w_1, s) = \phi^B(w_2, t),$$

we write

$$D_A(w_1) - D_B(w_2) = -s\alpha_1(w_1) + t\alpha_2(w_2).$$

Since these three vectors are linearly dependent, the following equation can be presented by

$$\sigma(w_1, w_2) = det\{\alpha_1(w_1), \alpha_2(w_2), [D_A(w_1) - D_B(w_2)]\}.$$

Theorem 5. Let $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ be the timelike ruled surfaces corresponding to two different curves on the dual Lorentz unit sphere at $\mathbb{R}_1^3$. If

$$\sigma(w_1, w_2) = det\{\alpha_1(w_1), \alpha_2(w_2), [D_A(w_1) - D_B(w_2)]\} = 0,$$

then the parameter curves $l_A(w_1)$ and $l_B(w_2)$ intersect.

For $\sigma(w_1, w_2) = 0$, the parameter curves $l_A(w_1)$ and $l_B(w_2)$ intersect. However, there are some points in the solution set of the equation $\sigma(w_1, w_2) = 0$ that is not at the intersection of these two timelike ruled surfaces. The intersection includes these points as the timelike base curves of the timelike ruled surfaces $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ are parallel. Thus, in the case of parallelism of the spacelike direction vectors of two timelike ruled surfaces, there may be some points that are not at the intersection of two timelike ruled surfaces.

According to the parallelism of these three vectors with each other, the bivariate functions are represented as follows, respectively:

$$\Gamma(w_1, w_2) = \|\alpha_1(w_1) \times_L \alpha_2(w_2)\|^2{}_L,$$

$$\gamma_1(w_1, w_2) = \|\ \alpha_1(w_1) \times_L\ [D_A(w_1) - D_B(w_2)]\|^2{}_L,$$

$$\gamma_2(w_1, w_2) = \|\ \alpha_2(w_2) \times_L\ [D_A(w_1) - D_B(w_2)]\|^2{}_L.$$

Theorem 6. Let $l_A(w_1)$ and $l_B(w_2)$ be the parameter curves of the timelike ruled surfaces $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ corresponding to two different curves on the dual Lorentz unit sphere at $\mathbb{R}^3_1$. In this case, the parameter curves $l_A(w_1)$ and $l_B(w_2)$ intersect if and only if

$$\Gamma(w_1, w_2) = \gamma_1(w_1, w_2) = \gamma_2(w_1, w_2) = 0.$$

Example 1: Let $\alpha_1(w_1) = (\sqrt{2}\sin w_1, \sqrt{2}\cos w_1, 1)$, $\alpha_1^*(w_1) = (\sqrt{2}\cos w_1, -\sqrt{2}\sin w_1, 0)$, $\alpha_2(w_2) = (-\sqrt{2}\cos w_2, \sqrt{2}\sin w_2, 1)$ and $\alpha_2^*(w_2) = (\sqrt{2}\sin w_2, \sqrt{2}\cos w_2, 0)$ be spacelike vectors in $\mathbb{R}^3_1$. Since $\langle \alpha_1(w_1), \alpha_1(w_1)\rangle_L = 1$ and $\langle \alpha_1(w_1), \alpha_1^*(w_1)\rangle_L = 0$, the $\bar{\alpha}_1 = \alpha_1 + \varepsilon\alpha_1^*$ is on the dual Lorentz unit sphere. Similarly, the $\bar{\alpha}_2 = \alpha_2 + \varepsilon\alpha_2^*$ is on the dual Lorentz unit sphere. The timelike ruled surface corresponding to the $\bar{\alpha}_1$ dual curve is

$$\phi^A(w_1, s) = (\sqrt{2}\sin w_1, -\sqrt{2}\cos w_1, 2) + s(\sqrt{2}\sin w_1, \sqrt{2}\cos w_1, 1).$$

Here, the timelike base curve and spacelike direction of this timelike ruled surface are

$$D_A(w_1) = (\sqrt{2}\sin w_1, -\sqrt{2}\cos w_1, 2)$$

and

$$\alpha_1(w_1) = (\sqrt{2}\sin w_1, \sqrt{2}\cos w_1, 1).$$

The timelike ruled surface corresponding to the $\bar{\alpha}_2$ dual curve is

$$\phi^B(w_2, t) = (-\sqrt{2}\cos w_2, -\sqrt{2}\sin w_2, 2) + t(-\sqrt{2}\cos w_2, \sqrt{2}\sin w_2, 1).$$

