ORIGINAL PAPER

SOME HERMITE-HADAMARD TYPE INEQUALITIES INVOLVING FRACTIONAL INTEGRAL OPERATORS

LOREDANA CIURDARIU[1]

Manuscript received: 15.09.2022; Accepted paper: 11.11.2022;
Published online: 30.12.2022.

Abstract. *The aim of this article is to give new generalized Hermite-Hadamard type inequalities involving the Riemann-Liouville fractional integral for functions whose absolute values of third derivatives are s-convex in the second sense. In order to do that an integral identity for three times differentiable mapping involving fractional integral operators is established. Several consequences are then presented in some special cases.*
Keywords: *Hermite-Hadamard inequality; Holder inequality; convex functions.*

1. INTRODUCTION

Convexity is an essential notion used in mathematics, mathematical statistics, convex programming, approximation theory and many other fields. In the recent years many variants of convex functions have been defined and are intensively studied by many researchers in applied sciences. Convex functions are very important in the formulation of new inequalities.

The Hermite-Hadamard inequality [1], discoverrd in 1893, is one of them, which describe lower and upper bound of the integral mean of a convex function over an interval $[a, b]$. This inequality was first generalized by Fejèr in [2]. Nowadays this inequality is generalized by defining new classes of convex functions which are clearly related to classical convex functions and by establishing similar results

The Hermite-Hadamard inequality is considered one of the most useful inequalities in mathematical analysis having a lot of generalizations, extensions refinements and many applications, see for example [3-15].

The aim of this paper is to establish type Hermite-Hadamard like inequalities for functions whose third derivative in absolute values are s-convex in the second sense, by using as a main tool a new integral identity involving the Riemann-Liouville fractional integral. Several consequences are pointed out for some special cases as setting $x = \frac{a+b}{2}$ or $s = 1$.

2. MATERIALS AND METHODS

Let $f: I \to \boldsymbol{R}$, be a convex function on the interval I in the set of real numbers $\boldsymbol{R}$. Then for any $a, b \in I, a < b$, $f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$, and this is the classical Hermite-Hadamard inequality [1].

[1] Politehnica University of Timisoara, Department of Mathematics, 300006 Timisoara, Romania.
E-mail: lgtirtirau@yahoo.com

In the following the necessary definitions and results which will be used below will be stated.

Definition 2.1. A function $f: [0,\infty) \to \boldsymbol{R}$ is called s-convex in the second sense if $f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y)$, for all $x, y \in [0,\infty)$, $t \in [0,1]$ and $s \epsilon (0,1]$.

The definition of s-convexity in the second sense was given in Breckner's article [16].

A hierarchy of convexity and several classical inequalities was presented in [17].

Definition 2.2. ([18]) Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \ge 0$ are defined by

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively where $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du$ is the Gamma function of Euler and $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$. In the case when $\alpha = 1$ the fractional integral becomes the classical integral.

The following class of functions was defined formally by Raina in [19],

$$\mathcal{F}_{\rho,\lambda}^{\sigma}(x) = \mathcal{F}_{\rho,\lambda}^{\sigma(0),\sigma(1),\ldots}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \quad (\rho, \lambda > 0; |x| < R),$$

where the coefficients $\sigma(k)$ $(k \in \boldsymbol{N})$ is a bounded sequence of positive real numbers and R is the set of real numbers and Γ is the function Gamma of Euler, see also [18]. The following left-sided and right-sided fractional integral operators were respectively defined as below, by Raina [19] and Agarwal et al. [20],

$$\left(\mathcal{J}_{\rho,\lambda,a^+;w}^{\sigma}\varphi\right)(x) = \int_a^x (x-t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}[w(x-t)^{\rho}]\varphi(t)dt \quad (x > a > 0) \tag{1}$$

$$\left(\mathcal{J}_{\rho,\lambda,b^-;w}^{\sigma}\varphi\right)(x) = \int_x^b (t-x)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}[w(t-x)^{\rho}]\varphi(t)dt \quad (0 < x < b), \tag{2}$$

