ORIGINAL PAPER

MULTIPLICATIVELY PREINVEX *P*-FUNCTIONS

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Manuscript received: 25.12.2022; Accepted paper: 29.01.2023;
Published online: 30.03.2023.

Abstract. *In this study, using both Hölder's integral inequality and Hölder-İşcan integral inequality, some inequalities of Hermite Hadamard type obtained for multiplicatively preinvex P-functions are given. Afterwards, the result obtained with Hölder-İşcan inequality has been shown to be better than that obtained with Hölder's integral inequality.*

Keywords: *convex function; preinvex function; multiplicatively preinvex P-function; Hermite-Hadamard type inequality.*

1. PRELIMINARIES

A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then the function f is said to be concave on interval $I \neq \emptyset$. Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences. One of the most important integral inequalities for convex functions is the Hermite-Hadamard (H-H) integral inequality. The classical H-H inequality provides estimates of the mean value of a continuous convex function $f: [a, b] \to \mathbb{R}$. The following double inequality is well known as the H-H inequality in the literature.

Definition 1. Let $\mathrm{f}: [\mathrm{a}, \mathrm{b}] \to \mathbb{R}$ be a convex function, then the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

is known as the Hermite-Hadamard inequality.

Some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors (e.g., [1-4]) and the Authors obtained a new refinement of the Hermite-Hadamard inequality for convex functions.

Definition 2. A nonnegative function $\mathrm{f}: \mathrm{I} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be P-function if the inequality

$$f(tx + (1-t)y) \leq f(x) + f(y)$$

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holds for all $x, y \in I$ and $t \in (0,1)$.

We will denote by $P(I)$ the set of P-functions on the interval I. Note that $P(I)$ contain all nonnegative convex and quasi-convex functions. In [5], Dragomir et al. proved the following inequality of Hadamard type for class of P-functions.

Theorem 1. Let $f \in P(I)$, $a, b \in I$ with $a < b$ and $f \in L[a, b]$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_a^b f(x)dx \leq 2[f(a)+f(b)].$$

Barani et al in [6] used the following lemma to prove Theorems.

Lemma 1. Let $f: I \rightarrow (0,\infty)$ be a differentiable mapping $a, a+\eta(b,a) \in I$ with $a < a + \eta(b,a)$. If $f' \in L_1[a, a+\eta(b,a)]$, then

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - \frac{f(a)+f\big(a+\eta(b,a)\big)}{2}$$
$$= \frac{\eta(b,a)}{2}\int_0^1 (1-2t)f'\big(a+t\eta(b,a)\big)dt.$$

In [7], Noor et al used the following lemma to prove Theorems.

Lemma 2. Let $f: I \rightarrow (0,\infty)$ be a differentiable mapping $a, a+\eta(b,a) \in I$ with $a < a + \eta(b,a)$. If $f' \in L_1[a, a+\eta(b,a)]$, then

$$\frac{1}{\eta(b,a)} f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{[\eta(b,a)]^2}\int_a^{a+\eta(b,a)} f(x)dx$$
$$= \int_0^{\frac{1}{2}} tf\big(a+t\eta(b,a)\big)dt + \int_{\frac{1}{2}}^1 (t-1)f\big(a+t\eta(b,a)\big)dt.$$

Definition 3. [8] Let K be a non-empty subset in $\mathbb{R}^n$ and $\eta: K \times K \rightarrow \mathbb{R}^n$. Let $x \in K$, then the set K is said to be invex at x with respect to $\eta(\cdot,\cdot)$, if

$$x + t\eta(y,x) \in K, \forall x, y \in K \text{and} t \in [0,1].$$

K is said to be an invex set with respect to $\eta(\cdot,\cdot)$ if K is invex at each $x \in K$. The invex set K is also called η-connected set.

It is true that every convex set is also an invex set with respect to $\eta(y,x) = y - x$, but the converse is not necessarily true, see [9,10] and the references therein. For the sake of simplicity, we always assume that $K = [x, x + t\eta(y,x)]$, unless otherwise specified [11].

