ORIGINAL PAPER

ON AN EXPANSION OF POST QUANTUM ANALYSIS

HAMZA MENKEN[1], BURÇAK HARNUPDALI[2]

Manuscript received: 10.08.2022; Accepted paper: 04.02.2023;
Published online: 30.03.2023.

Abstract. *In the present work we give an extension of (p,q)- analysis. As an extension of (p,q)-analysis, the (r,p,q)-analysis is introduced. We define some elementary concepts of this analysis such as (r,p,q)-numbers, (r,p,q)-derivative, (r,p,q)-exponential functions, (r,p,q)-antiderivative and (r,p,q)-integral. We obtain some properties of the polynomial* $(x-a)^n$ *(r,p,q)-Taylor formula, (r,p,q)-binomial coefficients, divided differences and some relations between (r,p,q)-derivative, (r,p,q)-exponential functions, (r,p,q)-integral and finally, the fundamental theorem of (r,p,q)-analysis are examined in details.*

Keywords: *(p,q) analysis; (p,q)-derivative; (p,q)-integral; extension of (p,q) analysis.*

1. INTRODUCTION

To understand the quantum analysis or q-analysis, it is necessary to know the classical analysis well. Although classical analysis and q-analysis are not exactly the same; they are not disconnected from each other. Recently, there has been increased interest q-analysis due to the high interest in the mathematics of quantum computing models.

The q-analysis has emerged as a link between mathematics and physics. Number theory, combinatorics, orthogonal polynomials, fundamental hyper-geometric functions, and other sciences have numerous applications in different fields of mathematics and physics, such as quantum theory, quantum mechanics, and relativity theory [1-3].

The q-analysis was first described by Euler. Euler proved the pentagonal number theorem

$$1+\sum_{m=1}^{\infty}(-1)^m\left(q^{\frac{m(3m-1)}{2}}+q^{\frac{m(3m+1)}{2}}\right)=\prod_{m=1}^{\infty}(1-q^m)\,,0<|q|<1$$

the first example of q-analysis, in 1750, and this is also the first example of the theta-function [2]. In 1866, 11 years after Euler's death, Gauss proved the equality

$$1+\sum_{m=1}^{\infty}q^{\binom{m+1}{2}}=\prod_{m=1}^{\infty}\frac{1-q^{2m}}{1-q^{2m-1}},|q|<1$$

which is another example of q-analysis [2]

In addition, Euler defined the q -derivative operator and the first form of the q -binomial theorem, which would be defined more than a century later [2]. The q-derivative

[1] Mersin University, Department of Mathematics, 33343 Mersin, Turkey. E-mail: hmenken@mersin.edu.tr.
[2] Mersin University, Graduate School of Natural and Applied Sciences, 33343 Mersin, Turkey.
E-mail: burcakharnupdali@gmail.com.

was first described by Euler, then Heine [4], and then by F. H. Jackson [5] in 1908 [see 1, 6]. Jackson was the first to systematically develop q-analysis [2].

In 1869, Edward Heine's student Thomae [7, 8] defined the q-integral on [0,1] by

$$\int_0^1 f(x)d_q x = (1-q)\sum_{n=0}^{\infty} q^n f(q^n)\,, 0 < q < 1$$

and in 1910, Jackson [9] defined the general q-integral on [a, b] by

$$\int_a^b f(x)d_q x = \int_0^b f(x)d_q x - \int_0^a f(x)d_q x$$

After Jackson's definitions of q-derivative and q-integral, studies have continued. With the addition of the p parameter to the q-analysis, a step was taken to the (p,q)-analysis. (p,q)-analysis was first and independently handled by Chakrabarti and Jagannathan [10], Brodimas et al. [11], Wachs and White [12], and Arik et al. [13].

Chakrabarti and Jagannathan described the (p,q)-analysis to generalize or combine various forms of q-oscillator algebras well known in the physics literature related to the symbol theory of single-parameter quantum algebras [10]. Brodimas et al. defined the (p,q)-number to derive the Bose representation of these operators by utilizing q-analysis by performing a Bargmann differential of the (p,q)-algebraic generation and destruction operator [11]. Wachs and White used the (p,q)-number in the mathematical literature to obtain the (p,q)-Stirling numbers that produces the common distribution of statistical pairs [12]. Arik et al. used the (p,q)-number to investigate Fibonacci oscillators [13].

Since 1991, q-analysis has been developed by many mathematicians and physicists in different research fields. For example, (p,q)-hypergeometric functions were defined by Burban and Klimyk and the relationships between basic hypergeometric functions, (p,q)-hypergeometric functions and (p,q)-hypergeometric functions were investigated [14]. The binomial coefficients (p,q)-analogue were developed by Corcino and some properties parallel to the known binomial coefficients and q-binomial coefficients were determined [15]. Some properties of q-derivative and q-integration were investigated by Sadjang [16]. Some connections between the (p,q)-derivative operator and divided differences were given by Araci et al., and the (p,q)-analogue of the Leibnitz rule was investigated with the help of divided differences [17]. These studies provided good ideas for the development of (p,q)-analysis in combinatorics, number theory, and other fields of mathematics and physics.

In this study, an extension of the (p,q)-analysis is considered and their various properties are given. We first first introduced the (r,p,q)-analysis by involving a parameter r to the (p,q)-analysis, and some elementary concepts of analysis such a (r,p,q)-numbers, (r,p,q)-derivative, (r,p,q)-exonential functions, (r,p,q)-antiderivative, and (r,p,q)-integral were defined. We obtain some properties of polynomial $(x-a)^n$, (r,p,q)-Taylor formula, (r,p,q)-binomial coefficients, divided differences and some relations between (r,p,q)-derivative, (r,p,q)-exponential functions, (r,p,q)-integral and finally, the fundamental theorem of (r,p,q)-analysis are examined in detail.

In this article, some elementary concepts of the (r,p,q)-analysis are defined. The (r,p,q)-numbers, (r,p,q)-differential, (r,p,q)-derivative, (r,p,q)-Taylor formula and fundamental theorem of (r,p,q)-analysis are studied and their properties are obtained. Firtly we recall some information about the q-analysis and (p,q)-analysis.