where $\rho, \lambda > 0, w \in \boldsymbol{R}$ and $\varphi(t)$ is such that the integral on the right side exists. More integral inequalities involving this operator can be found in [20]. It is known that if $\mathfrak{M} = \mathcal{F}_{\rho,\lambda+1}^{\sigma}[w(b-a)^{\rho}] < \infty$, then $\left(\mathcal{J}_{\rho,\lambda,a^+;w}^{\sigma}\varphi\right)(x)$ and $\left(\mathcal{J}_{\rho,\lambda,b^-;w}^{\sigma}\varphi\right)(x)$ are bounded integral operators on $L(a,b)$ and moreover, for $\varphi \in L(a,b)$, $||\left(\mathcal{J}_{\rho,\lambda,a^+;w}^{\sigma}\varphi\right)(x)||_1 \le \mathfrak{M}(b-a)^{\lambda} ||\varphi||_1$ and $||\left(\mathcal{J}_{\rho,\lambda,b^-;w}^{\sigma}\varphi\right)(x)||_1 \le \mathfrak{M}(b-a)^{\lambda} ||\varphi||_1$, where $||\varphi||_p = \left(\int_a^b |\varphi(t)|^p dt\right)^{\frac{1}{p}}$. The classical Riemann-Liouville fractional integrals $J_{a^+}^{\alpha}$ and $J_{b^-}^{\alpha}$ of order λ can be obtained from (1) and (2) by taking $\lambda = \alpha$, $\sigma(0) = 1$, $w = 0$.

Lemma 2.1. ([18]) Let $f: [a,b] \to \boldsymbol{R}$ be a twice differentiable mapping on $[a,b]$ with $a < b$ and $\lambda > 0$. If $f'' \in L[a,b]$ then the following equality holds:

$$\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{\lambda}}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a^{+};w}f\right)(b)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b^{-};w}f\right)(a)\right]=$$
$$=\frac{(b-a)^2}{2}\int_0^1\left\{t\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(b-a)^{\rho}]-t^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(b-a)^{\rho}t^{\rho}]\right\}\left[f''(ta+(1-t)b)+\right.$$
$$\left.f''\big((1-t)a+tb\big)\right]dt\,.$$

Next lemma is the main result given in [21] and used for establishing new Hermite-Hadamard's inequalities for differentiable convex functions.

Lemma 2.2. Let $f:I^0\to\boldsymbol{R}$ be a differentiable mapping on $I^0, a, b\in I^0$ with $a<b$. If $f''\in L[a,b]$ then for all $x\in I^0$ the following equality holds:

$$\frac{(b-x)f(b)+(x-a)f(a)+f(x)(b-a)}{2(b-a)}-\frac{1}{b-a}\int_a^b f(t)dt=$$
$$=\frac{1}{2(b-a)}\int_0^1(t-t^2)\big((x-a)^3f''(tx+(1-t)a)-(x-b)^3f''(tx+(1-t)b)\big)dt.$$

The integral identities from Lemma 2.1. and Lemma 2.2. are the starting point for result from Lemma 3.1. and this identity is connected with Hermite-Hadamard's inequalities for different type of convex functions which will be presented below.

3. RESULTS AND DISCUSSION

3.1. RESULTS

In this section there are presented new Hermite-Hadamard type inequalities for three times differentiable convex functions and also some consequences.

Lemma 3.1. Let $f:[a,b]\to\boldsymbol{R}$ be a three times differentiable mapping on $[a,b]$ with $a<b$ and $\lambda>0$. If $f'''\in L[a,b]$ and $x\in(a,b)$ then the following equality holds:

$$f'(x)\left\{\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(x-a)^{\rho}]-2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]}{(x-a)^2}-\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(b-x)^{\rho}]-2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]}{(x-b)^2}\right\}+$$
$$\frac{2[f(x)-f(a)]}{(x-a)^3}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]-\frac{2[f(x)-f(b)]}{(x-b)^3}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]-$$
$$f(x)\left\{\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]}{(x-a)^3}-\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]}{(x-b)^3}\right\}+\frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x^{-};w}f\right)(a)}{(x-a)^{\lambda+3}}+\frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x^{+};w}f\right)(b)}{(b-x)^{\lambda+3}}=$$
$$=\int_0^1\left\{t^2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]-t^{\lambda+2}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}t^{\rho}]\right\}f'''(tx+(1-t)a)dt-$$
$$-\int_0^1\left\{t^2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]-t^{\lambda+2}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}t^{\rho}]\right\}f'''(tx+(1-t)b)dt. \qquad (1)$$