Definition 4. [8] A function $f: K \rightarrow \mathbb{R}$ on an invex set $K \subseteq \mathbb{R}$ is said to be preinvex with respect to $\eta(\cdot,\cdot)$, if

$$f(x + t\eta(y,x)) \leq (1-t)f(x) + tf(y), \forall x, y \in K, t \in [0,1].$$

It is to be noted that every convex function is preinvex with respect to the map $\eta(y,x) = y - x$ but the converse is not true see for instance.

Definition 5. [9] Let $S \subseteq \mathbb{R}$ be an open invex subset with respect to the mapping $\eta(\cdot,\cdot): S \times S \to \mathbb{R}$. We say that the function satisfies the Condition C if, for any $x, y \in S$ and any $t \in [0,1]$,

$$\eta(y, y + t\eta(x,y)) = -t\eta(x,y) \quad (1)$$

$$\eta(x, y + t\eta(x,y)) = (1-t)\eta(x,y). \quad (2)$$

In [12], Noor has obtained the following Hermite-Hadamard type inequalities for the preinvex functions.

Theorem 2. [12] Let $f: [x, x+\eta(y,x)] \to (0,\infty)$ be a preinvex function on the interval of the real numbers K° (the interior of K) and $x, y \in K^\circ$ with $\eta(y,x) > 0$. Then the following inequalities holds

$$f\left(\frac{2x+\eta(y,x)}{2}\right) \leq \frac{1}{\eta(y,x)} \int_x^{x+\eta(y,x)} f(x)dx \leq \frac{f(x)+f(y)}{2}$$

Definition 6. [13] Let $I \neq \emptyset$ be an interval in $\mathbb{R}$. The function $f: I \to [0,\infty)$ is said to be multiplicatively P-function (or log-P-function), if the inequality

$$f(tx + (1-t)y) \leq f(x)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

We will denote by $MP(I)$ the class of all multiplicatively P-functions on interval I. Clearly, $f: I \to [0,\infty)$ is multiplicatively P-function if and only if $\log f$ is P-function. In recent years, many mathematicians have been studying about preinvexity and types of preinvexity. A lot of efforts have been made by many mathematicians to generalize the classical convexity, (see, [8, 14-16] and references therein).

In [13], Kadakal obtained the following Hermite-Hadamard type inequalities for multiplicatively P-functions:

Theorem 3. Let the function $f: I \to [1,\infty)$,be a multiplicatively P-function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$, then the following inequalities hold:

$$\text{i) } f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)f(a+b-x)dx \leq [f(a)f(b)]^2$$

$$\text{ii) } f\left(\frac{a+b}{2}\right) \leq f(a)f(b) \frac{1}{b-a} \int_a^b f(x)dx \leq [f(a)f(b)]^2$$

Theorem 4. (Hölder Inequality for Integrals [17]) Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a,b]$ then

$$\int_a^b |f(x)g(x)|dx \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}},$$

with equality holding if and only if $A|f(x)|^p = B|g(x)|^q$ almost everywhere, where A and B are constants.

Theorem 5. (Hölder-İşcan Integral Inequality [18]) Let $p > 1$ and $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a,b]$ then

$$i)\ \int_a^b |f(x)g(x)|dx \leq \frac{1}{b-a}\left\{\left(\int_a^b (b-x)|f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)|g(x)|^q dx\right)^{\frac{1}{q}} + \left(\int_a^b (x-a)|f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)|g(x)|^q dx\right)^{\frac{1}{q}}\right\}$$

$$ii)\ \frac{1}{b-a}\left\{\left(\int_a^b (b-x)|f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)|g(x)|^q dx\right)^{\frac{1}{q}} + \left(\int_a^b (x-a)|f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)|g(x)|^q dx\right)^{\frac{1}{q}}\right\} \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}}.$$

2. DEFINITION OF MULTIPLICATIVELY PREINVEX P-FUNCTIONS AND THEIR SOME PROPERTIES

The main purpose of this paper is to establish new estimations and refinements of the Hermite-Hadamard type inequalities for functions whose derivatives in absolute value are multiplicatively preinvex P-function. In this section, we begin by setting some algebraic properties for multiplicatively preinvex P-functions.