Here, the timelike base curve and spacelike direction of this timelike ruled surface are, respectively,

$$D_B(w_2) = (-\sqrt{2}\cos w_2, -\sqrt{2}\sin w_2, 2)$$

and

$$\alpha_2(w_2) = (-\sqrt{2}\cos w_2, \sqrt{2}\sin w_2, 1).$$

Let's examine the intersection of two timelike ruled surfaces corresponding to two different curves on the dual Lorentz unit sphere. Assume that $\phi^A(w_1, s) = \phi^B(w_2, t)$, the following equation can be written by

$$\sqrt{2}(\sin w_1 + \cos w_2\,, \sin w_2 - \cos w_1\,, 0) = -s(\sqrt{2}\sin w_1\,, \sqrt{2}\cos w_1\,, 1) + t(-\sqrt{2}\cos w_2\,, \sqrt{2}\sin w_2\,, 1).$$

Here, $D_A(w_1) - D_B(w_2)$ is written as a linear combination of vectors $\alpha_1(w_1)$ and $\alpha_2(w_2)$. Since these three vectors are linearly dependent, $\sigma(w_1, w_2) = 0$. Since $\sigma(w_1, w_2) = 0$, the parameter curves $l_A(w_1)$ and $l_B(w_2)$ intersect. If $\Gamma(w_1, w_2), \gamma_1(w_1, w_2)$ and $\gamma_2(w_1, w_2)$ are calculated, we obtain

$$\Gamma(w_1, w_2) = 4\sin(w_1 - w_2)[\sin(w_1 - w_2) + 1],$$

$$\gamma_1(w_1, w_2) = -4\{(\cos(w_1 + w_2) + \sin 2w_1)^2 - (\sin(w_1 - w_2) + 1)\},$$

$$\gamma_2(w_1, w_2) = -4\{(\cos(w_1 + w_2) - \sin 2w_2)^2 - (\sin(w_1 - w_2) + 1)\}.$$

For the $\sigma(w_1, w_2) = 0$, the solution of the $(w_1, w_2) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\Gamma(w_1, w_2) = \gamma_1(w_1, w_2) = \gamma_2(w_1, w_2) = 0$. Therefore, the parameter curves $l_A(w_1)$ and $l_B(w_2)$ intersect. As a result, $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ intersect.

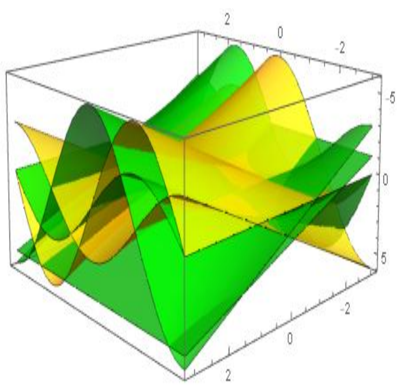

Figure 1. The intersection of two timelike ruled surfaces

3.2. INTERSECTION OF TWO SPACELIKE RULED SURFACES CORRESPONDING TO CURVES ON THE LORENTZ UNIT SPHERE

Let $\bar{\alpha}_1 = \alpha_1 + \varepsilon\alpha_1^*$ and $\bar{\alpha}_2 = \alpha_2 + \varepsilon\alpha_2^*$ be curves on the dual Lorentz unit sphere $\mathbb{S}_1^2$. With the help of E. Study mapping, let $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ be the spacelike ruled surfaces corresponding to these curves given above in $\mathbb{R}_1^3$. These spacelike ruled surfaces are denoted as follows:

$$\phi^A(w_1, s) = \alpha_1(w_1) \times_L \alpha_1^*(w_1) + s\alpha_1(w_1) \tag{10}$$

and

$$\phi^B(w_2, t) = \alpha_2(w_2) \times_L \alpha_2^*(w_2) + t\alpha_2(w_2). \tag{11}$$