2. QUANTUM AND POST QUANTUM ANALYSİS

2.1. QUANTUM ANALYSIS

Let $q \in \mathbb{R}$, $q \neq 1$. q-extension (or q-analogue) of a ineger number $n \in \mathbb{Z}$ is defined by

$$[n]_q = \frac{q^n - 1}{q - 1}.$$

Here, it follows that

$$\lim_{q\to 1}[n]_q = \lim_{q\to 1}(1 + q + \cdots + q^{n-1}) = n.$$

The following formulas are valid for any real (or complex) numbers n and m (see [16, 18, 19]):

$$[n+m]_q = [n]_q + q^n[m]_q = [m]_q + q^m[n]_q$$

$$[n-m]_q = [n]_q - q^{(n-m)}[m]_q = q^{-m}(\,[n]_q - [m]_q)$$

$$[nm]_q = [n]_{q^m}[m]_q = [m]_{q^n}[n]_q$$

$$\left[\frac{n}{m}\right]_q = \frac{[\mathrm{n}]_{q^{1/m}}}{[\mathrm{m}]_{q^{1/m}}} = \frac{[n]_q}{[\mathrm{m}]_{q^{n/m}}}$$

The q-differential of any function $f(x)$ is defined by

$$d_q f(x) = f(qx) - f(x)$$

and the q-derivative of $f(x)$ is defined by

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x}, \; x \neq 0 .$$

The q-analogue of $n!$ is defined by

$$[n]_q ! = [n]_q [n-1]_q ... [2]_q [1]_q$$

and 0! =1 (see [20, 21]).

The q-analogue of the $(x-a)^n$ polynomial is defined as $(x-a)_q^0 = 1$ to

$$(x-a)_q^n = (x-a)(x-qa)\cdots(x-q^{n-1}a) .$$

For any polynomial $f(x)$ of degree N and any number c, the q-Taylor expansion of $f(x)$ is given by formula

$$f(x) = \sum_{j=0}^{N} \left(D_q^j f\right)(c) \frac{(x-c)_q^n}{[j]_q !} .$$

Let's take $f(x) = x^n$ and $c = 1$, where n is a positive integer. For $j \leq n$ hence the q-Taylor formula for x^n around $c = 1$ is

$$x^n = \sum_{j=0}^{n} \frac{[n]_q [n-1]_q \ldots [n-j+1]_q}{[j]_q!} (x-1)_q^j .$$

Using the q-Taylor formula, taking $f(x) = (x-a)_q^n$ around $x = 0$, is

$$(x-a)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\frac{k(k-1)}{2}} x^{n-k} a^k .$$

The q-binomial coefficients for n, k positive integers are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} = \begin{bmatrix} n \\ n-k \end{bmatrix}_q ,$$

with $\begin{bmatrix} n \\ n \end{bmatrix}_q = \begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ for $n < k$.

Theorem 2.1. (Fundamental Theorem of q-analysis) $F(x)$ is an antiderivative of $f(x)$ and $F(x)$ is

$$\int_a^b f(x) d_q x = F(b) - F(a)$$

$0 \leq a < b \leq \infty$ if $x = 0$ is continuous.

Proposition 2.1. If $f'(x)$ exists around $x = 0$ and is continuous at $x = 0$, then $f'(x)$ is the classical derivative of $f(x)$, is

$$\int_a^b D_q f(x) d_q x = f(b) - f(a) .$$

2.2. POST QUANTUM ANALYSIS

The (p, q)-analoge of n is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} .$$

Hence, $[n]_{p,q} = [n]_{q,p}$. For $p \to 1$, the (p,q)-number $[n]_{p,q}$ turns into the q –number $[n]_q$. Some formulas for sum, difference, product and quotient of (p, q)-numbers are

$$[n+m]_{p,q} = p^n [m]_{p,q} + q^m [n]_{p,q} = p^m [n]_{p,q} + q^n [m]_{p,q}$$

$$[n-m]_{p,q} = q^{-m}\left([n]_{p,q} + p^{n-m}[m]_{p,q}\right) = p^{-m}\left([n]_{p,q} + q^{n-m}[m]_{p,q}\right)$$

$$[nm]_{p,q} = [n]_{p^m,q^m}[m]_{p,q} = [m]_{p^n,q^n}[n]_{p,q}$$

$$\begin{bmatrix} \frac{n}{m} \end{bmatrix}_{p,q} = \frac{[n]_{p^{\frac{1}{m}},q^{\frac{1}{m}}}}{[m]_{p^{\frac{1}{m}},q^{\frac{1}{m}}}} = \frac{[n]_{p,q}}{[m]_{p^{\frac{n}{m}},q^{\frac{n}{m}}}}$$

where n and m are real or complex numbers [16, 22].

Let $f(x)$ be an arbitrary function. The (p,q)-differential of $f(x)$ is defined by

$$d_{p,q}f(x) = f(px) - f(qx)$$

and the (p,q)-derivative of $f(x)$ is defined by

$$D_{p,q}f(x) = \frac{f(px)-f(qx)}{(p-q)x},\ (x \neq 0).$$

The (p,q)-analogue of factorial of n is defined by

$$[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q}\cdots[2]_{p,q}[1]_{p,q}$$

and $[0]_{p,q}! = 1$. The (p,q)-analogue of the $(x-a)^n$ polynomial is defined as $(x-a)_{p,q}^0 = 1$ to

$$(x-a)_{p,q}^n = (x-a)(px-qa)\cdots(p^{n-1}x-q^{n-1}a).$$

For any polynomial $f(x)$ of degree N and any number a, the (p,q)-Taylor expansion of $f(x)$ is

$$f(x) = \sum_{k=0}^{N} p^{-\binom{k}{2}} \frac{(D_{p,q}^k f)(ap^{-k})}{[k]_{p,q}!}(x-a)_{p,q}^k.$$

which based on the formula

$$x^n = \sum_{k=0}^{n} p^{-\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (ap^{-k})^{n-k}(x-a)_{p,q}^k$$

The (p,q)-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!} = \begin{bmatrix} n \\ n-k \end{bmatrix}_{p,q}.$$

The (p,q)-binomial coefficients for $k > n$, the initial conditions $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = 0$ and $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{p,q} = 1$ and the triangle recursion relationship is satisfied as

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} = p^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q}$$

and

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} = q^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + p^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q} .$$

Theorem 2.1. (Fundamental Theorem of $(\boldsymbol{p}, \boldsymbol{q})$-analysis) $F(x)$ is an antiderivative of $f(x)$ and $F(x)$ is

$$\int_a^b f(x) d_{p,q} x = F(b) - F(a),$$

$0 \leq a < b \leq \infty$ if $x = 0$ is continuous.