Definition 7. Let $I \neq \emptyset$ be an interval in $\mathbb{R}$. The function $f: I \to (0,\infty)$ is said to be multiplicatively preinvex P-function (or log-P-function), if the inequality

$$f(a+t\eta(b,a)) \leq f(a)f(b) \tag{3}$$

holds for all $a,b \in I$ and $t \in [0,1]$.

We will denote by $MPP(I)$ the class of all multiplicatively preinvex P-functions on interval I. Clearly, $f: I \to (0,\infty)$ is multiplicatively preinvex P-function if and only if $\log f$ is P-preinvex function. Moreover, if we take $\eta(b,a)=b-a$ in the inequality (3), then we obtain multiplicatively P-functions.

Theorem 6. Let $f: I \to \mathbb{R}$ and $c \in \mathbb{R}$. If f is a multiplicatively preinvex P-function, then the following inequality holds:

Proof: For $x,y \in I$ and $t \in [0,1]$, we get

$$(cf)(x+t\eta(y,x)) = cf(x+t\eta(y,x)) \leq cf(x)f(y).$$

Theorem 7. Let $f,g: I \to \mathbb{R}$. If f and g are multiplicatively preinvex P-functions. If f and g are non-negative functions, then the function $f+g$ is multiplicatively preinvex P-function.

Proof: For $x, y \in I$ and $t \in [0,1]$, we get

$$\begin{aligned}(f+g)(x+t\eta(y,x)) &= f(x+t\eta(y,x)) + g(x+t\eta(y,x))\\ &\le f(x)f(y) + g(x)g(y)\\ &\le f(x)f(y)+g(x)g(y)+f(x)g(y)+f(y)g(x)\\ &= f(x)[f(y)+g(y)] + g(x)[f(y)+g(y)]\\ &= [f(x)+g(x)][f(y)+g(y)]\\ &= [(f+g)(x)][(f+g)(y)].\end{aligned}$$

Theorem 8. Let $\mathrm{f, g: I \to \mathbb{R}}$. If f and g are multiplicatively preinvex P-functions, then fg is a multiplicatively preinvex P-function.

Proof: For $x, y \in I$ and $t \in [0,1]$, we get

$$\begin{aligned}(fg)(x+t\eta(y,x)) = f(x+t\eta(y,x))g(x+t\eta(y,x)) &\le [f(x)f(y)][g(x)g(y)]\\ &= [f(x)g(x)][f(y)g(y)]\\ &= [(fg)(x)][(fg)(y)].\end{aligned}$$

Theorem 9. Let $\mathrm{f, g: I \to \mathbb{R}}$. If f is multiplicatively preinvex P-function and decreasing and g is a super-multiplicatively function, then gof is multiplicatively preinvex P-function.

Proof: For $x, y \in 1, \infty)$ and $t \in [0,1]$, we obtain

$$\begin{aligned}(gof)(x+t\eta(y,x)) = g\big(f(x+t\eta(y,x))\big) \le g(f(x)f(y)) &\le g(f(x))g(f(y))\\ &= (gof)(x)(gof)(y).\end{aligned}$$

3. HERMITE-HADAMARD TYPE INEQUALITIES FOR MULTIPLICATIVELY PREINVEX *P*-FUNCTIONS

The goal of this paper is to develop concept of the multiplicatively preinvex *P*-functions and to establish some inequalities of Hermite-Hadamard type for these classes of functions.