Here, the spacelike base curves of the spacelike ruled surfaces $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ are

$$D_A = \alpha_1(w_1) \times_L \alpha_1^*(w_1)$$

and

$$D_B = \alpha_2(w_2) \times_L \alpha_2^*(w_2).$$

The w_1-parameter curve of $\phi^A(w_1, s)$ with a constant s_0-parameter is presented by the following equation:

$$l_A(w_1) = \phi^A(w_1, s_0).$$

Similarly, the w_2-parameter curve of $\phi^B(w_2, t)$ with a constant t_0-parameter is presented by the following equation:

$$l_B(w_2) = \phi^B(w_2, t_0).$$

As $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ intersect,

$$\phi^A(w_1, s) = \phi^B(w_2, t),$$

we write

$$D_A(w_1) - D_B(w_2) = -s\alpha_1(w_1) + t\alpha_2(w_2).$$

Since these three vectors are linearly dependent, the following equation can be presented by

$$\sigma(w_1, w_2) = det\{\alpha_1(w_1), \alpha_2(w_2), [D_A(w_1) - D_B(w_2)]\}.$$

Theorem 7. Let $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ be the spacelike ruled surfaces corresponding to two different curves on the dual Lorentz unit sphere at $\mathbb{R}_1^3$. If

$$\sigma(w_1, w_2) = det\{\alpha_1(w_1), \alpha_2(w_2), [D_A(w_1) - D_B(w_2)]\} = 0,$$

then the parameter curves $l_A(w_1)$ and $l_B(w_2)$ intersect.

Theorem 8. Let $l_A(w_1)$ and $l_B(w_2)$ be the parameter curves of the spacelike ruled surfaces $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ corresponding to two different curves on the dual Lorentz unit sphere at $\mathbb{R}_1^3$. In this case, the parameter curves $l_A(w_1)$ and $l_B(w_2)$ intersect if and only if

$$\Gamma(w_1, w_2) = \gamma_1(w_1, w_2) = \gamma_2(w_1, w_2) = 0.$$

Example 2: Let $\alpha_1(w_1) = (\sin w_1, \cos w_1, 0)$ and $\alpha_1^*(w_1) = (-\cos w_1, \sin w_1, 2)$ be spacelike and timelike vectors in $\mathbb{R}_1^3$, respectively. Since $\langle \alpha_1(w_1), \alpha_1(w_1) \rangle_L = 1$ and $\langle \alpha_1(w_1), \alpha_1^*(w_1) \rangle_L = 0$, the $\bar{\alpha}_1 = \alpha_1 + \varepsilon\alpha_1^*$ is on the dual Lorentz unit sphere. The spacelike ruled surface corresponding to $\bar{\alpha}_1$ is

$$\phi^A(w_1, s) = (2\cos w_1, 2\sin w_1, -1) + s(\sin w_1, \cos w_1, 0).$$

Here, the spacelike base curve and spacelike direction of this spacelike ruled surface are, respectively,

$$D_A(w_1) = (2\cos w_1, 2\sin w_1, -1)$$

and

$$\alpha_1(w_1) = (\sin w_1, \cos w_1, 0).$$

Let $\alpha_2(w_2) = (\cos w_2, \sin w_2, 0)$ and $\alpha_2^*(w_2) = (-\sin w_2, \cos w_2, 2)$ be spacelike and timelike vectors in $\mathbb{R}_1^3$, respectively. Since $\langle \alpha_2(w_2), \alpha_2(w_2) \rangle_L = 1$ and $\langle \alpha_2(w_2), \alpha_2^*(w_2) \rangle_L = 0$, the $\bar{\alpha}_2 = \alpha_2 + \varepsilon\alpha_2^*$ is on the dual Lorentz unit sphere. The spacelike ruled surface corresponding to the $\bar{\alpha}_2$ is

$$\phi^B(w_2,t) = (2\sin w_2, 2\cos w_2, -1) + t(\cos w_2, \sin w_2, 0).$$

Here, the spacelike base curve and spacelike direction of this spacelike ruled surface are, respectively,

$$D_B(w_2) = (2\sin w_2, 2\cos w_2, -1)$$

and

$$\alpha_2(w_2) = (\cos w_2, \sin w_2, 0).$$

Let's examine the intersection of two spacelike ruled surfaces corresponding to two different curves on the dual Lorentz unit sphere.

