*ORIGINAL PAPER*

# OSCULATING CURVES ACCORDING TO THE EXTENDED DARBOUX FRAME IN $E^4$

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***Abstract.*** *In this paper, we define osculating curves according to the extended Darboux frame (ED-frame), in Euclidean 4-space. Then, according to Case 1 and Case 2 ED-frame fields, the necessary and sufficient condition for a unit speed curve lying on the hypersurface to be congruent to an osculating curve is obtained. For these necessary and sufficient conditions, the solutions of the differential equations have been investigated. Finally, an application has been made on the subject.*

***Keywords:*** *osculating curve; extended Darboux frame; geodesic curvature; geodesic torsion.*

## 1. INTRODUCTION

Some special curves in differential geometry are quite remarkable for researchers. One of them is an osculating curve. After Chen [1] defined the rectifying curve in Euclidean space, Ilarslan and Nesovic [2] using the definition of the rectifying curve, defined the osculating curve as a curve in Euclidean 4-space whose position vector is always lied in the orthogonal complement of the first binormal vector field. This definition given for the osculating curve has been used in some studies in Euclidean space [2-4].

The frame fields are very useful for defining curves and examining properties. One of these frame fields is the Frenet frame fields. The Frenet frame along a curve is a moving (right-handed) coordinate system determined by the tangent line and curvature [5, 6]. Another important frame field is known as Darboux frame [7]. In addition to these frame fields, a new frame fields has been introduced to the literature. This frame fields was defined by Düldül et al. and named as extended Darboux frame field (ED-frame field) [8]. By using the newly defined frame fields, some special curves in Euclidean space are defined and their properties are investigated [9, 10].

In this study, we define osculating curves according to the extended Darboux frame in Euclidean 4-space and characterize such curves in terms of their curvature functions.

## 2. PRELIMINARIES

**Definition 2.1** Let $x=\sum_{i=1}^{4} x_i e_i$, $y=\sum_{i=1}^{4} y_i e_i$, $z=\sum_{i=1}^{4} z e_i$ be vectors in Euclidean 4-space $E^4$, where $\{e_i\}$, $1 \leq i \leq 4$ is the standart basis vectors of $E^4$. The vector product of three vector is given by [11]

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$$x \otimes y \otimes z = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}.$$

Let $\mathcal{M}$ be an orientable hypersurface and the curve $\gamma$ lies on $\mathcal{M}$. On the other hand, if the unit tangent vector field of the curve is $T$, the unit normal vector field of the hypersurface restricted to the curve $\gamma$ is $N$ and the unit normal field vector of $\mathcal{M}$ is $\mathcal{N}$, then $T$ is given as $T = \gamma'(s)$ and $N(s) = \mathcal{N}(\gamma(s))$ [8].

**Case 1.** Let $\{N, T, \gamma''\}$ be linearly independent. In this case, the orthonormal set $\{N, T, E\}$ with $E = \frac{\gamma'' - \langle \gamma'', N \rangle N}{\|\gamma'' - \langle \gamma'', N \rangle N\|}$, is obtained [8].

**Case 2.** Let $\{N, T, \gamma''\}$ be linearly dependent. In this case, the orthonormal set $\{N, T, E\}$ with $E = \frac{\gamma'' - \langle \gamma'', N \rangle N - \langle \gamma''', T \rangle T}{\gamma'' - \langle \gamma'', N \rangle N - \langle \gamma''', T \rangle T}$, is obtained [8].

If $D = N \otimes T \otimes E$, then we obtain orthonormal frame field $\{T, E, D, N\}$ along the curve $\gamma$, [5]. One can easily see that the vector fields $E$ and $D$ are tangent to $\mathcal{M}$. Also, $\{T, E, D\}$ spans the tangent hyperplane of the hypersurface at the point $\gamma$ (s) [8].