Theorem 10. Let the function $\mathrm{f: I \to [1, \infty)}$,be a multiplicatively preinvex P-function and $\mathrm{a, b \in I}$ with $\mathrm{a} < a + \eta(b,a)$. If $\mathrm{f \in L[a, b]}$, then the following inequalities hold:

$$\text{i) } f\left(\frac{2a+\eta(b,a)}{2}\right) \le \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)f(a+b-x)dx \le [f(a)f(b)]^2$$

$$\text{ii) } f\left(\frac{2a+\eta(b,a)}{2}\right) \le f(a)f(b)\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \le [f(a)f(b)]^2$$

Proof: i) Since the function f is a multiplicatively preinvex P-function, we write the following inequality:

$$\begin{aligned}f\left(\frac{2a+\eta(b,a)}{2}\right) &= f\left(\frac{[a+(1-t)\eta(b,a)]+[a+t\eta(b,a)]}{2}\right)\\ &\le f(a+(1-t)\eta(b,a))f(a+t\eta(b,a))\\ &\le [f(a)f(b)]^2.\end{aligned}$$

By integrating this inequality on $[0,1]$ and changing the variable as $x = a + t\eta(b,a)$, then

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \le \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) f(2a+\eta(b,a)-x)dx \le [f(a)f(b)]^2.$$

Moreover, changing the variable as $x = a + t\eta(b,a)$, a simple calculation give us that

$$\int_0^1 f(a+(1-t)\eta(b,a)) f(a+t\eta(b,a))dt$$
$$= \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) f(a+b-x)dx \le [f(a)f(b)]^2.$$

So, we get

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \le \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) f(a+b-x)dx \le [f(a)f(b)]^2.$$

ii) Similarly, as f is a multiplicatively preinvex P-function, we write the following:

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \le f(a+(1-t)\eta(b,a)) f(a+t\eta(b,a)) \le f(a)f(b) f(a+t\eta(b,a))$$

Here, by integrating this inequality on $[0,1]$ and changing the variable as $x = a + t\eta(b,a)$, then, we have

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \le f(a)f(b) \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx.$$

Since,

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \le f(a)f(b),$$

we obtain

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \le f(a)f(b) \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \le [f(a)f(b)]^2$$

This completes the proof of theorem.

Remark 1. Above Theorem (i) and (ii) can be written together as follows:

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \le \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) f(2a+\eta(b,a)-x)dx$$
$$\le f(a)f(b) \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx.$$

Proof: By integrating the following inequality on $[0,1]$, the desired result can be obtained:

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \le f\left(\frac{A_t + A_{1-t}}{2}\right) \le f(A_t) f(A_{1-t}) \le f(a)f(b)f(A_t),$$

where $A_t = ta + (1-t)b$.

Theorem 11. Let $f: I \to \mathbb{R}$ be a differentiable function on I° such that the function $|f'|$ is multiplicatively preinvex P-function. Suppose that $a, a+\eta(b,a) \in I$ with $a < a+\eta(b,a)$ and $f' \in L[a, a+\eta(b,a)]$. Then the following inequality holds:

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - \frac{f(a)+f\big(a+\eta(b,a)\big)}{2}\right| \leq \frac{\eta(b,a)|f'(a)||f'(b)|}{4}. \quad (4)$$

Proof: Using Lemma 1, since $|f'|$ is multiplicatively preinvex P-function, we obtain

$$\begin{aligned}\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - \frac{f(a)+f\big(a+\eta(b,a)\big)}{2}\right| &= \frac{\eta(b,a)}{2}\left|\frac{\eta(b,a)}{2}\int_0^1 (1-2t)f'\big(a+t\eta(b,a)\big)dt\right| \\ &\leq \frac{\eta(b,a)}{2}\int_0^1 |1-2t|\left|f'\big(a+t\eta(b,a)\big)\right|dt \\ &\leq \frac{\eta(b,a)}{2}|f'(a)||f'(b)|\int_0^1 |1-2t|dt \\ &= \frac{\eta(b,a)|f'(a)||f'(b)|}{4},\end{aligned}$$

where $\int_0^1 |1-2t|dt = \frac{1}{2}$. This completes the proof of theorem.

Corollary 1. If we take $\eta(b,a) = b-a$ in the inequality (4), then we have the following inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}\right| \leq \frac{(b-a)|f'(a)||f'(b)|}{4} \quad (5)$$

Remark 2. The inequality (5) coincides with that in [13].