Assume that $\phi^A(w_1,s) = \phi^B(w_2,t)$, the following equation can be written by

$$2(\cos w_1 - \sin w_2, \sin w_1 - \cos w_2, 0) = -s(\sin w_1, \cos w_1, 0) + t(\cos w_2, \sin w_2, 0).$$

Here, $\mathrm{D_A}(\mathrm{w}_1) - \mathrm{D_B}(\mathrm{w}_2)$ is written as a linear combination of vectors $\alpha_1(\mathrm{w}_1)$ and $\alpha_2(\mathrm{w}_2)$. Since these three vectors are linearly dependent, $\sigma(\mathrm{w}_1,\mathrm{w}_2) = 0$. Since $\sigma(\mathrm{w}_1,\mathrm{w}_2) = 0$, the parameter curves $\mathrm{l_A}(\mathrm{w}_1)$ and $\mathrm{l_B}(\mathrm{w}_2)$ intersect. If $\Gamma(\mathrm{w}_1,\mathrm{w}_2)$, $\gamma_1(\mathrm{w}_1,\mathrm{w}_2)$ and $\gamma_2(\mathrm{w}_1,\mathrm{w}_2)$ are calculated, we obtain

$$\Gamma(w_1,w_2) = -\{\cos(w_1+w_2)\}^2,$$

$$\gamma_1(w_1,w_2) = -4\{\cos 2w_1 + \sin(w_1-w_2)\}^2,$$

$$\gamma_2(w_1,w_2) = -4\{\cos 2w_2 + \sin(w_2-w_1)\}^2.$$

For the, $\sigma(\mathrm{w}_1,\mathrm{w}_2) = 0$, the solution of the $(\mathrm{w}_1,\mathrm{w}_2) = \left(\frac{\pi}{4},\frac{\pi}{4}\right), \Gamma(\mathrm{w}_1,\mathrm{w}_2) = \gamma_1(\mathrm{w}_1,\mathrm{w}_2) = \gamma_2(\mathrm{w}_1,\mathrm{w}_2) = 0$. Therefore, the parameter curves $\mathrm{l_A}(\mathrm{w}_1)$ and $\mathrm{l_B}(\mathrm{w}_2)$ intersect. As a result, $\phi^A(\mathrm{w}_1,\mathrm{s})$ and $\phi^B(\mathrm{w}_2,\mathrm{t})$ intersect.

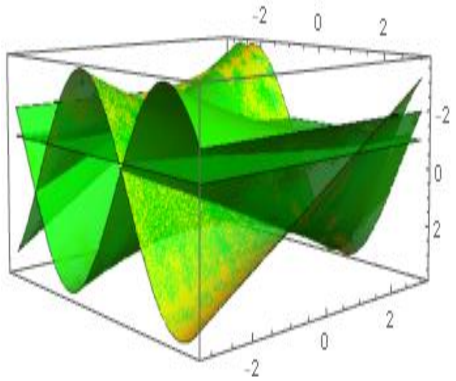

Figure 2. The intersection of two spacelike ruled surfaces

3.3. INTERSECTION OF SPACELIKE AND TIMELİKE RULED SURFACES CORRESPONDING TO CURVES ON THE LORENTZ UNIT SPHERE

Let $\bar{\alpha}_1 = \alpha_1 + \varepsilon\alpha_1^*$ and $\bar{\alpha}_2 = \alpha_2 + \varepsilon\alpha_2^*$ be curves on the dual Lorentz unit sphere $\mathbb{S}_1^2$. With the help of E. Study mapping, let $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ be the spacelike and the timelike ruled surfaces corresponding to these curves given above in $\mathbb{R}_1^3$, respectively. These spacelike and timelike ruled surfaces are denoted as follows:

$$\phi^A(w_1, s) = \alpha_1(w_1) \times_L \alpha_1^*(w_1) + s\alpha_1(w_1) \tag{12}$$

and

$$\phi^B(w_2, t) = \alpha_2(w_2) \times_L \alpha_2^*(w_2) + t\alpha_2(w_2). \tag{13}$$