Proof: It will be used the following notations,

$$I_1=\int_0^1 t^2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]\,f'''(tx+(1-t)a)dt,$$

$I_2=\int_0^1 t^2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]\,f'''(tx+(1-t)b)dt,$

$I_3=\int_0^1 t^{\lambda+2}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}t^{\rho}]\,f'''(tx+(1-t)a)dt,$

$I_4=\int_0^1 t^{\lambda+2}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}t^{\rho}]\,f'''(tx+(1-t)b)dt.$

Integrating by parts I_1 and then changing the variables $u=tx+(1-t)a$ we have,

$$I_1=\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]\{\frac{f''(x)}{x-a}-\frac{2}{x-a}[\frac{tf'(tx+(1-t)a)}{x-a}\,|_0^1-\int_0^1\frac{f'(tx+(1-t)a)}{x-a}dt]\}=$$

$$=\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]\left\{\frac{f''(x)}{x-a}-\frac{2f'(x)}{(x-a)^2}+\frac{2[f(x)-f(a)]}{(x-a)^3}\right\}.$$

Analogously, for I_2 we get, $I_2=\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]\left\{\frac{f''(x)}{x-b}-\frac{2f'(x)}{(x-b)^2}+\frac{2[f(x)-f(b)]}{(x-b)^3}\right\}.$
For I_3 by the same method, we obtain sucessively,

$$I_3=\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]\frac{f''(x)}{x-a}-\int_0^1 t^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(x-a)^{\rho}t^{\rho}]\frac{f''(tx+(1-t)a)}{x-a}dt=$$
$$=\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]\frac{f''(x)}{x-a}-\{t^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(x-a)^{\rho}t^{\rho}]\frac{f'(tx+(1-t)a)}{(x-a)^2}\,|_0^1-$$
$$-\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}t^{\rho}]\frac{f'(tx+(1-t)a)}{x-a}dt\}=$$
$$=\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]\frac{f''(x)}{x-a}-\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(x-a)^{\rho}]\frac{f'(x)}{(x-a)^2}$$
$$+\frac{1}{(x-a)^2}\{t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}t^{\rho}]\frac{f(tx+(1-t)a)}{x-a}\,|_0^1--\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(x-a)^{\rho}t^{\rho}]\frac{f(tx+(1-t)a)}{x-a}dt\}=$$

$$I_3=\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]\frac{f''(x)}{x-a}-\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(x-a)^{\rho}]\frac{f'(x)}{(x-a)^2}+\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]\frac{f(x)}{(x-a)^3}-$$
$$-\frac{1}{(x-a)^{\lambda}}\int_a^x (u-a)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(x-a)^{\rho}]f(u)du.$$

From here,

$$I_3=\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]\frac{f''(x)}{x-a}-\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(x-a)^{\rho}]\frac{f'(x)}{(x-a)^2}+\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]\frac{f(x)}{(x-a)^3}-\frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x^-;w}f\right)(a)}{(x-a)^{\lambda+3}},$$

and

$$I_4=\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]\frac{f''(x)}{x-b}-\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(b-x)^{\rho}]\frac{f'(x)}{(x-b)^2}+\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]\frac{f(x)}{(x-b)^3}+\frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x^+;w}f\right)(b)}{(b-x)^{\lambda+3}}.$$