The corresponding version for powers of the absolute value of the derivative is incorporated in the following result.

Theorem 12. Let $f: I \to \mathbb{R}$ be a differentiable function on I°. Assume $q \in \mathbb{R}$, $q > 1$, is such that the function $|f'|^q$ is multiplicatively preinvex P-function. Suppose that $a, a+\eta(b,a) \in I$ with $a < a+\eta(b,a)$ and $f' \in L[a, a+\eta(b,a)]$. Then the following inequality holds:

$$\begin{aligned}&\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - \frac{f(a)+f\big(a+\eta(b,a)\big)}{2}\right| \\ &\leq \frac{\eta(b,a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}|f'(a)||f'(b)|,\end{aligned} \quad (6)$$

where $\frac{1}{p}+\frac{1}{q} = 1$.

Proof: Let $a, b \in I$. By assumption, Hölder's integral inequality, Lemma 1 and the inequality

$$\left|f'\big(a+\eta(b,a)\big)\right|^q \leq |f'(a)|^q|f'(b)|^q,$$

we have

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - \frac{f(a)+f\big(a+\eta(b,a)\big)}{2}\right|$$

$$\le \frac{\eta(b,a)}{2}\int_0^1 |1-2t|\left|f'\left(a+t\eta(b,a)\right)\right|dt$$
$$\le \frac{\eta(b,a)}{2}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(a+t\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$= \frac{\eta(b,a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(a+t\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\le \frac{\eta(b,a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 |f'(a)|^q|f'(b)|^q dt\right)^{\frac{1}{q}}$$
$$= \frac{\eta(b,a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}|f'(a)||f'(b)|,$$

where $\int_0^1 |1-2t|^p dt = \frac{1}{p+1}$.

Corollary 2. If we take $\eta(b,a) = b-a$ in the inequality (6), then we have the following inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}\right| \le \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}|f'(a)||f'(b)| \tag{7}$$

Remark 3. The inequality (7) coincides with that in [13].

Theorem 13. Let $f: I \to \mathbb{R}$ be a differentiable function on I°. Assume that the function $|f'|$ is multiplicatively preinvex P-function. Suppose that $a, a+\eta(b,a) \in I$ with $a < a+\eta(b,a)$ and $f' \in L[a, a+\eta(b,a)]$. Then the following inequality holds:

$$\left|\frac{1}{\eta(b,a)}f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{[\eta(b,a)]^2}\int_a^{a+\eta(b,a)} f(x)dx\right| \le \frac{|f'(a)||f'(b)|}{4}. \tag{8}$$

Proof: Since the function $|f'|$ is a multiplicatively preinvex P-function, from Lemma 2 and the Hölder's integral inequality, we have

$$\left|\frac{1}{\eta(b,a)}f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{[\eta(b,a)]^2}\int_a^{a+\eta(b,a)} f(x)dx\right|$$
$$\le \int_0^{\frac{1}{2}} t\left|f'\left(a+t\eta(b,a)\right)\right|dt + \int_{\frac{1}{2}}^1 |t-1|\left|f'\left(a+t\eta(b,a)\right)\right|dt$$
$$\le |f'(a)||f'(b)|\int_0^{\frac{1}{2}} t dt + |f'(a)||f'(b)|\int_{\frac{1}{2}}^1 |t-1|dt$$
$$= \frac{|f'(a)||f'(b)|}{4}.$$

Corollary 3. If we take $\eta(b,a) = b-a$ in the inequality (8), then we have the following inequality:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}|f'(a)||f'(b)| \tag{9}$$

Remark 4. The inequality (9) coincides with that in [13].