Here, the spacelike base curve of $\phi^A(w_1, s)$ is

$$D_A = \alpha_1(w_1) \times_L \alpha_1^*(w_1),$$

and the timelike base curve of $\phi^B(w_2, t)$ is

$$D_B = \alpha_2(w_2) \times_L \alpha_2^*(w_2).$$

The w_1-parameter curve of $\phi^A(w_1, s)$ with a constant s_0-parameter is presented by the following equation:

$$l_A(w_1) = \phi^A(w_1, s_0).$$

Similarly, the w_2-parameter curve of $\phi^B(w_2, t)$ with a constant t_0-parameter is presented by the following equation:

$$l_B(w_2) = \phi^B(w_2, t_0).$$

As $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ intersect,

$$\phi^A(w_1, s) = \phi^B(w_2, t),$$

we write

$$D_A(w_1) - D_B(w_2) = -s\alpha_1(w_1) + t\alpha_2(w_2).$$

Since these three vectors are linearly dependent, the following equation can be presented

$$\sigma(w_1, w_2) = det\{\alpha_1(w_1), \alpha_2(w_2), [D_A(w_1) - D_B(w_2)]\}.$$

Theorem 9. Let $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ be the spacelike and timelike ruled surfaces corresponding to two different curves on the dual Lorentz unit sphere at $\mathbb{R}_1^3$. If

$$\sigma(w_1, w_2) = det\{\alpha_1(w_1), \alpha_2(w_2), [D_A(w_1) - D_B(w_2)]\} = 0,$$

then the parameter curves $l_A(w_1)$ and $l_B(w_2)$ intersect.

Theorem 10. Let $l_A(w_1)$ and $l_B(w_2)$ be the parameter curves of the spacelike ruled surface $\phi^A(w_1, s)$ and the timelike ruled surface $\phi^B(w_2, t)$ corresponding to two different curves on the dual Lorentz unit sphere at $\mathbb{R}^3_1$, respectively. In this case, the parameter curves $l_A(w_1)$ and $l_B(w_2)$ intersect if and only if

$$\Gamma(w_1, w_2) = \gamma_1(w_1, w_2) = \gamma_2(w_1, w_2) = 0.$$

Example 3: Let $\alpha_1(w_1) = (\sin w_1, \cos w_1, 0)$ and $\alpha_1^*(w_1) = (\cos w_1, -\sin w_1, \sqrt{2})$ be spacelike and timelike vectors in $\mathbb{R}^3_1$, respectively. Since $\langle \alpha_1(w_1), \alpha_1(w_1) \rangle_L = 1$ and $\langle \alpha_1(w_1), \alpha_1^*(w_1) \rangle_L = 0$, the $\bar{\alpha}_1 = \alpha_1 + \varepsilon \alpha_1^*$ is on the dual Lorentz unit sphere. The spacelike ruled surface corresponding to $\bar{\alpha}_1$ is

$$\phi^A(w_1, s) = (\sqrt{2}\cos w_1, \sqrt{2}\sin w_1, 1) + s(\sin w_1, \cos w_1, 0).$$

Here, the spacelike base curve and spacelike direction of this spacelike ruled surface are, respectively,

$$D_A(w_1) = (\sqrt{2}\cos w_1, \sqrt{2}\sin w_1, 1)$$

and

$$\alpha_1(w_1) = (\sin w_1, \cos w_1, 0).$$

Let $\alpha_2(w_2) = (-\cos w_2, \sin w_2, 0)$ and $\alpha_2^*(w_2) = (\sin w_2, \cos w_2, 0)$ be spacelike vectors in $\mathbb{R}^3_1$. Since $\langle \alpha_2(w_2), \alpha_2(w_2) \rangle_L = 1$ and $\langle \alpha_2(w_2), \alpha_2^*(w_2) \rangle_L = 0$, the $\bar{\alpha}_2 = \alpha_2 + \varepsilon \alpha_2^*$ is on the dual Lorentz unit sphere. The timelike ruled surface corresponding to $\bar{\alpha}_2$ is

$$\phi^B(w_2, t) = (0,0,1) + t(-\cos w_2, \sin w_2, 0).$$

$$\phi^B(w_2, t) = (0,0,1) + t(-\cos w_2, \sin w_2, 0).$$

Here, the timelike base curve and spacelike direction of this timelike ruled surface are, respectively,