Proposition 2.1. If $f'(x)$ exists around $x = 0$ and is continuous at $x = 0$, then $f'(x)$ is the classical derivative of $f(x)$, is

$$\int_a^b D_{p,q} f(x) d_{p,q} x = f(b) - f(a) .$$

3. MAIN RESULTS

In the present work we introduce the (r, p, q)-analysis as an extension of the (p, q)-analysis. Fisrtly, we define the (r, p, q)-numbers, (r, p, q)-differential and (r, p, q)-derivative, and we give their properties. We derive the (r, p, q)-Taylor formula and fundamental theorem of (r, p, q)-analysis.

3.1. (r, p, q)-ANALYSIS AS AN EXTENSION OF (p, q)-ANALYSIS

Assume that $0 < q < p < 1$, $r \in \mathbb{R}$ and $n \in \mathbb{Z}^+$.

Definition 3.1.1 The (r, p, q)-analogue of $n \in \mathbb{Z}^+$ is defined by

$$[n]_{r,p,q} = \begin{cases} \dfrac{[p]_r^n - [q]_r^n}{[p]_r - [q]_r} & , r \neq 1 \\ \dfrac{p^n - q^n}{p - q} & , r = 1 \end{cases} \tag{3.1.1}$$

We note that the numbers r, p and q can be chosen as real or complex numbers. Here, the r-analogue of p is defined by

$$[p]_r = \frac{r^p - 1}{r - 1}$$

and for $r = 1$, $[p]_r$ turns into the number p. Similarly, the r-analogue of q is given by

$$[q]_r = \frac{r^q - 1}{r - 1}$$

for $r = 1$, $[q]_r$ turns into q. For $r \to 1$, the $[n]_{r,p,q}$ (r,p,q)-number turns into $[n]_{p,q}$ (p,q)-number. A few examples of (r,p,q)-numbers are

$$[0]_{r,p,q} = 0,\ [1]_{r,p,q} = 1,\ [2]_{r,p,q} = [p]_r + [q]_r,\ [3]_{r,p,q} = [p]_r^2 + [p]_r [q]_r + [q]_r^2 .$$

Now, we examine the sum and difference formulas of (r,p,q)-numbers.

$$\begin{aligned}[n+m]_{r,p,q} &= \frac{[p]_r^{n+m} - [q]_r^{n+m}}{[p]_r - [q]_r} \\ &= \frac{[p]_r^{n+m} - [p]_r^n [q]_r^m + [p]_r^n [q]_r^m - [q]_r^{n+m}}{[p]_r - [q]_r} \\ &= [p]_r^n [m]_{r,p,q} + [q]_r^m [n]_{r,p,q}\end{aligned}$$

Similarly, we have

$$\begin{aligned}[n+m]_{r,p,q} &= \frac{[p]_r^{n+m} - [q]_r^{n+m}}{[p]_r - [q]_r} \\ &= \frac{[p]_r^{n+m} - [p]_r^m [q]_r^n + [p]_r^m [q]_r^n - [q]_r^{n+m}}{[p]_r - [q]_r} \\ &= [p]_r^m [n]_{r,p,q} + [q]_r^n [m]_{r,p,q}\end{aligned}$$

Taking $-m$ instead of m in the sum formula above, we have

$$[n-m]_{r,p,q} = [p]_r^n [-m]_{r,p,q} + [q]_r^{-m} [n]_{r,p,q}$$

and by

$$[-n]_{r,p,q} = \frac{[p]_r^{-n} - [q]_r^{-n}}{[p]_r - [q]_r} = -\left([p]_r [q]_r\right)^{-n} [n]_{r,p,q}$$

we reach to the equality

$$[n-m]_{r,p,q} = [q]_r^{-m} \left([n]_{r,p,q} - [p]_r^{n-m} [m]_{r,p,q}\right).$$

3.1.1. $(\mathbf{r},\mathbf{p},\mathbf{q})$-differential and $(\mathbf{r},\mathbf{p},\mathbf{q})$-derivative

Definition 3.1.2. Let f be a function defined on a subset of complex numbers. The (r,p,q)-differential of the f function is defined by

$$d_{r,p,q} f(x) = f\left([p]_r x\right) - f\left([q]_r x\right) \tag{3.1.2}$$

For any function $f(x)$ and $g(x)$ we have

$$d_{r,p,q}\left(f(x)g(x)\right) = f\left([p]_r x\right)g\left([p]_r x\right) - f\left([q]_r x\right)g\left([q]_r x\right)$$
$$= f\left([p]_r x\right)g\left([p]_r x\right) - f\left([q]_r x\right)g\left([q]_r x\right) - f\left([p]_r x\right)g\left([q]_r x\right) + f\left([p]_r x\right)g\left([q]_r x\right)$$
$$= f\left([p]_r x\right)d_{r,p,q}g(x) + g\left([q]_r x\right)d_{r,p,q}f(x)$$

and thus,

$$d_{r,p,q}\left(f(x)g(x)\right) = f\left([p]_r x\right)d_{r,p,q}g(x) + g\left([q]_r x\right)d_{r,p,q}f(x) \tag{3.1.3}$$

are obtained. Similarly,

$$d_{r,p,q}\left(f(x)g(x)\right) = g\left([p]_r x\right)d_{r,p,q}f(x) + f\left([q]_r x\right)d_{r,p,q}g(x) \tag{3.1.4}$$

Definition 3.1.3. The (r,p,q)-derivative of the f function is defined by

$$D_{r,p,q} f(x) = \frac{d_{r,p,q} f(x)}{d_{r,p,q} x} = \frac{f\left([p]_r x\right) - f\left([q]_r x\right)}{[p]_r x - [q]_r x},\ (x \neq 0). \tag{3.1.5}$$