Theorem 14. Let $f: I \to \mathbb{R}$ be a differentiable function on $I°$. Assume $q \in \mathbb{R}$, $q > 1$, is such that the function $|f'|^q$ is multiplicatively preinvex P-function. Suppose that $a, a+\eta(b,a) \in I$ with $a < a+\eta(b,a)$ and $f' \in L[a, a+\eta(b,a)]$. Then the following inequality holds:

$$\begin{aligned}&\left|\frac{1}{\eta(b,a)}f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{1}{[\eta(b,a)]^2}\int_a^{a+\eta(b,a)} f(x)dx\right| \\ &\leq \frac{|f'(a)||f'(b)|}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}},\end{aligned} \tag{10}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof: Since the function $|f'|^q$ is a multiplicatively preinvex *P*-function, from Lemma 2 and the Hölder's integral inequality, we have

$$\begin{aligned}&\left|\frac{1}{\eta(b,a)}f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{1}{[\eta(b,a)]^2}\int_a^{a+\eta(b,a)} f(x)dx\right| \\ &\leq \int_0^{\frac{1}{2}} t\left|f'\left(a+t\eta(b,a)\right)\right|dt + \int_{\frac{1}{2}}^1 |t-1|\left|f'\left(a+t\eta(b,a)\right)\right|dt \\ &\leq \left(\int_0^{\frac{1}{2}} |t|^p dtdt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'\left(a+t\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}} \\ &+\left(\int_{\frac{1}{2}}^1 |t-1|^p dtdt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'\left(a+t\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}} \\ &= \frac{|f'(a)||f'(b)|}{2^{\frac{1}{q}}}\left[\left(\int_0^{\frac{1}{2}}|t|^p dt\right)^{\frac{1}{p}}+\left(\int_{\frac{1}{2}}^1|t-1|^p dt\right)^{\frac{1}{p}}\right] \\ &= \frac{|f'(a)||f'(b)|}{2^{\frac{1}{q}}}\left[\left(\frac{1}{(p+1)2^{p+1}}\right)^{\frac{1}{p}}+\left(\frac{1}{(p+1)2^{p+1}}\right)^{\frac{1}{p}}\right] \\ &= \frac{|f'(a)||f'(b)|}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\end{aligned}$$

where $\int_0^{\frac{1}{2}} t^p dt = \int_{\frac{1}{2}}^1 |t-1|^p dt = \frac{1}{(p+1)2^{p+1}}$.

Corollary 4. If we take $\eta(b,a) = b-a$ in the inequality (10), then we have the following inequality:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{2}|f'(a)||f'(b)|\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \quad (11)$$

Remark 5. The inequality (11) coincides with that in [13].

Now, using the Hölder-İşcan integral inequality we will prove the following theorem, which is a better result than the inequality (10)

Theorem 15. Let $f: I \to \mathbb{R}$ be a differentiable function on $I°$. Assume $q \in \mathbb{R}$, $q > 1$, is such that the function $|f'|^q$ is multiplicatively preinvex P-function. Suppose that $a, a+\eta(b,a) \in I$ with $a < a+\eta(b,a)$ and $f' \in L[a, a+\eta(b,a)]$. Then the following inequality holds:

$$\begin{aligned}&\left|\frac{1}{\eta(b,a)}f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{1}{[\eta(b,a)]^2}\int_a^{a+\eta(b,a)} f(x)dx\right| \\ &\le |f'(a)||f'(b)|\left(\frac{1}{2}\right)^{1+\frac{1}{q}}\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left[1+\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\right],\end{aligned} \quad (12)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof: Since the function $|f'|^q$ is a multiplicatively preinvex *P*-function, from Lemma 2 and the Hölder-İşcan integral inequality, we have

$$\begin{aligned}&\left|\frac{1}{\eta(b,a)}f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{1}{[\eta(b,a)]^2}\int_a^{a+\eta(b,a)} f(x)dx\right| \\
&\le \int_0^{\frac{1}{2}} t\left|f'\left(a+t\eta(b,a)\right)\right|dt + \int_{\frac{1}{2}}^1 |t-1|\left|f'\left(a+t\eta(b,a)\right)\right|dt \\
&\le 2\left[\left(\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)|t|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)\left|f'\left(a+t\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\qquad \left.+\left(\int_0^{\frac{1}{2}} t|t|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}} t\left|f'\left(a+t\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&+2\left[\left(\int_{\frac{1}{2}}^1 (1-t)|t-1|^p dt\right)^{\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 (1-t)\left|f'\left(a+t\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1\left(t-\frac{1}{2}\right)|t-1|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left(t-\frac{1}{2}\right)\left|f'\left(a+t\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&\le 2|f'(a)||f'(b)|\left(\frac{1}{8}\right)^{\frac{1}{q}}\left[\left(\frac{1}{(p+1)(p+2)2^{p+2}}\right)^{\frac{1}{p}}+\left(\frac{1}{(p+2)2^{p+2}}\right)^{\frac{1}{p}}\right]\end{aligned}$$