Let $\kappa_n$ be the normal curvature of the hypersurface in the direction of the tangent vector $T$, $\kappa_g^i$ and $\tau_g^i$ be the geodesic curvatures and the geodesic torsion of order i (i=1,2), respectively [8]. The derivative equations for Case 1 and Case 2 are given with

$$\begin{bmatrix} T' \\ E' \\ D' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g^1 & 0 & \kappa_n \\ -\kappa_g^1 & 0 & \kappa_g^2 & \tau_g^1 \\ 0 & -\kappa_g^2 & 0 & \tau_g^2 \\ -\kappa_n & -\tau_g^1 & -\tau_g^2 & 0 \end{bmatrix} \begin{bmatrix} T \\ E \\ D \\ N \end{bmatrix}, \tag{1}$$

and

$$\begin{bmatrix} T' \\ E' \\ D' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \kappa_n \\ 0 & 0 & \kappa_g^2 & \tau_g^1 \\ 0 & -\kappa_g^2 & 0 & 0 \\ -\kappa_n & -\tau_g^1 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ E \\ D \\ N \end{bmatrix}. \tag{2}$$

On the other hand, the following statements hold [8]:

$$\langle T', N \rangle = \kappa_n,\ \langle E', N \rangle = \tau_g^1,\ \langle D', N \rangle = \tau_g^2,\ \langle T', E \rangle = \kappa_g^1 \text{ and } \langle E', D \rangle = \kappa_g^2. \tag{3}$$

## 3. OSCULATING CURVES ACCORDING TO ED-FRAME IN EUCLIDEAN 4-SPACE $E^4$

In this section, we define the osculating curves according to the extended Darboux frame in Euclidean 4-space. And then, we find the relationship between the curvatures for any unit speed curve which lies on the orientable hypersurface $\mathcal{M}$ to be congruent to this osculating curve in $E^4$.

**Definition 3.1** Let $\alpha: I \subset \mathbb{R} \to \mathcal{M}$ be a unit speed curve on an oriented hypersurface $\mathcal{M}$ in Euclidean 4-space and $\{T, E, D, N\}$ denote the ED-frame field of $\alpha(s)$. Then we define the osculating curve according to the ED-frame in the Euclidean space $E^4$ as a curve whose position vector always lies in the orthogonal complement $D^{\perp}$, and we express it with

$$\alpha(s) = \lambda(s)T(s) + \mu_1(s)E(s) + \mu_2(s)N(s) \tag{4}$$

for some differentiable functions $\lambda(s), \mu_1(s)$ and $\mu_2(s)$ of $s \in I \subset \mathbb{R}$.

**Theorem 3.1.** Let $\alpha: I \subset \mathbb{R} \to \mathcal{M}$ be a unit speed curve on an oriented hypersurface $\mathcal{M}$ in Euclidean 4-space and $\{T, E, D, N\}$ denote the ED-frame field of $\alpha(s)$. Then $\alpha(s)$ is congruent to a osculating curve if and only if

$$\frac{d}{ds}\left\{\left(\frac{1}{\kappa_g^1}\right)\left[\left(\frac{\tau_g^2}{\kappa_g^2}\right)' + A.\left(\frac{\tau_g^2}{\kappa_g^2}\right)\right]\right\} + \frac{1}{\kappa_g^1}\left[(A' + A^2)\left(\frac{\tau_g^2}{\kappa_g^2}\right) - A\tau_g^1\right] + \kappa_g^1\left(\frac{\tau_g^2}{\kappa_g^2}\right) - \frac{\tau_g^1}{\kappa_g^1} + \kappa_n = -\frac{1}{c_1}e^{-\int A ds} \qquad \text{(Case 1)}$$

where $c_1 \epsilon\ \mathbb{R} - \{0\}$, $\kappa_g^1, \kappa_g^2 \neq 0$, and $A = A(s) = \dfrac{\kappa_n(s)\tau_g^1(s) + \frac{\kappa_g^1(s)\tau_g^1(s)\tau_g^2(s)}{\kappa_g^2(s)} - \kappa_n(s)\left(\frac{\tau_g^2(s)}{\kappa_g^2(s)}\right)'}{\kappa_n(s)\frac{\tau_g^2(s)}{\kappa_g^2(s)} - \kappa_g^1(s)}$.

$$\frac{d}{ds}\left\{\frac{\tau_g^1}{\kappa_n}\right\} = -\frac{c_3}{c_2}\kappa_n, \qquad \text{(Case 2)}$$

where $c_2 \in \mathbb{R} - \{0\}, c_3 \epsilon\ \mathbb{R}$, $\kappa_n \neq 0$.