Substracting I_3 from I_1 and I_4 from I_2, we have,

$$I_1 - I_3 = \frac{f'(x)}{(x-a)^2}\left\{\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(x-a)^{\rho}] - 2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]\right\} + \frac{2[f(x)-f(a)]}{(x-a)^3}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}] - \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]\frac{f(x)}{(x-a)^3} + \frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x^-;w}f\right)(a)}{(x-a)^{\lambda+3}}$$

and

$$I_2 - I_4 = \frac{f'(x)}{(x-b)^2}\left\{\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(b-x)^{\rho}] - 2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]\right\} + \frac{2[f(x)-f(b)]}{(x-b)^3}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}] - \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]\frac{f(x)}{(x-b)^3} - \frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x^+;w}f\right)(b)}{(b-x)^{\lambda+3}}.$$

Computing now the expression $I_1 - I_3 - (I_2 - I_4)$ the desired equality will be obtained.

This lemma can be used for obtaining the following inequalities via fractional integral operator for three time differentiable functions whose value are s-convex in the second sense.

Theorem 3.2. Let $f:[a,b] \to \boldsymbol{R}$ be a three times differentiable function on the interval $[a,b]$ with $a < b$, $\lambda > 0$. If $|f'''|$ is s-convex in the second sense on (a,b) then we have the following inequality:

$$\left|f'(x)\left\{\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(x-a)^{\rho}] - 2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]}{(x-a)^2} - \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(b-x)^{\rho}] - 2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]}{(x-b)^2}\right\}\right.$$
$$+\frac{2[f(x)-f(a)]}{(x-a)^3}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}] - \frac{2[f(x)-f(b)]}{(x-b)^3}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]$$
$$\left.-f(x)\left\{\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]}{(x-a)^3} - \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]}{(x-b)^3}\right\} + \frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x^-;w}f\right)(a)}{(x-a)^{\lambda+3}} + \frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x^+;w}f\right)(b)}{(b-x)^{\lambda+3}}\right|$$
$$\leq \left|f'''(x)\right|\left\{\mathcal{F}^{\sigma_{1,s}}_{\rho,\lambda+3}[w(x-a)^{\rho}] + \mathcal{F}^{\sigma_{1,s}}_{\rho,\lambda+3}[w(b-x)^{\rho}]\right\} + \left|f'''(a)\right|$$
$$\left\{\mathcal{F}^{\sigma_{2,s}}_{\rho,\lambda+3}[w(x-a)^{\rho}] + \left|f'''(b)\right|\mathcal{F}^{\sigma_{2,s}}_{\rho,\lambda+3}[w(b-x)^{\rho}]\right\},$$

where

$$\sigma_{1,s}(k) = \sigma(k)\frac{\lambda+\rho k}{(s+3)(\rho k+\lambda+s+3)}$$

and $\sigma_{2,s}(k) = \sigma(k)[B(3,s+1) - B(\rho k+\lambda+3, s+1)]$, $\rho,\lambda > 0, w \in \boldsymbol{R}, s \in (0,1]$. The function $B(.,.)$ is the beta function of Euler.

Proof: It will be used here Lemma 3.1 and the properties of modulus. We will obtain then,

$$|I_1 - I_3 - (I_2 - I_4)| =$$
$$= \left|f'(x)\left\{\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(x-a)^{\rho}] - 2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]}{(x-a)^2} - \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(b-x)^{\rho}] - 2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]}{(x-b)^2}\right\}\right.$$
$$+\frac{2[f(x)-f(a)]}{(x-a)^3}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}] - \frac{2[f(x)-f(b)]}{(x-b)^3}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]$$
$$-f(x)\left\{\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]}{(x-a)^3} - \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]}{(x-b)^3}\right\}$$

$$+\frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x^-;w}f\right)(a)}{(x-a)^{\lambda+3}}$$

$$+\left.\frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x^+;w}f\right)(b)}{(b-x)^{\lambda+3}}\right|\leq\int_0^1\left|t^2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]-t^{\lambda+2}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}t^{\rho}]\right|\left|f'''(tx\right.$$