For $x = 0$, we have $D_{r,p,q} f(x) = f'(0)$. Let $f(x)$ and $g(x)$ be any two functions. The (r,p,q)-derivative of the product of the functions $f(x)$ and $g(x)$, using (3.1.3) and (3.1.5), is as follows:

$$D_{r,p,q}\left(f(x)g(x)\right) = \frac{d_{r,p,q}\left(f(x)g(x)\right)}{d_{r,p,q}x} = \frac{f\left([p]_r x\right)d_{r,p,q}g(x) + g\left([q]_r x\right)d_{r,p,q}f(x)}{d_{r,p,q}x}$$
$$= f\left([p]_r x\right)D_{r,p,q}g(x) + g\left([q]_r x\right)D_{r,p,q}f(x) \tag{3.1.6}$$

and similarly,

$$D_{r,p,q}\left(f(x)g(x)\right) = g\left([p]_r x\right)D_{r,p,q}f(x) + f\left([q]_r x\right)D_{r,p,q}g(x) \tag{3.1.7}$$

Let's look at how the (r,p,q)-derivative of the quotient of functions $f(x)$ and $g(x)$ is defined.

$$f(x) = f(x)$$
$$g(x)\frac{f(x)}{g(x)} = f(x)$$

Let's take the (r,p,q)-derivative of both sides of the equality,

$$D_{r,p,q}\left(g(x)\frac{f(x)}{g(x)}\right) = D_{r,p,q}f(x).$$

Using (3.1.6), we find the following result with $g([q]_r x) \neq 0$ and $g([p]_r x) \neq 0$.

$$g\left([p]_r x\right) D_{r,p,q}\left(\frac{f(x)}{g(x)}\right) + \frac{f\left([q]_r x\right)}{g\left([q]_r x\right)} D_{r,p,q} g(x) = D_{r,p,q} f(x)$$

$$D_{r,p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{D_{r,p,q} f(x) - \dfrac{f\left([q]_r x\right)}{g\left([q]_r x\right)} D_{r,p,q} g(x)}{g\left([p]_r x\right)}$$

$$D_{r,p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{g\left([q]_r x\right) D_{r,p,q} f(x) - f\left([q]_r x\right) D_{r,p,q} g(x)}{g\left([q]_r x\right) g\left([p]_r x\right)}$$

Lemma 3.1.1. Let the functions $f: D \to \mathbb{C}$ and $g: D \to \mathbb{C}$ be (r, p, q)-differentiable on the order of n. Then,

$$D^n_{r,p,q}(fg)(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{r,p,q} D^k_{r,p,q}(f)(x[p]^{n-k}_r) D^{n-k}_{r,p,q}(g)(x[q]^k_r).$$

Proof: Lemma is provided for $n = 1$. Assuma that $m \geq 1$ and the lemma is true for $n = m$. Let's show that it is true for $n = m + 1$ using $n = m$ and the equality (3.4.4).

$$D^{m+1}_{r,p,q}(fg)(x) = D_{r,p,q}\left(D^m_{r,p,q}(fg)(x)\right)$$

$$= D_{r,p,q}\left(\sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_{r,p,q} D^k_{r,p,q}(f)\left(x[p]^{m-k}_r\right) D^{m-k}_{r,p,q}(g)\left(x[q]^k_r\right)\right)$$

$$= \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_{r,p,q} \Big[[p]^{m-k}_r D^{k+1}_{r,p,q}(f)\left(x[p]^{m-k}_r\right) D^{m-k}_{r,p,q}(g)\left(x[q]^{k+1}_r\right)$$

$$+ D^k_{r,p,q}(f)\left(x[p]^{m+1-k}_r\right) D^{m+1-k}_{r,p,q}(g)\left(x[q]^k_r\right)[q]^k_r \Big]$$

$$= \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_{r,p,q} D^k_{r,p,q}(f)\left(x[p]^{m+1-k}_r\right) D^{m+1-k}_{r,p,q}(g)\left(x[q]^k_r\right)[q]^k_r$$

$$+ \sum_{k=1}^{m+1} \begin{bmatrix} m \\ k-1 \end{bmatrix}_{r,p,q} [p]^{m+1-k}_r D^k_{r,p,q}(f)\left(x[p]^{m+1-k}_r\right) D^{m+1-k}_{r,p,q}(g)\left(x[q]^k_r\right)$$

$$= f\left(x[p]^{m+1}_r\right) D^{m+1}_{r,p,q}(g)(x)$$

$$= \sum_{k=1}^{m} \left\{ [q]^k_r \begin{bmatrix} m \\ k \end{bmatrix}_{r,p,q} + [p]^{m+1-k}_r \begin{bmatrix} m \\ k-1 \end{bmatrix}_{r,p,q} \right\} D^k_{r,p,q}(f)\left(x[p]^{m+1-k}_r\right) D^{m+1-k}_{r,p,q}(g)\left(x[q]^k_r\right)$$

$$+ D^{m+1}_{r,p,q}(f)(x) g\left(x[q]^{m+1}_r\right)$$

$$= \sum_{k=0}^{m+1} \begin{bmatrix} m \\ k+1 \end{bmatrix}_{r,p,q} D^k_{r,p,q}(f)\left(x[p]^{m+1-k}_r\right) D^{m+1-k}_{r,p,q}(g)\left(x[q]^k_r\right)$$

and the proof is complete.

3.2. (r,p,q)-ANALOGUE AND (r,p,q)-DERIVATIVE OF $(x-a)^n$

First, let's recall the general Taylor formula for polynomials.

Theorem 3.2.1. Let a be any number and D be a linear operator in the space of polynomials. Let $\left(P_0(x), P_1(x), \ldots\right)$ be the sequence of polynomials satisfying the following three conditions:

1. $P_0(a)=1$ and $P_n(a)=0 \;\; n\geq 1$
2. $derP_n(x)=n$
3. $dP_n(x)=P_{n-1}(x)$ for $\forall n\geq 1$ and $D(1)=0$

The general Taylor formula for any polynomial $f(x)$ of degree N is:

$$f(x)=\sum_{n=0}^{N}\left(D^n f\right)(a)P_n(x) \text{ [30].}$$

Let's take the linear operator $D_{r,p,q}$ instead of the linear operator D in the theorem above. If we choose $a=0$, we can find polynomials $P_n(x)$ satisfying the the conditions above. Let's look at the question of how to write a polynomial $P_n(x)$ if $a\neq 0$.