*Proof:* Let $\alpha: I \subset \mathbb{R} \to \mathcal{M}$ be a unit speed curve on an oriented hypersurface $\mathcal{M}$ in Euclidean 4-space and $\{T, E, D, N\}$ denote the ED-frame field of $\alpha(s)$. If the derivative of both sides of equation (4) with respect to s is taken and the derivative equation (1) is applied, the following expression for Case 1 is obtained that

$$\alpha'(s) = \left(\lambda'(s) - \mu_1(s)\kappa_g^1(s) - \mu_2(s)\kappa_n(s)\right)T(s) + \left(\mu_1'(s) + \lambda(s)\kappa_g^1(s) - \mu_2(s)\tau_g^1(s)\right)E(s)$$
$$+ \left(\mu_1(s)\kappa_g^2(s) - \mu_2(s)\tau_g^2(s)\right)D(s) + \left(\mu_2'(s) + \lambda(s)\kappa_n(s) + \mu_1(s)\tau_g^1(s)\right)N(s). \text{ (Case 1)}$$

We know that $\alpha'(s) = T(s)$. So, using the equality of both sides, we get the following expressions for the coeffcients of $T(s)$, $E(s)$, $D(s)$ and $N(s)$.

**Case 1.**

$$\lambda'(s) - \mu_1(s)\kappa_g^1(s) - \mu_2(s)\kappa_n(s) = 1, \tag{5}$$

$$\mu_1'(s) + \lambda(s)\kappa_g^1(s) - \mu_2(s)\tau_g^1(s) = 0, \tag{6}$$

$$\mu_1(s)\kappa_g^2(s) - \mu_2(s)\tau_g^2(s) = 0, \tag{7}$$

$$\mu_2'(s) + \lambda(s)\kappa_n(s) + \mu_1(s)\tau_g^1(s) = 0. \tag{8}$$

If the equations (6) and (8) are solved together, and using the equation $\mu_1(s) = \mu_2(s)\frac{\tau_g^2(s)}{\kappa_g^2(s)}$, then we obtain

$$\frac{\mu_2'(s)}{\mu_2(s)}=\frac{\kappa_n(s)\tau_g^1(s)+\frac{\kappa_g^1(s)\tau_g^1(s)\tau_g^2(s)}{\kappa_g^2(s)}-\kappa_n(s)\left(\frac{\tau_g^2(s)}{\kappa_g^2(s)}\right)'}{\kappa_n(s)\frac{\tau_g^2(s)}{\kappa_g^2(s)}-\kappa_g^1(s)}.$$

Let's denote

$$A(s)=\frac{\kappa_n(s)\tau_g^1(s)+\frac{\kappa_g^1(s)\tau_g^1(s)\tau_g^2(s)}{\kappa_g^2(s)}-\kappa_n(s)\left(\frac{\tau_g^2(s)}{\kappa_g^2(s)}\right)'}{\kappa_n(s)\frac{\tau_g^2(s)}{\kappa_g^2(s)}-\kappa_g^1(s)}.$$