$$\left.+(1-t)a)\right|dt$$

$$+\int_0^1\left|t^{\lambda+2}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}t^{\rho}]-t^2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]\right||f'''(tx+(1-t)b)|dt\leq$$

$$\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(x-a)^{\rho k}}{\Gamma(\rho k+\lambda+3)}\int_0^1\left(t^2-t^{\rho k+\lambda+2}\right)|f'''(tx+(1-t)a)|dt+$$

$$+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-x)^{\rho k}}{\Gamma(\rho k+\lambda+3)}\int_0^1\left(t^2-t^{\rho k+\lambda+2}\right)|f'''(tx+(1-t)b)|dt.$$

Taking into account that $|f'''|$ is s-convex in the second sense, we get

$$|I_1-I_3-(I_2-I_4)|\leq\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(x-a)^{\rho k}}{\Gamma(\rho k+\lambda+3)}\int_0^1\left(t^2-t^{\rho k+\lambda+2}\right)[t^s|f'''(x)|+(1-t)^s|f'''(a)|]dt+$$

$$+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-x)^{\rho k}}{\Gamma(\rho k+\lambda+3)}\int_0^1\left(t^2-t^{\rho k+\lambda+2}\right)[t^s|f'''(x)|+(1-t)^s|f'''(b)|]\,dt=$$

$$=\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(x-a)^{\rho k}}{\Gamma(\rho k+\lambda+3)}\{|f'''(x)|\frac{(\rho k+\lambda)}{(s+3)(\rho k+\lambda+s+3)}+|f'''(a)|[B(3,s+1)-B(\rho k+\lambda+3,s+1)]\}$$

$$+$$

$$+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-x)^{\rho k}}{\Gamma(\rho k+\lambda+3)}\left\{|f'''(x)\left|\frac{(\rho k+\lambda)}{(s+3)(\rho k+\lambda+s+3)}+|f'''(b)|[B(3,s+1)-B(\rho k+\lambda+3,s+1)]\right.\right\}$$

$$=$$

$$=\left|f'''(x)\right|\left\{\mathcal{F}^{\sigma_{1,s}}_{\rho,\lambda+3}[w(x-a)^{\rho}]+\mathcal{F}^{\sigma_{1,s}}_{\rho,\lambda+3}[w(b-x)^{\rho}]\right\}+\left|f'''(a)\right|$$
$$\mathcal{F}^{\sigma_{2,s}}_{\rho,\lambda+3}[w(x-a)^{\rho}]+\left|f'''(b)\right|$$
$$\mathcal{F}^{\sigma_{2,s}}_{\rho,\lambda+3}[w(b-x)^{\rho}].$$

Corollary 3.3. If it will be considered in previous theorem $s=1$, then we have,

$$|f'(x)\left\{\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(x-a)^{\rho}]-2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]}{(x-a)^2}-\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(b-x)^{\rho}]-2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]}{(x-b)^2}\right\}+$$
$$\frac{2[f(x)-f(a)]}{(x-a)^3}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]-\frac{2[f(x)-f(b)]}{(x-b)^3}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]-f(x)\left\{\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]}{(x-a)^3}-\right.$$
$$\left.\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]}{(x-b)^3}\right\}+\frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x^-;w}f\right)(a)}{(x-a)^{\lambda+3}}+\frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x^+;w}f\right)(b)}{(b-x)^{\lambda+3}}|\leq\left|f'''(x)\right|\left\{\mathcal{F}^{\sigma_{1,1}}_{\rho,\lambda+3}[w(x-a)^{\rho}]+\mathcal{F}^{\sigma_{1,1}}_{\rho,\lambda+3}[w(b-\right.$$
$$\left.x)^{\rho}]\right\}+\left|f'''(a)\right|\mathcal{F}^{\sigma_{2,1}}_{\rho,\lambda+3}[w(x-a)^{\rho}]+\left|f'''(b)\right|\mathcal{F}^{\sigma_{2,1}}_{\rho,\lambda+3}[w(b-x)^{\rho}],$$ where $\sigma_{1,1}(k)=\sigma(k)\frac{\lambda+\rho k}{4(\rho k+\lambda+4)}$

and

$$\sigma_{2,1}(k)=\sigma(k)[B(3,2)-B(\rho k+\lambda+3,2)],\ \rho,\lambda>0,w\in\boldsymbol{R}.$$