Let's find the polynomial $P_n(x)$ with the help of the theorem above by choosing $a\neq 0$ and $D=D_{r,p,q}$. It should be $P_0(x)=1$, $D_{r,p,q}P_1(x)=P_0(x)$ and $P_1([p]_r a)=0$.

$$D_{r,p,q}P_1(x)=P_0(x)\Rightarrow \frac{P_1\left([q]_r x\right)-P_1\left([p]_r x\right)}{[q]_r x-[p]_r x}=1$$

$$P_1\left([q]_r x\right)-P_1\left([p]_r x\right)=[q]_r x-[p]_r x$$

For $x=a$, $P_1\left([q]_r a\right)-P_1\left([p]_r a\right)=[q]_r a-[p]_r a$. $P_1([p]_r a)=0$ should be $P_1\left([q]_r a\right)=[q]_r a-[p]_r a$. Then it is found as

$$P_1(x)=x-a\,.$$

Now let's find $P_2(x)$. It should be $D_{r,p,q}P_2(x)=P_1(x)$ and $P_2([p]_r a)=0$.

$$D_{r,p,q}P_2(x)=\frac{P_2\left([q]_r x\right)-P_2\left([p]_r x\right)}{[q]_r x-[p]_r x}=x-a$$

$$P_2\left([q]_r x\right)-P_2\left([p]_r x\right)=(x-a)\left([q]_r x-[p]_r x\right)$$

For $x=a$, $P_2\left([q]_r a\right)-P_2\left([p]_r a\right)=0$ and must be $P_2\left([q]_r a\right)=0$. Therefore, it must be $P_2(x)=(x-a)\left([p]_r x-[q]_r a\right)$. Let's try this.

$$D_{r,p,q}P_2(x)=\frac{P_2\left([q]_r x\right)-P_2\left([p]_r x\right)}{[q]_r x-[p]_r x}=x-a$$

must be. Accordingly,

$$\begin{aligned}D_{r,p,q}P_2(x)&=\frac{\left([q]_r x-a\right)\left([p]_r [q]_r x-[q]_r a\right)-\left([p]_r x-a\right)\left([p]_r^2 x-[q]_r a\right)}{\left([q]_r-[p]_r\right)x}\\&=\frac{\left([p]_r x-a\right)\left([q]_r^2 x-[q]_r a-[p]_r^2 x+[q]_r a\right)}{\left([q]_r-[p]_r\right)x}\\&=\frac{\left([p]_r x-a\right)\left([q]_r^2-[p]_r^2\right)x}{\left([q]_r-[p]_r\right)x}\\&=\left([p]_r x-a\right)[2]_{r,p,q}.\end{aligned}$$

whereas $D_{r,p,q}P_2(x)=(x-a)$ should have been. In that case, it is found as

$$P_2(x)=\frac{(x-a)\left([p]_r x-[q]_r a\right)}{[2]_{r,p,q}}$$

by providing the equality. Continuing in this way, we can generalize the polynomial $P_n(x)$ for $a\neq 0$ as follows:

$$P_n(x)=\frac{(x-a)\left([p]_r x-[q]_r a\right)\left([p]_r^2 x-[q]_r^2 a\right)\ldots\left([p]_r^{n-1} x-[q]_r^{n-1} a\right)}{[n]_{r,p,q}!}.$$

Definition 3.2.1. The (r,p,q)-analogue of $n!$ is identified by

$$[n]_{r,p,q}!=\begin{cases}1 & n=0\\ [n]_{r,p,q}[n-1]_{r,p,q}\ldots[2]_{r,p,q}[1]_{r,p,q} & n\geq 1\end{cases}$$

Definition 3.2.2. The (r,p,q)-analogue of the $(x-a)^n$ polynomial is defined by

$$(x-a)^n_{r,p,q}=\begin{cases}1 & n=0\\ (x-a)\left([p]_r x-[q]_r a\right)\ldots\left([p]_r^{n-1} x-[q]_r^{n-1} a\right) & n\geq 1\end{cases}$$

3.2.1. Some propositions of polynomials $(x-a)^n$

Proposition 3.2.1. For $n\geq 1$, the following statement is true:

$$D_{r,p,q}\left(x-a\right)_{r,p,q}^{n}=\left[n\right]_{r,p,q}\left(\left[p\right]_{r}x-a\right)_{r,p,q}^{n-1} \tag{3.2.1}$$

and $D_{r,p,q}\left(x-a\right)_{r,p,q}^{0}=0$.

Proof: Let's do the proof by inductive method. Since $D_{r,p,q}\left(x-a\right)_{r,p,q}=1=\left[1\right]_{r,p,q}\left(\left[p\right]_{r}x-a\right)_{r,p,q}^{0}$ for $n=1$, the proposition is true for $n=1$. Suppose (3.2.1) is true for some value of k, $D_{r,p,q}\left(x-a\right)_{r,p,q}^{k}=\left[k\right]_{r,p,q}\left(\left[p\right]_{r}x-a\right)_{r,p,q}^{k-1}$. Using this definition for (r,p,q)-power, we can write

$$\left(x-a\right)_{r,p,q}^{k+1}=\left(x-a\right)_{r,p,q}^{k}\left(\left[p\right]_{r}^{k}x-\left[q\right]_{r}^{k}a\right)_{r,p,q}.$$

Let us show that (3.2.1) is true for the value $k+1$. From the equality (3.1.7) we obtain

$$\begin{aligned}D_{r,p,q}\left(x-a\right)_{r,p,q}^{k+1}&=\left[p\right]_{r}^{k}\left(\left[p\right]_{r}x-a\right)_{r,p,q}^{k}+\left(\left[p\right]_{r}^{k}\left[q\right]_{r}x-\left[q\right]_{r}^{k}a\right)\left[k\right]_{r,p,q}\left(\left[p\right]_{r}x-a\right)_{r,p,q}^{k-1}\\&=\left[p\right]_{r}^{k}\left(\left[p\right]_{r}x-a\right)_{r,p,q}^{k}+\left[q\right]_{r}\left(\left[p\right]_{r}^{k}x-\left[q\right]_{r}^{k-1}a\right)\left[k\right]_{r,p,q}\left(\left[p\right]_{r}x-a\right)_{r,p,q}^{k-1}\\&=\left(\left[p\right]_{r}x-a\right)_{r,p,q}^{k}\left(\left[p\right]_{r}^{k}+\left[q\right]_{r}\left[k\right]_{r,p,q}\right)=\left[k+1\right]_{r,p,q}\left(\left[p\right]_{r}x-a\right)_{r,p,q}^{k}\end{aligned}$$

Hence, the proof is completed.