If we integrate the equation $\frac{\mu_2'(s)}{\mu_2(s)}=A(s)$, we find

$$\mu_2(s)=c_1e^{\int A(s)ds}. \tag{9}$$

On the other hand, one can easily see that

$$\mu_1(s)=c_1\frac{\tau_g^2(s)}{\kappa_g^2(s)}e^{\int A(s)ds}. \tag{10}$$

Considering equation (6), the differentiable function $\lambda(s)$ can be expressed in terms of curvatures $\kappa_g^1(s),\kappa_g^2(s),\tau_g^1(s)$ and $\tau_g^2(s)$ as follows:

$$\lambda(s)=\frac{c_1}{\kappa_g^1(s)}\left[e^{\int A(s)ds}\,\tau_g^1(s)-\left(\frac{\tau_g^2(s)}{\kappa_g^2(s)}e^{\int A(s)ds}\right)'\right] \tag{11}$$

ff equations (9)-(11) are written in equation (5) and the resulting expression is edited, we obtained that for Case 1

$$\frac{d}{ds}\left\{\left(\frac{1}{\kappa_g^1}\right)\left[\left(\frac{\tau_g^2}{\kappa_g^2}\right)'+A.\left(\frac{\tau_g^2}{\kappa_g^2}\right)\right]\right\}+\frac{1}{\kappa_g^1}\left[(A'+A^2)\left(\frac{\tau_g^2}{\kappa_g^2}\right)-A\tau_g^1\right]+\kappa_g^1\left(\frac{\tau_g^2}{\kappa_g^2}\right)-\frac{\tau_g^1}{\kappa_g^1}+\kappa_n = -\frac{1}{c_1}e^{-\int Ads} \tag{12}$$

where $c_1\in\ \mathbb{R}-\{0\}\,,\kappa_g^1,\kappa_g^2\neq 0$

Now let's examine Case 2. It is clear that the following equations can be easily obtained if what is done for Case1 is done according to Case 2.

$$\alpha'(s)=\big(\lambda'(s)-\mu_2(s)\kappa_n(s)\big)T(s)+\left(\mu_1'(s)-\mu_2(s)\tau_g^1(s)\right)E(s)$$

$$+\mu_1(s)\kappa_g^2(s)D(s)+\left(\mu_2'(s)+\lambda(s)\kappa_n(s)+\mu_1(s)\tau_g^1(s)\right)N(s).$$

**Case 2.**

$$\lambda'(s)-\mu_2(s)\kappa_n(s)=1, \tag{13}$$

$$\mu_1'(s)-\mu_2(s)\tau_g^1(s)=0, \tag{14}$$

$$\mu_1(s)\kappa_g^2(s) = 0, \tag{15}$$

$$\mu_2'(s) + \lambda(s)\kappa_n(s) + \mu_1(s)\tau_g^1(s) = 0. \tag{16}$$

Starting from (15), we integrate these expressions with respect to s to obtain the coefficient functions

$$\mu_1(s) = c_2,\, c_2 \in \mathbb{R}. \tag{17}$$

Similarly, the integration of (14) yield

$$\mu_2(s) = c_3,\, c_3 \in \mathbb{R}. \tag{18}$$

From equation (16), we get that

$$\lambda(s) = -c_2 \frac{\tau_g^1(s)}{\kappa_g^1(s)}. \tag{19}$$

If equations (17)-(19) are written in equation (13) and the resulting expression is edited, we obtained that for Case 1

$$\frac{d}{ds}\left\{\frac{\tau_g^1}{\kappa_n}\right\} = -\frac{c_3}{c_2}\kappa_n \tag{20}$$

where $c_2 \in \mathbb{R} - \{0\}, c_3 \epsilon\, \mathbb{R}\,, \kappa_n \neq 0.$

Conversely, consider an arbitrary unit speed curve on an oriented hypersurface $\mathcal{M}$ in Euclidean 4-space for which the curvature functions satisfy the relations (12) and (20). Then, we consider the vector $X \epsilon E^4$ defined by

$$X(s) = \alpha(s) - \frac{c_1}{\kappa_g^1(s)}\left[e^{\int A(s)ds}\,\tau_g^1(s) - \left(\frac{\tau_g^2(s)}{\kappa_g^2(s)}e^{\int A(s)ds}\right)'\right]T(s) - c_1\frac{\tau_g^2(s)}{\kappa_g^2(s)}e^{\int A(s)ds}E(s)$$
$$-c_1 e^{\int A(s)ds}N(s), \text{ (Case 1)}$$

$$X(s) = \alpha(s) + c_2\frac{\tau_g^1(s)}{\kappa_g^1(s)}T(s) - c_2E(s) - c_3N(s). \text{ (Case 2)}$$

It can be seen that $X(s) = 0$ through the relations (1), (2), (12) and (20). Thus, X is a constant vector. This implies that $\alpha$ is congruent to an osculating curve.