Corollary 3.4. If in previous theorem we take $x = \frac{a+b}{2}$, then we have,

$$|\frac{16}{(b-a)^3}\left\{f\left(\frac{a+b}{2}\right)[2\mathcal{F}^{\sigma}_{\rho,\lambda+3}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right] - \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]] - [f(a)+f(b)]\mathcal{F}^{\sigma}_{\rho,\lambda+3}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]\right\} + \frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\frac{a+b}{2}^-;w}f\right)(a) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,\frac{a+b}{2}^+;w}f\right)(b)}{\left(\frac{b-a}{2}\right)^{\lambda+3}}| \le$$

$$2\left|f'''\left(\frac{a+b}{2}\right)\right|\mathcal{F}^{\sigma_{1,s}}_{\rho,\lambda+3}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right] + (|f'''(a)|+|f'''(b)|)\mathcal{F}^{\sigma_{2,s}}_{\rho,\lambda+3}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right],$$

where

$$\sigma_{1,s}(k) = \sigma(k)\frac{\lambda+\rho k}{(s+3)(\rho k+\lambda+s+3)}$$

and

$$\sigma_{2,s}(k) = \sigma(k)[B(3,s+1) - B(\rho k+\lambda+3, s+1)],\ \rho,\lambda > 0, w \in \boldsymbol{R}, s \in (0,1].$$

Theorem 3.5. Let $f:[a,b] \to \boldsymbol{R}$ be a three times differentiable function on the interval $[a,b]$ with $a < b$. If $|f'''|^q$ is s-convex in the second sense on (a,b) and $q > 1$, where $\frac{1}{p}+\frac{1}{q} = 1$, then the following inequality holds:

$$|f'(x)\left\{\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(x-a)^{\rho}] - 2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]}{(x-a)^2} - \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(b-x)^{\rho}] - 2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]}{(x-b)^2}\right\}$$
$$+\frac{2[f(x)-f(a)]}{(x-a)^3}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}] - \frac{2[f(x)-f(b)]}{(x-b)^3}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]$$
$$-f(x)\left\{\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]}{(x-a)^3} - \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]}{(x-b)^3}\right\} + \frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x^-;w}f\right)(a)}{(x-a)^{\lambda+3}}$$
$$+\frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x^+;w}f\right)(b)}{(b-x)^{\lambda+3}}|$$
$$\le \frac{\left[|f'''(x)|^q + |f'''(a)|^q\right]^{\frac{1}{q}}}{(s+1)^{\frac{1}{q}}}\mathcal{F}^{\sigma_3}_{\rho,\lambda+3}[w(x-a)^{\rho}]$$
$$+\frac{\left[|f'''(x)|^q + |f'''(b)|^q\right]^{\frac{1}{q}}}{(s+1)^{\frac{1}{q}}}\mathcal{F}^{\sigma_3}_{\rho,\lambda+3}[w(b-x)^{\rho}],$$

where $\sigma_3(k) = \sigma(k)\frac{B^{\frac{1}{p}}(\frac{3}{\rho k+\lambda},p)}{(\rho k+\lambda)^{\frac{1}{p}}}$, $\rho,\lambda > 0, w \in \boldsymbol{R}$, $s \in (0,1]$ and $B(.,.)$ is the beta function of Euler.