Proposition 3.2.2. Let λ be a complex number and $n\geq 1$ an integer. The relation holds:

$$D_{r,p,q}\left(\lambda x-a\right)_{r,p,q}^{n}=\lambda\left[n\right]_{r,p,q}\left(\lambda\left[p\right]_{r}x-a\right)_{r,p,q}^{n-1}.$$

Proof: The proof can be done by induction on n.

Proposition 3.2.3. Let $0\leq k\leq n$ and with $n\in\mathbb{Z}$, $n\geq 1$. So it is

$$D_{r,p,q}^{k}(x-a)_{r,p,q}^{n}=[p]_{r}^{\mathrm{k(k-1)/2}}\frac{[\mathrm{n}]_{r,p,q}!}{[\mathrm{n-k}]_{r,p,q}!}([\mathrm{p}]_{r}^{k}x-a)_{r,p,q}^{n-k}. \tag{3.2.2}$$

3.3. (r,p,q)-TAYLOR FORMULA

In this section, two Taylor formulas for polynomials are given in the light of (r,p,q)-analysis and some results are investigated. The first of the Taylor formulas is as follows.

Theorem 3.3.1. For any polynomial $f(x)$ of degree N and any number a, the Taylor's formula is given by

$$f(x)=\sum_{k=0}^{N}[p]_r^{-k(k-1)/2}\frac{\left(D_{r,p,q}^{k}f\right)(a[p]_r^{-k})}{[k]_{r,p,q}!}(x-a)_{r,p,q}^{k} \tag{3.3.1}$$

Proof: Let f be a polynomial of degree N. In this case

$$f(x)=\sum_{j=0}^{N}c_j\left(x-a\right)_{r,p,q}^{j} \tag{3.3.2}$$

is provided. k is an integer such that $0\leq k\leq N$. Then, applying $D_{r,p,q}^{k}$ and using equalities (3.3.2) and (3.2.2) we have the formula

$$\left(D_{r,p,q}^{k}f\right)(x)=\sum_{j=k}^{N}c_j[p]_r^{\mathrm{k(k-1)/2}}\frac{[j]_{r,p,q}!}{[j-k]_{r,p,q}!}([p]_r^{k}x-a)_{r,p,q}^{j-k}.$$

If $x=a[p]_r^{-k}$ is chosen in the above equality,

$$\left(D_{r,p,q}^{k}f\right)(a[p]_r^{-k})=\mathrm{c}_k[k]_{r,p,q}!\,[p]_r^{\mathrm{k(k-1)/2}}$$

is found. Hence, it is

$$c_j=[p]_r^{-\mathrm{k(k-1)/2}}\frac{\left(D_{r,p,q}^{k}f\right)(a[p]_r^{-k})}{[k]_{r,p,q}!}.$$

Thus ends the proof.

Corollary 3.3.1. The following statement is true:

$$x^n=\sum_{k=0}^{n}[p]_r^{-k(k-1)/2}\begin{bmatrix}n\\k\end{bmatrix}_{r,p,q}(a[p]_r^{-k})^{n-k}(x-a)_{r,p,q}^{k}\ .$$

Theorem 3.3.2. For any polynomial $f(x)$ of degree N and any number a, theTayor's formula is given by:

$$f(x)=\sum_{k=0}^{N}(-1)^k[q]_r^{-\mathrm{k(k-1)/2}}\frac{\left(D_{r,p,q}^{k}f\right)(a[q]_r^{-k})}{[k]_{r,p,q}!}(a-x)_{r,p,q}^{k}.$$

Proof: The proof can be proved similarly to the proof of Theorem 3.3.1.

Corollary 3.3.2. The following equality is valid:

$$x^n=\sum_{k=0}^{n}(-1)^k[q]_r^{-\mathrm{k(k-1)/2}}\begin{bmatrix}n\\k\end{bmatrix}_{r,p,q}(a[q]_r^{-k})^{n-k}(a-x)_{r,p,q}^{k} \tag{3.3.3}$$

Corollary 3.3.3. The following equalities are valid:

$$\left(x-b\right)_{r,p,q}^{n}=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{r,p,q}\left(a-b\right)_{r,p,q}^{n-k}\left(x-a\right)_{r,p,q}^{k} \tag{3.3.4}$$

$$\left(b-x\right)_{r,p,q}^{n}=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{r,p,q}\left(b-a\right)_{r,p,q}^{n-k}\left(a-x\right)_{r,p,q}^{k} \tag{3.3.5}$$

We note that by taking ab instead of b in equalities (3.3.4) and (3.3.5), we obtain the following expressions:

$$\left(x-ab\right)_{r,p,q}^{n}=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{r,p,q}a^{n-k}\left(1-b\right)_{r,p,q}^{n-k}\left(x-a\right)_{r,p,q}^{k} \tag{3.3.6}$$

$$\left(ab-x\right)_{r,p,q}^{n}=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{r,p,q}a^{n-k}\left(b-1\right)_{r,p,q}^{n-k}\left(a-x\right)_{r,p,q}^{k} \tag{3.3.7}$$

3.4. (r,p,q)-BINOMIAL COEFFICIENTS

Definition 3.4.1. The (r,p,q)-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{r,p,q}=\frac{[n]_{r,p,q}!}{[n-k]_{r,p,q}![k]_{r,p,q}!}, \tag{3.4.1}$$

with $[n]_{r,p,q}!=[n]_{r,p,q}[n-1]_{r,p,q}\cdots[2]_{r,p,q}[1]_{r,p,q}\quad(n>0)$ and $[0]_{r,p,q}!=1$.