**Corollarly 3.1.** Let $\alpha: I \subset \mathbb{R} \to \mathcal{M}$ be a unit speed line of curvature $(\tau_g^1(s) = \tau_g^2(s) = 0)$ on an oriented hypersurface $\mathcal{M}$ in Euclidean 4-space and $\{T, E, D, N\}$ denote Case 1 ED-frame field of $\alpha(s)$. Then $\alpha(s)$ is congruent to an osculating curve if and only if

$$\frac{d}{ds}\left(\frac{1}{\kappa_g^1}\right) + \kappa_n = -\frac{1}{c_1}$$

where $c_1 \epsilon\, \mathbb{R} - \{0\}\,, \kappa_g^1 \neq 0.$

*Proof:* The proof is obvious from theorem (4).

**Corollarly 3.2.** Let $\alpha: I \subset \mathbb{R} \to \mathcal{M}$ be a unit speed line of curvature $(\tau_g^1(s) = \tau_g^2(s) = 0)$ on an oriented hypersurface $\mathcal{M}$ in Euclidean 4-space and $\{T, E, D, N\}$ denote Case 2 ED-frame field of $\alpha(s)$. Then $\alpha(s)$ is congruent to an osculating curve if and only if $\kappa_n$=0. (It means that osculating curve is an asymptotic curve).

**Theorem 3.2.** Let $\alpha: I \subset \mathbb{R} \to \mathcal{M}$ be a unit speed curve on an oriented hypersurface $\mathcal{M}$ in Euclidean 4-space and $\{T, E, D, N\}$ denote Case 1 ED-frame field of $\alpha(s)$. The differential equation

$$\frac{d}{ds}\left\{\left(\frac{1}{\kappa_g^1}\right)\left[\left(\frac{\tau_g^2}{\kappa_g^2}\right)' + A.\left(\frac{\tau_g^2}{\kappa_g^2}\right)\right]\right\} + \frac{1}{\kappa_g^1}\left[(A' + A^2)\left(\frac{\tau_g^2}{\kappa_g^2}\right) - A\tau_g^1\right] + \kappa_g^1\left(\frac{\tau_g^2}{\kappa_g^2}\right) - \frac{\tau_g^1}{\kappa_g^1} + \kappa_n = -\frac{1}{c_1}e^{-\int A ds}$$

where $c_1 \in \mathbb{R} - \{0\}$, $\kappa_g^1, \kappa_g^2 \neq 0$ with the initial conditions $\frac{\tau_g^2(s_0)}{\kappa_g^2(s_0)} = \delta_0$, $\left(\frac{\tau_g^2(0)}{\kappa_g^2(0)}\right)' = \delta_1$ for $s_0 \in I \subset \mathbb{R}$ has a unique solution on an open interval $I \subset \mathbb{R}$ if the functions $f_5(s)$, $f_6(s)$ and $f_7(s)$ are continuous on $I \subset \mathbb{R}$ . This equation has a general solution of the form

$$\frac{\tau_g^2(s)}{\kappa_g^2(s)} = k_1\left(\frac{\tau_g^2}{\kappa_g^2}\right)_1(s) + k_2\left(\frac{\tau_g^2}{\kappa_g^2}\right)_2(s)$$

where $\left(\frac{\tau_g^2}{\kappa_g^2}\right)_1, \left(\frac{\tau_g^2}{\kappa_g^2}\right)_2(s)$ form the fundamental set of solutions for the homogeneous equation