$$D_B(w_2) = (0,0,1)$$

and

$$\alpha_2(w_2) = (-\cos w_2, \sin w_2, 0).$$

Let's examine the intersection of spacelike and timelike ruled surfaces corresponding to two different curves on the dual Lorentz unit sphere. Assume that $\phi^A(w_1, s) = \phi^B(w_2, t)$. The following equation can be written by

$$(\sqrt{2}\cos w_1, \sqrt{2}\sin w_1, 0) = -s(\sin w_1, \cos w_1, 0) + t(-\cos w_2, \sin w_2, 0).$$

Here, $D_A(w_1) - D_B(w_2)$ is written as a linear combination of vectors $\alpha_1(w_1)$ and $\alpha_2(w_2)$. Since these three vectors are linearly dependent, $\sigma(w_1, w_2) = 0$. Since $\sigma(w_1, w_2) = 0$, the parameter curves $l_A(w_1)$ and $l_B(w_2)$ intersect. If $\Gamma(w_1, w_2), \gamma_1(w_1, w_2)$ and $\gamma_2(w_1, w_2)$ are calculated, we obtain

$$\Gamma(w_1, w_2) = -\{\cos(w_1 - w_2)\}^2,$$

$$\gamma_1(w_1, w_2) = -2\{\cos(2w_1)\}^2,$$

$$\gamma_2(w_1, w_2) = -2\{\sin(w_1 + w_2)\}^2.$$

For the $\sigma(w_1, w_2) = 0$, the solution of the $(w_1, w_2) = \left(\frac{\pi}{4}, -\frac{\pi}{4}\right), \Gamma(w_1, w_2) = \gamma_1(w_1, w_2) = \gamma_2(w_1, w_2) = 0$. Therefore, the parameter curves $l_A(w_1)$ and $l_B(w_2)$ intersect. As a result, $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ intersect.

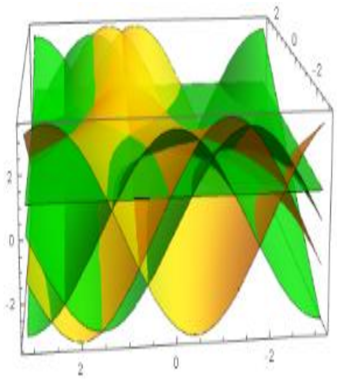

Figure 3. The intersection of spacelike and timelike ruled surfaces

4. INTERSECTION OF TWO RULED SURFACES CORRESPONDING TO CURVES ON THE HYPERBOLIC UNIT SPHERE

In this section, the intersections of two timelike ruled surfaces corresponding to two different curves on the dual hyperbolic unit sphere $\mathbb{H}^2$ are examined and some examples are given to support the main results.

4.1. INTERSECTION OF TWO TIMELIKE RULED SURFACES CORRESPONDING TO CURVES ON THE HYPERBOLIC UNIT SPHERE

Let $\bar{\alpha}_1 = \alpha_1 + \varepsilon\alpha_1^*$ and $\bar{\alpha}_2 = \alpha_2 + \varepsilon\alpha_2^*$ be curves on the dual hyperbolic unit sphere $\mathbb{H}^2$. With the help of E. Study mapping, let $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ be the timelike ruled surfaces corresponding to these curves given above in $\mathbb{R}_1^3$. These timelike ruled surfaces are denoted as follows:

$$\phi^A(w_1, s) = \alpha_1(w_1) \times_L \alpha_1^*(w_1) + s\alpha_1(w_1) \tag{14}$$

and

$$\phi^B(w_2, t) = \alpha_2(w_2) \times_L \alpha_2^*(w_2) + t\alpha_2(w_2). \tag{15}$$