If $r=1$, (p,q)-binomial coefficients are obtained. In this case, the properties of (r,p,q)-binomial coefficients are similar to the properties of (p,q)-binomial coefficients. Some properties of (r,p,q)-binomial coefficients with n,k non-negative integers and $k\le n$ are as follows:

$$\begin{gathered}\begin{bmatrix} n \\ k \end{bmatrix}_{r,p,q}=\frac{[n]_{r,p,q}!}{[n-k]_{r,p,q}![k]_{r,p,q}!},\\ \begin{bmatrix} n \\ k \end{bmatrix}_{r,p,q}=\begin{bmatrix} n \\ k \end{bmatrix}_{r,q,p},\\ \begin{bmatrix} n \\ k \end{bmatrix}_{r,p,q}=\begin{bmatrix} n \\ n-k \end{bmatrix}_{r,p,q},\\ \begin{bmatrix} n \\ k \end{bmatrix}_{r,p,q}=\frac{[n]_{r,p,q}}{[k]_{r,p,q}}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{r,p,q}.\end{gathered} \tag{3.4.2}$$

First, let's give the triangle recursion relationship to calculate (r,p,q)-binomial coefficients.

Theorem 3.4.1. The (r,p,q)-binomial coefficients are given by

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{r,p,q} = [p]_r^k \begin{bmatrix} n \\ k \end{bmatrix}_{r,p,q} + [q]_r^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{r,p,q} \tag{3.4.3}$$

and

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{r,p,q} = [q]_r^k \begin{bmatrix} n \\ k \end{bmatrix}_{r,p,q} + [p]_r^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{r,p,q} \tag{3.4.4}$$

with initial conditions $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{r,p,q} = 1$ and $\begin{bmatrix} n \\ k \end{bmatrix}_{r,p,q} = 0$ for $n<k$.

Proof. Let $0<k<n$.

$$\begin{aligned}
[n+1]_{r,p,q} &= [p]_r^n + [p]_r^{n-1}[q]_r + \cdots + [p]_r^k [q]_r^{n-k} + [p]_r^{k-1}[q]_r^{n-k+1} + \cdots + [p]_r [q]_r^{n-1} + [q]_r^n \\
&= [p]_r^k \left([p]_r^{n-k} + [p]_r^{n-k-1}[q]_r + \cdots + [q]_r^{n-k} \right) + [q]_r^{n-k+1} \left([p]_r^{k-1} + [p]_r^{k-2}[q]_r + \cdots + [q]_r^{k-1} \right) \\
&= [p]_r^k [n-k+1]_{r,p,q} + [q]_r^{n-k+1} [k]_{r,p,q}
\end{aligned}$$

so that,

$$\begin{aligned}
\begin{bmatrix} n+1 \\ k \end{bmatrix}_{r,p,q} &= \frac{[n+1]_{r,p,q}!}{[n-k+1]_{r,p,q}![k]_{r,p,q}!} = \frac{[n+1]_{r,p,q}[n]_{r,p,q}!}{[n-k+1]_{r,p,q}![k]_{r,p,q}!} \\
&= \frac{\left([p]_r^k [n-k+1]_{r,p,q} + [q]_r^{n-k+1}[k]_{r,p,q} \right)[n]_{r,p,q}!}{[n-k+1]_{r,p,q}![k]_{r,p,q}!} \\
&= [p]_r^k \frac{[n-k+1]_{r,p,q}[n]_{r,p,q}!}{[n-k+1]_{r,p,q}![k]_{r,p,q}!} + [q]_r^{n-k+1} \frac{[k]_{r,p,q}[n]_{r,p,q}!}{[n-k+1]_{r,p,q}![k]_{r,p,q}!} \\
&= [p]_r^k \begin{bmatrix} n \\ k \end{bmatrix}_{r,p,q} + [q]_r^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{r,p,q}
\end{aligned}$$

Thus, the equality (3.4.3) is proved. The proof of the equality (3.4.4) can be proved similarly.

Theorem 3.4.2. The (r,p,q)-binomial coefficients are given by

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{r,p,q} = \sum_{j=k}^{n} [p]_r^{(n-j)(k+1)} [q]_r^{j-k} \begin{bmatrix} j \\ k \end{bmatrix}_{r,p,q}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{r,p,q} = \sum_{j=0}^{n-k} (-1)^j [p]_r^{-(j+1)(n-k)+\binom{j+1}{2}} [q]_r^{jk+\binom{j+1}{2}} \begin{bmatrix} n+1 \\ k+j+1 \end{bmatrix}_{r,p,q}.$$

Theorem 3.4.3. (r,p,q)-binomial coefficients are

$$\sum_{j=0}^{n-1}\left([p]_r^j + x[q]_r^j\right) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{r,p,q} [p]_r^{\binom{n-k}{2}} [q]_r^{\binom{k}{2}} x^k \tag{3.4.5}$$

Proof: The proof can be done by inductive method.

Result 3.4.1. For $n \geq 1$, the following equality is valid:

$$\sum_{k\,çift}\begin{bmatrix} n \\ k \end{bmatrix}_{r,p,q} [p]_r^{\binom{n-k}{2}} [q]_r^{\binom{k}{2}} = \sum_{k\,tek}\begin{bmatrix} n \\ k \end{bmatrix}_{r,p,q} [p]_r^{\binom{n-k}{2}} [q]_r^{\binom{k}{2}} \tag{3.4.6}$$

Proof: Taking as $x = -1$ in Theorem 3.4.3, we obtain

$$\sum_{j=0}^{n}\left([p]_r^j - [q]_r^j\right) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{r,p,q} [p]_r^{\binom{n-k}{2}} [q]_r^{\binom{k}{2}} (-1)^k .$$

The first element of the $\sum_{j=0}^{n-1}\left([p]_r^j - [q]_r^j\right)$ expansion equals zero when $j = 0$, therefore

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{r,p,q} [p]_r^{\binom{n-k}{2}} [q]_r^{\binom{k}{2}} (-1)^k = 0 .$$

This corresponds to equality (3.4.6).