Proof: From Lemma 3.1, Holder's inequality and the s-convexity in the second sense of $|f'''|^q$ we obtain the following inequality,

$$|f'(x)\left\{\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(x-a)^{\rho}] - 2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]}{(x-a)^2} - \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(b-x)^{\rho}] - 2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]}{(x-b)^2}\right\} + \frac{2[f(x)-f(a)]}{(x-a)^3}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}] - \frac{2[f(x)-f(b)]}{(x-b)^3}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}] - f(x)\left\{\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]}{(x-a)^3} - \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]}{(x-b)^3}\right\} + \frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x^-;w}f\right)(a)}{(x-a)^{\lambda+3}} + \frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x^+;w}f\right)(b)}{(b-x)^{\lambda+3}}| \le \int_0^1\left|t^2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}] - t^{\lambda+2}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}t^{\rho}]\right||f'''(tx+(1-t)a) + \int_0^1|t^{\lambda+2}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}t^{\rho}] - t^2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]||f'''(tx+(1-t)b)|dt \le \sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(x-a)^{\rho k}}{\Gamma(\rho k+\lambda+3)}\int_0^1\left(t^2 - t^{\rho k+\lambda+2}\right)|f'''(tx+(1-t)a)|dt +$$

$$+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-x)^{\rho k}}{\Gamma(\rho k+\lambda+3)}\int_0^1\left(t^2-t^{\rho k+\lambda+2}\right)|f'''(tx+(1-t)b)|dt\le$$

$$\le\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(x-a)^{\rho k}}{\Gamma(\rho k+\lambda+3)}\left(\int_0^1\left(t^2-t^{\rho k+\lambda+2}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f'''(tx+(1-t)a)|^q dt\right)^{\frac{1}{q}}+$$

$$+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-x)^{\rho k}}{\Gamma(\rho k+\lambda+3)}\left(\int_0^1\left(t^2-t^{\rho k+\lambda+2}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f'''(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}}\le$$

$$\le\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(x-a)^{\rho k}}{\Gamma(\rho k+\lambda+3)}\left(\int_0^1 t^{2p}\left(1-t^{\rho k+\lambda}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1[t^s|f'''(x)|^q+(1-t)^s|f'''(a)|^q]dt\right)^{\frac{1}{q}}+$$

$$+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-x)^{\rho k}}{\Gamma(\rho k+\lambda+3)}\left(\int_0^1 t^{2p}\left(1-t^{\rho k+\lambda}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1[t^s|f'''(x)|^q+(1-t)^s|f'''(b)|^q]dt\right)^{\frac{1}{q}}=$$

$$=\frac{[|f'''(x)|^q+|f'''(a)|^q]^{\frac{1}{q}}}{(s+1)^{\frac{1}{q}}}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(x-a)^{\rho k}}{\Gamma(\rho k+\lambda+3)}\frac{B^{\frac{1}{p}}(\frac{3}{\rho k+\lambda},p)}{(\rho k+\lambda)^{\frac{1}{p}}}$$

$$+\frac{[|f'''(x)|^q+|f'''(b)|^q]^{\frac{1}{q}}}{(s+1)^{\frac{1}{q}}}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-x)^{\rho k}}{\Gamma(\rho k+\lambda+3)}\frac{B^{\frac{1}{p}}(\frac{3}{\rho k+\lambda},p)}{(\rho k+\lambda)^{\frac{1}{p}}}=$$

$$=\frac{[|f'''(x)|^q+|f'''(a)|^q]^{\frac{1}{q}}}{(s+1)^{\frac{1}{q}}}\mathcal{F}^{\sigma_3}_{\rho,\lambda+3}[w(x-a)^\rho]+\frac{[|f'''(x)|^q+|f'''(b)|^q]^{\frac{1}{q}}}{(s+1)^{\frac{1}{q}}}\mathcal{F}^{\sigma_3}_{\rho,\lambda+3}[w(b-x)^\rho].$$