$$f_1(s)\left(\frac{\tau_g^2}{\kappa_g^2}\right)''(s) + f_2(s)\left(\frac{\tau_g^2}{\kappa_g^2}\right)'(s) + f_3(s)\left(\frac{\tau_g^2}{\kappa_g^2}\right)(s) + f_4(s)$$

satisfying the condition

$$W\left(\left(\frac{\tau_g^2}{\kappa_g^2}\right)_1, \left(\frac{\tau_g^2}{\kappa_g^2}\right)_2\right) = \begin{vmatrix} \left(\frac{\tau_g^2}{\kappa_g^2}\right)_1 & \left(\frac{\tau_g^2}{\kappa_g^2}\right)_2 \\ \left(\frac{\tau_g^2}{\kappa_g^2}\right)_1' & \left(\frac{\tau_g^2}{\kappa_g^2}\right)_2' \end{vmatrix} \neq 0.$$

*Proof:* Let $y = y(s) = \frac{\tau_g^2(s)}{\kappa_g^2(s)}$, $u = u(s) = \kappa_g^1(s)$, $v = v(s) = A'(s)$, $z = z(s) = \tau_g^1(s)$, $h = h(s) = \frac{\tau_g^1(s)}{\kappa_g^1(s)}$, $g = g(s) = \kappa_n$, $d = -\frac{1}{c_1}$ and $w = w(s) = e^{-\int A ds}$. Then the differential equation (12) can be written as follows:

$$\left(-\frac{u}{u^2}\right)y'' + \left(\frac{1}{u} + \frac{A}{u^2}\right)y' + \left(-\frac{A}{u^2} - \frac{Au'}{u^2} + \frac{v}{u} + \frac{A^2}{u} + u\right)y + \left(g - h - \frac{A}{u}z - dw\right) = 0.$$

Let

$$f_1(s) = -\frac{u}{u^2},\ f_2(s) = \frac{1}{u} + \frac{A}{u^2},\ f_3(s) = -\frac{A}{u^2} - \frac{Au'}{u^2} + \frac{v}{u} + \frac{A^2}{u} + u \text{ and } f_4(s) = g - h - \frac{A}{u}z - dw.$$

Then we get the following nonhomogeneous linear differential equation

$$f_1(s)y'' + f_2(s)y' + f_3(s)y + f_4(s) = 0.$$

From the last lineer differential equation, we obtain the initial value problem:

$$y'' + f_5(s)y' + f_6(s)y + f_7(s) = 0, y(s_0) = \delta_0, y'(s_0) = \delta_1 \text{ for } s_0 \in I \subset \mathbb{R},$$

where $f_5(s) = \frac{f_2(s)}{f_1(s)}$, $f_6(s) = \frac{f_3(s)}{f_1(s)}$ and $f_4(s) = \frac{f_4(s)}{f_1(s)}$.

Let choose the functions $f_5(s)$, $f_6(s)$ and $f_7(s)$ continuously on $I \subset \mathbb{R}$ . Thus the differential equation has a general solution of the form

$$\frac{\tau_g^2(s)}{\kappa_g^2(s)} = k_1 \left(\frac{\tau_g^2}{\kappa_g^2}\right)_1 (s) + k_2 \left(\frac{\tau_g^2}{\kappa_g^2}\right)_2 (s)$$

where $\left(\frac{\tau_g^2}{\kappa_g^2}\right)_1, \left(\frac{\tau_g^2}{\kappa_g^2}\right)_2 (s)$ form the fundamental set of solutions for the homogeneous equation

$$f_1(s)\left(\frac{\tau_g^2}{\kappa_g^2}\right)'' (s) + f_2(s)\left(\frac{\tau_g^2}{\kappa_g^2}\right)' (s) + f_3(s)\left(\frac{\tau_g^2}{\kappa_g^2}\right)(s) + f_4(s)$$

satisfying the condition $W\left(\left(\frac{\tau_g^2}{\kappa_g^2}\right)_1, \left(\frac{\tau_g^2}{\kappa_g^2}\right)_2\right) = \begin{vmatrix} \left(\frac{\tau_g^2}{\kappa_g^2}\right)_1 & \left(\frac{\tau_g^2}{\kappa_g^2}\right)_2 \\ \left(\frac{\tau_g^2}{\kappa_g^2}\right)_1' & \left(\frac{\tau_g^2}{\kappa_g^2}\right)_2' \end{vmatrix} \neq 0.$