3.5. FUNDAMENTAL THEOREM OF (r,p,q)-ANALYSIS

Theorem 3.5.1. (The Fundamental Theorem of (r,p,q)-Analysis). $F(x)$, is an antiderivative of $f(x)$' and if $F(x)$, is continuous at $x = 0$ and $0 \leq a < b \leq \infty$,

$$\int_a^b f(x)\, d_{r,p,q}\, x = F(b) - F(a). \tag{3.5.1}$$

Proof: Since $F(x)$ is continuous at $x = 0$, $F(x)$ can be written as

$$F(x) = \left([p]_r - [q]_r\right) x \sum_{j=0}^{\infty} \frac{[q]_r^j}{[p]_r^{j+1}} f\left(\frac{[q]_r^j}{[p]_r^{j+1}} x\right) + F(0) .$$

Since

$$\int_0^a f(x)\, d_{r,p,q}\, x = \left([p]_r - [q]_r\right) a \sum_{j=0}^{\infty} \frac{[q]_r^j}{[p]_r^{j+1}} f\left(\frac{[q]_r^j}{[p]_r^{j+1}} x\right),$$

it can be written as

$$\int_0^a f(x)\, d_{r,p,q}\, x = F(a) - F(0) .$$

Similarly, for a finite b, it can be written as

$$\int_0^b f(x)d_{r,p,q}x = F(b) - F(0).$$

Thus, we obtain

$$\int_a^b f(x)d_{r,p,q}x = F(b) - F(a).$$

Proposition 3.5.1. If there is $f'(x)$ in a neighborhood of $x=0$ and it is continuous at the point $x=0$, then we have

$$\int_a^b D_{r,p,q} f(x)d_{r,p,q}x = f(b) - f(a) \tag{3.5.2}$$

Proof: Using L'Hospital's rule, we have

$$\begin{aligned}\lim_{x\to 0} D_{r,p,q} f(x) &= \lim_{x\to 0} \frac{f\left([p]_r x\right) - f\left([q]_r x\right)}{\left([p]_r - [q]_r\right)x} \\ &= \lim_{x\to 0} \frac{[p]_r f'\left([p]_r x\right) - [q]_r f'\left([q]_r x\right)}{[p]_r - [q]_r} \\ &= \frac{[p]_r - [q]_r}{[p]_r - [q]_r} f'(0) = f'(0)\end{aligned}$$

Therefore, it follows from the equality $\left(D_{r,p,q} f\right)(0) = f'(0)$ and (3.5.2) that $D_{r,p,q} f(x)$ is continuous at the point $x=0$.

4. CONCLUSIONS

In this paper, the basic concepts of (r,p,q)-analysis as an extension of $(\boldsymbol{p},\boldsymbol{q})$-analysis are defined and examined. We obtain results that include the concepts of (r,p,q)-analogue of a number *n*, and (r,p,q)-differential and derivative of a function *f(x)*, (r,p,q)-analogue and properties of the $(x-a)^n$ polynomial.

Some results containing the (r,p,q)-analogue and (r,p,q)-Taylor formula, (r,p,q)-binomial coefficients, and the fundamental theorem and properties of (r,p,q)-analysis are obtained.

REFERENCES

[1] Annaby, M.H., Zeinab, M.S., *q-Fractional Calculus and Equations*, Springer, Berlin, Germany, 2012.

[2] Ernst, T., *History of q-Calculus and a New Method*, U. U. D. M. Report 2000, Uppsala University, 1999.

[3] Gauchman, H., *Computers Mathematics with Applications*, **47**(2-3), 281, 2004.

[4] Heine, E., *Handbuch Der Kugelfunctionen, Theorie Und Anwendugen*, Druck und Verlag von G. Reimer, Berlin, 1878.

[5] Jackson, F.H., *Transactions of the Royal Society of Edinburgh Earth Science*, **46**(2), 253, 1909.

[6] Goldman, J., Rota, G.C., *The number of subspaces of a vector space*, Technical Report AD0683687, University of Harvard, United States, 1969.

[7] Thomae, J., *Journal Für die Reine und Angewandte Mathematik*, **70**, 258, 1869.

[8] Thomae, J., *Mathematicshe Annalen*, **2**, 427, 1870.

[9] Jackson, F.H., *Pure and Applied Mathematics Quarterly*, **41**, 193, 1910.

[10] Chakrabarti, R., Jagannathan, R., *Journal of Physics a Mathematical and General*, **24**(13), L711, 1991.

[11] Brodimas, G., Jannussis, A., Mignani, R., Two parameter quantum groups, *Universita di Roma Preprint*, 820, 1991.

[12] Wachs, M., White, D., *Journal of Combinatorial Theory, Serias A*, **56**, 27, 1991.

[13] Arik, M., Demircan, E., Turgut, T., Ekinci, L., Mungan, M., *Zeitschrift für Physik C Particles and Fields*, **55**(1), 89, 1992.

[14] Burban, M., Klimyk, A., *UIntegral Transforms and Special Functions*, **2**, 15, 1994.

[15] Corcino, R. B., *Electronic Journal of Combinatorial Number Theory*, **8**, A29, 2008.

[16] Sadjang, P. N., *Journal of Difference Equations and Applications*, **23** (9), 1562, 2017.

[17] Araci, S., Duran, U., Acikgoz, M., Srivastava, H.M., *Journal of Inequalities and Applications*, **2016**, 301, 2016.

[18] Duran, U., Acikgoz, M., Araci, S., *TWMS Journal of Pure and Applied Mathematics*, **8** (2), 198, 2017.

[19] Jagannathan, R., Rao, K.S., Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series, arXiv preprint: https://arxiv.org/abs/math/0602613v1, 2006.

[20] Duran, U., Master Thesis *Post Quantum Calculus*, University of Gaziantep, 2016.

[21] Kac, V., Cheung, P., *Quantum Calculus*, Springer Verlag, New York, 2002.

[22] Sadjang, P., Duran, U., *Miskolc Mathematical Notes*, **20**(2), 1185, 2019

[23] Sadjang, P. N., *Results in Mathematics*, **73**, 39, 2018.