Here, the timelike base curves of the timelike ruled surfaces $\phi^A(w_1, s)$ and $\phi^B(w_2, t)$ are

$$D_A = \alpha_1(w_1) \times_L \alpha_1^*(w_1)$$

and

$$D_B = \alpha_2(w_2) \times_L \alpha_2^*(w_2).$$

The w_1-parameter curve of $\phi^A(w_1, s)$ with a constant s_0-parameter is presented by the following equation:

$$l_A(w_1) = \phi^A(w_1, s_0).$$

Similarly, the w_2-parameter curve of $\phi^B(w_2,t)$ with a constant t_0-parameter is presented by the following equation:

$$l_B(w_2)=\phi^B(w_2,t_0).$$

As $\phi^A(w_1,s)$ and $\phi^B(w_2,t)$ intersect,

$$\phi^A(w_1,s)=\phi^B(w_2,t),$$

we write

$$D_A(w_1)-D_B(w_2)=-s\alpha_1(w_1)+t\alpha_2(w_2)$$

Since these three vectors are linearly dependent, the following equation can be presented

$$\sigma(w_1,w_2)=det\{\alpha_1(w_1),\alpha_2(w_2),[D_A(w_1)-D_B(w_2)]\}.$$

Theorem 11. Let $\phi^A(w_1,s)$ and $\phi^B(w_2,t)$ be the timelike ruled surfaces corresponding to two different curves on the dual hyperbolic unit sphere at $\mathbb{R}_1^3$. If

$$\sigma(w_1,w_2)=det\{\alpha_1(w_1),\alpha_2(w_2),[D_A(w_1)-D_B(w_2)]\}=0,$$

then the parameter curves $l_A(w_1)$ and $l_B(w_2)$ intersect.

Theorem 12. Let $l_A(w_1)$ and $l_B(w_2)$ be the parameter curves of the timelike ruled surfaces $\phi^A(w_1,s)$ and $\phi^B(w_2,t)$ corresponding to two different curves on the dual hyperbolic unit sphere at $\mathbb{R}_1^3$. In this case, the parameter curves $l_A(w_1)$ and $l_B(w_2)$ intersect if and only if

$$\Gamma(w_1,w_2)=\gamma_1(w_1,w_2)=\gamma_2(w_1,w_2)=0.$$

Example 4: Let $\alpha_1(w_1)=\left(\sin w_1,\cos w_1,\sqrt{2}\right)$ and $\alpha_1^*(w_1)=(\cos w_1,-\sin w_1,0)$ be timelike and spacelike vectors in $\mathbb{R}_1^3$, respectively. Since $\langle\alpha_1(w_1),\alpha_1(w_1)\rangle_L=-1$ and $\langle\alpha_1(w_1),\alpha_1^*(w_1)\rangle_L=0$, the $\bar{\alpha}_1=\alpha_1+\varepsilon\alpha_1^*$ is on the dual Hyperbolic unit sphere. The timelike ruled surface corresponding to $\bar{\alpha}_1$ is

$$\phi^A(w_1,s)=\left(\sqrt{2}\sin w_1,-\sqrt{2}\cos w_1,1\right)+s\left(\sin w_1,\cos w_1,\sqrt{2}\right).$$

Here, the spacelike base curve and timelike direction of this timelike ruled surface are

$$D_A(w_1)=\left(\sqrt{2}\sin w_1,-\sqrt{2}\cos w_1,1\right)$$

and

$$\alpha_1(w_1)=\left(\sin w_1,\cos w_1,\sqrt{2}\right).$$

Let $\alpha_2(w_2)=\left(-\cos w_2,\sin w_2,\sqrt{2}\right)$ and $\alpha_2^*(w_2)=(\sin w_2,\cos w_2,0)$ be timelike and spacelike vectors in $\mathbb{R}_1^3$, respectively. Since $\langle\alpha_2(w_2),\alpha_2(w_2)\rangle_L=-1$ and $\langle\alpha_2(w_2),\alpha_2^*(w_2)\rangle_L=0$, the $\bar{\alpha}_2=\alpha_2+\varepsilon\alpha_2^*$ is on the dual hyperbolic unit sphere. The timelike ruled surface corresponding to $\bar{\alpha}_2$ is

$$\phi^B(w_2,t)=\left(-\sqrt{2}\cos w_2,-\sqrt{2}\sin w_2,1\right)+t\left(-\cos w_2,\sin w_2,\sqrt{2}\right).$$