**Theorem 3.3.** Let $\alpha: I \subset \mathbb{R} \to \mathcal{M}$ be a unit speed curve on an oriented hypersurface $\mathcal{M}$ in Euclidean 4-space and $\{T, E, D, N\}$ denote Case 2 ED-frame field of $\alpha(s)$. The curvatures $\kappa_n(s)$, $\tau_g^2(s)$ satisfy the following equality

$$\frac{\tau_g^2(s)}{\kappa_g^2(s)} = -\frac{c_3}{c_2}\int \kappa_n(s)ds.$$

*Proof:* Assume that $\alpha: I \subset \mathbb{R} \to \mathcal{M}$ be a unit speed curve on an oriented hypersurface $\mathcal{M}$ in Euclidean 4-space. If we take the integral both side of the equation (20), then we get

$$\frac{\tau_g^2(s)}{\kappa_g^2(s)} = -\frac{c_3}{c_2}\int \kappa_n(s)ds.$$

**Theorem 3.4.** Let $\alpha: I \subset \mathbb{R} \to \mathcal{M}$ be a unit speed curve on an oriented hypersurface $\mathcal{M}$ in Euclidean 4-space and $\{T, E, D, N\}$ denote the ED-frame field of $\alpha(s)$. If $\alpha$ is an osculating curve, then the following statements hold:

**i)** $\langle \alpha(s), T(s)\rangle = \frac{c_1}{\kappa_g^1(s)}\left[e^{\int A(s)ds}\, \tau_g^1(s) - \left(\frac{\tau_g^2(s)}{\kappa_g^2(s)} e^{\int A(s)ds}\right)'\right]$, (Case 1)

where $\kappa_g^1(s)$, $\kappa_g^2(s) \neq 0$,

$$\langle \alpha(s), T(s)\rangle = -c_2 \frac{\tau_g^1(s)}{\kappa_g^1(s)}, \text{ (Case 2) where } \kappa_g^1(s) \neq 0.$$

**ii)** $\langle \alpha(s), E(s)\rangle = c_1 \frac{\tau_g^2(s)}{\kappa_g^2(s)}$, (Case 1)

where $\kappa_g^2(s) \neq 0$.

$\langle \alpha(s), E(s)\rangle = c_2$, (Case 2)

where $c_2 \in \mathbb{R}$.

**iii)** $\langle \alpha(s), N(s)\rangle = c_1 e^{\int A(s)ds}$, (Case 1)

$\langle \alpha(s), N(s)\rangle = c_3$, (Case 2)

where $c_3 \in \mathbb{R}$.

*Proof:* By using the relations (5)-(8), (13)-(16), we get

$$\alpha(s) = \frac{c_1}{\kappa_g^1(s)}\left[e^{\int A(s)ds}\,\tau_g^1(s) - \left(\frac{\tau_g^2(s)}{\kappa_g^2(s)} e^{\int A(s)ds}\right)'\right] T(s) + c_1 \frac{\tau_g^2(s)}{\kappa_g^2(s)} e^{\int A(s)ds} E(s)$$

$$+ c_1 e^{\int A(s)ds} N(s), \text{ (Case 1)}$$

$$\alpha(s) = -c_2 \frac{\tau_g^1(s)}{\kappa_g^1(s)} T(s) + c_2 E(s) + c_3 N(s). \text{ (Case 2)}$$