$$+2|f'(a)||f'(b)|\left(\frac{1}{8}\right)^{\frac{1}{q}}\left[\left(\frac{1}{(p+2)2^{p+2}}\right)^{\frac{1}{p}}+\left(\frac{1}{(p+1)(p+2)2^{p+2}}\right)^{\frac{1}{p}}\right]$$

$$=4|f'(a)||f'(b)|\left(\frac{1}{8}\right)^{\frac{1}{q}}\left(\frac{1}{(p+2)2^{p+2}}\right)^{\frac{1}{p}}\left[1+\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\right]$$

$$=|f'(a)||f'(b)|\left(\frac{1}{2}\right)^{1+\frac{1}{q}}\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left[1+\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\right],$$

where

$$\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)|t|^p dt=\frac{1}{(p+1)(p+2)2^{p+2}},\qquad \int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)dt=\frac{1}{8}$$

$$\int_0^{\frac{1}{2}}t|t|^p dt=\frac{1}{(p+2)2^{p+2}},\qquad \int_0^{\frac{1}{2}}tdt=\frac{1}{8}$$

$$\int_{\frac{1}{2}}^{1}(1-t)|t-1|^p dt=\frac{1}{(p+2)2^{p+2}},\qquad \int_{\frac{1}{2}}^{1}(1-t)dt=\frac{1}{8}$$

$$\int_{\frac{1}{2}}^{1}\left(t-\frac{1}{2}\right)|t-1|^p dt=\frac{1}{(p+1)(p+2)2^{p+2}}\qquad \int_{\frac{1}{2}}^{1}\left(t-\frac{1}{2}\right)dt=\frac{1}{8}.$$

Corollary 5. If we take $\eta(b,a)=b-a$ in the inequality (12), then we have the following inequality:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq (b-a)|f'(a)||f'(b)|\left(\frac{1}{2}\right)^{1+\frac{1}{q}}\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left[1+\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\right].$$

Corollary 6. The inequality (12) is better than the inequality (10).

Proof: Let

$$M(p)=\left(\frac{1}{2}\right)^{1+\frac{1}{q}}\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left[1+\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\right],$$

$$N(p)=\frac{1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}.$$

It is enough to show that

$$\frac{M(p)}{N(p)}=\left(\frac{1}{2}\right)^{\frac{1}{q}}\left[\left(\frac{1}{p+2}\right)^{\frac{1}{p}}+\left(\frac{p+1}{p+2}\right)^{\frac{1}{p}}\right]\leq 1.$$

If we use concavity of the function

$$g:[0,\infty)\to\mathbb{R},\ g(x)=x^s,\ 0<s\leq 1$$

we obtain

$$\frac{M(p)}{N(p)} = 2^{1-\frac{1}{q}}\left[\frac{1}{2}\left(\frac{1}{p+2}\right)^{\frac{1}{p}} + \frac{1}{2}\left(\frac{p+1}{p+2}\right)^{\frac{1}{p}}\right] \leq 2^{\frac{1}{p}}\left(\frac{\frac{1}{p+2}+\frac{p+1}{p+2}}{2}\right)^{\frac{1}{p}} = 1.$$

So, $M(p) \leq N(p)$.

CONCLUSION

In this study, using both Hölder and Hölder-İşcan integral inequalities, some inequalities of Hermite Hadamard type integral inequalities obtained for multiplicatively preinvex P-functions are given. A similar method can be applied to other classes of convexity.

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