Here, the timelike base curve and spacelike direction of this timelike ruled surface are

$$D_B(w_2) = \left(-\sqrt{2}\cos w_2, -\sqrt{2}\sin w_2, \sqrt{2}\right)$$

and

$$\alpha_2(w_2) = \left(-\cos w_2, \sin w_2, \sqrt{2}\right).$$

Let's examine the intersection of two timelike ruled surfaces corresponding to two different curves on the dual Hyperbolic unit sphere. Assume that $\phi^A(w_1, s) = \phi^B(w_2, t)$. The following equation can be written by

$$\sqrt{2}(\sin w_1 + \cos w_2, \sin w_2 - \cos w_1, 0) = -s\left(\sin w_1, \cos w_1, \sqrt{2}\right) + t\left(-\cos w_2, \sin w_2, \sqrt{2}\right).$$

Here, $\mathrm{D_A}(\mathrm{w_1}) - \mathrm{D_B}(\mathrm{w_2})$ is written as a linear combination of vectors $\alpha_1(\mathrm{w_1})$ and $\alpha_2(\mathrm{w_2})$. Since these three vectors are linearly dependent, $\sigma(\mathrm{w_1}, \mathrm{w_2}) = 0$. Since $\sigma(\mathrm{w_1}, \mathrm{w_2}) = 0$, the parameter curves $\mathrm{l_A}(\mathrm{w_1})$ and $\mathrm{l_B}(\mathrm{w_2})$ intersect. If $\Gamma(\mathrm{w_1}, \mathrm{w_2}), \gamma_1(\mathrm{w_1}, \mathrm{w_2})$ and $\gamma_2(\mathrm{w_1}, \mathrm{w_2})$ are calculated, we obtain

$$\Gamma(w_1, w_2) = [\sin(w_1 - w_2) + 1].[\sin(w_1 - w_2) + 3],$$

$$\gamma_1(w_1, w_2) = -2\{(\cos(w_1 + w_2) + \sin(2w_1))^2 - 4(\sin(w_1 - w_2) + 1)\},$$

$$\gamma_2(w_1, w_2) = -2\{(\cos(w_1 + w_2) - \sin(2w_1))^2 - 4(\sin(w_1 - w_2) + 1)\}.$$

For the $\sigma(\mathrm{w_1}, \mathrm{w_2}) = 0$, the solution of the $(\mathrm{w_1}, \mathrm{w_2}) = \left(0, \frac{\pi}{2}\right), \Gamma(\mathrm{w_1}, \mathrm{w_2}) = \gamma_1(\mathrm{w_1}, \mathrm{w_2}) = \gamma_2(\mathrm{w_1}, \mathrm{w_2}) = 0$. Therefore, the parameter curves $\mathrm{l_A}(\mathrm{w_1})$ and $\mathrm{l_B}(\mathrm{w_2})$ intersect. As a result, $\phi^A(\mathrm{w_1}, \mathrm{s})$ and $\phi^B(\mathrm{w_2}, \mathrm{t})$ intersect.

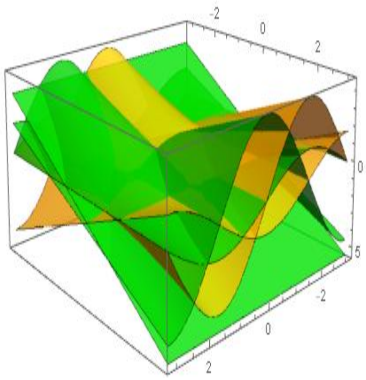

Figure 4. The intersection of timelike ruled surfaces.

5. CONCLUSIONS

In this paper, considering the E. Study mapping, the curve on the dual Lorentz unit sphere (or the dual hyperbolic unit sphere) corresponds to the ruled surfaces in Lorentzian space. At the same time, the intersection curve of two ruled surfaces in Lorentzian space is presented with the help of bivariate functions. Additionally, the intersection curve of two different ruled surfaces in Lorentzian space is investigated with E. Study mapping to two different curves taken on the dual Lorentz unit sphere (or the dual hyperbolic unit sphere). Afterwards, the intersection curve for the corresponding ruled surfaces is expressed by some main theorems. Consequently, these theorems are supported by many examples. Moreover, these results have important application areas in solid modeling and geometry.

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