From the last two equations, we obtain **i)**, **ii)** and **iii).**

## 4. APPLICATION

**Example 3.1.** Let

$$\alpha(s) = \left(sin\left(\frac{3s}{\sqrt{11}}\right), -sin\left(\sqrt{\frac{2}{11}}s\right), -cos\left(\frac{3s}{\sqrt{11}}\right), cos\left(\sqrt{\frac{2}{11}}s\right)\right)$$

be a unit speed curve on an hypersurface $\mathcal{M}$ ... $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 2$ in Euclidean 4-space. The unit normal vector of $\mathcal{M}$ along $\alpha$ is $N(s) = \frac{1}{\sqrt{2}}(\alpha(s))$. If we calculate the unit normal vector field we can find as follows:

$$T(s) = \left(\frac{3}{\sqrt{11}} cos\left(\frac{3s}{\sqrt{11}}\right), -\sqrt{\frac{2}{11}} cos\left(\sqrt{\frac{2}{11}}s\right), \frac{3}{\sqrt{11}} sin\left(\frac{3s}{\sqrt{11}}\right), -\sqrt{\frac{2}{11}} sin\left(\sqrt{\frac{2}{11}}s\right)\right).$$

The derivative of $T(s)$ is given by

$$T'(s) = \alpha''(s) = \left(-\frac{9}{11}sin\left(\frac{3s}{\sqrt{11}}\right), \frac{2}{11}sin\left(\sqrt{\frac{2}{11}}s\right), \frac{9}{11}cos\left(\frac{3s}{\sqrt{11}}\right), -\frac{2}{11}cos\left(\sqrt{\frac{2}{11}}s\right)\right).$$

Considering $\mathrm{T}'(\mathrm{s})$ and $\mathrm{N(s)}$, they appear to be linearly independent. Therefore, Case 1 applies.

$$E(s) = \left(-\frac{1}{\sqrt{2}}sin\left(\frac{3s}{\sqrt{11}}\right), -\frac{1}{\sqrt{2}}sin\left(\sqrt{\frac{2}{11}}s\right), \frac{1}{\sqrt{2}}cos\left(\frac{3s}{\sqrt{11}}\right), \frac{1}{\sqrt{2}}cos\left(\sqrt{\frac{2}{11}}s\right)\right),$$

and

$$D(s) = \left(-\frac{2}{\sqrt{11}}cos\left(\frac{3s}{\sqrt{11}}\right), -\frac{3}{\sqrt{11}}cos\left(\sqrt{\frac{2}{11}}s\right), -\frac{2}{\sqrt{11}}sin\left(\frac{3s}{\sqrt{11}}\right), -\frac{3}{\sqrt{11}}sin\left(\sqrt{\frac{2}{11}}s\right)\right).$$

If we use the equation (3) we get

$$\langle T', N\rangle = \kappa_n = -\tfrac{1}{\sqrt{2}}, \langle E', N\rangle = \tau_g^1 = 0, \langle D', N\rangle = \tau_g^2 = \tfrac{6-3\sqrt{2}}{11\sqrt{2}}, \langle T', E\rangle = \kappa_g^1 = \tfrac{7}{11\sqrt{2}}$$

and

$$\langle E', D\rangle = \kappa_g^2 = \frac{6-3\sqrt{2}}{11\sqrt{2}}.$$

If the equation (12) is used, it can be seen that for curve

$$\alpha(s) = \left(sin\left(\frac{3s}{\sqrt{11}}\right), -sin\left(\sqrt{\frac{2}{11}}s\right), -cos\left(\frac{3s}{\sqrt{11}}\right), cos\left(\sqrt{\frac{2}{11}}s\right)\right)$$

to be congruent to an osculating curve, the constant $c_1$ is found as $c_1 = \frac{11\sqrt{2}}{4}$.

## 5. CONCLUSION

In this paper, we study osculating curve according to the extended Darboux frame in Euclidean 4-space. We give the relationship between the curvatures for any unit speed curve which lies on the orientable hypersurface $\mathcal{M}$ to be congruent to this osculating curve in $E^4$. Finally, we present an application about the subject.

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