Corollary 3.6. If in previous theorem it is taken $x=\frac{a+b}{2}$ then we have,

$$\left|\frac{16}{(b-a)^3}\left\{f\left(\frac{a+b}{2}\right)[2\mathcal{F}^{\sigma}_{\rho,\lambda+3}\left[w\left(\frac{b-a}{2}\right)^\rho\right]-\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^\rho\right]]-[f(a)\right.\right.$$

$$\left.\left.+f(b)]\mathcal{F}^{\sigma}_{\rho,\lambda+3}\left[w\left(\frac{b-a}{2}\right)^\rho\right]\right\}+\frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\frac{a+b}{2}^-;w}f\right)(a)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\frac{a+b}{2}^+;w}f\right)(b)}{\left(\frac{b-a}{2}\right)^{\lambda+3}}\right|$$

$$\le\frac{\mathcal{F}^{\sigma_3}_{\rho,\lambda+3}\left[w\left(\frac{b-a}{2}\right)^\rho\right]}{(s+1)^{\frac{1}{q}}}\left\{[|f'''(\frac{a+b}{2})|^q+|f'''(a)|^q]^{\frac{1}{q}}\right.$$

$$\left.+[|f'''(\frac{a+b}{2})|^q+|f'''(b)|^q]^{\frac{1}{q}}\right\}$$

.

Theorem 3.7. Let $f:[a,b]\to\boldsymbol{R}$ be a three times differentiable function on the interval $[a,b]$ where $a<b$. If $|f'''|^q$ is s-convex in the second sense on (a,b) and $q>1$, where $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality for generalized fractional integral operators holds:

$$|f'(x)\left\{\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(x-a)^{\rho}]-2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]}{(x-a)^2}-\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(b-x)^{\rho}]-2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]}{(x-b)^2}\right\}$$
$$+\frac{2[f(x)-f(a)]}{(x-a)^3}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]-\frac{2[f(x)-f(b)]}{(x-b)^3}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]$$
$$-f(x)\left\{\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]}{(x-a)^3}-\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]}{(x-b)^3}\right\}+\frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x^-;w}f\right)(a)}{(x-a)^{\lambda+3}}$$
$$+\frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x^+;w}f\right)(b)}{(b-x)^{\lambda+3}}|\le \mathcal{F}^{\sigma_{3,s}}_{\rho,\lambda+3}[w(x-a)^{\rho}]+\mathcal{F}^{\sigma_{4,s}}_{\rho,\lambda+3}[w(b-x)^{\rho}],$$

where

$$\sigma_{3,s}(k)=\sigma(k)\left(\frac{\rho k+\lambda}{3(\rho k+\lambda+3)}\right)^{1-\frac{1}{q}}[|f'''(x)|^q\frac{\rho k+\lambda}{(s+3)(\rho k+\lambda+s+3)}+|f'''(a)|^q(B(3,s+1)-B(\rho k+\lambda$$
$$+3,s+1))]^{\frac{1}{q}}$$

,

$$\sigma_{4,s}(k)=\sigma(k)\left(\frac{\rho k+\lambda}{3(\rho k+\lambda+3)}\right)^{1-\frac{1}{q}}[|f'''(x)|^q\frac{\rho k+\lambda}{(s+3)(\rho k+\lambda+s+3)}+|f'''(b)|^q(B(3,s+1)-B(\rho k+\lambda$$
$$+3,s+1))]^{\frac{1}{q}}$$

$\rho,\lambda>0, w\in \boldsymbol{R}$, $s\in(0,1]$ and $B(.,.)$ is the beta function of Euler.

Proof: It will be used again the Lemma 3.1, and properties of modulus and we have,

$$|I_1-I_3-(I_2-I_4)|\le$$
$$\le|f'(x)\left\{\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(x-a)^{\rho}]-2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]}{(x-a)^2}-\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(b-x)^{\rho}]-2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]}{(x-b)^2}\right\}$$
$$+\frac{2[f(x)-f(a)]}{(x-a)^3}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]-\frac{2[f(x)-f(b)]}{(x-b)^3}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]$$
$$-f(x)\left\{\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]}{(x-a)^3}-\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]}{(x-b)^3}\right\}+\frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x^-;w}f\right)(a)}{(x-a)^{\lambda+3}}$$
$$+\frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x^+;w}f\right)(b)}{(b-x)^{\lambda+3}}|$$
$$\le\int_0^1\left|t^2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]-t^{\lambda+2}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}t^{\rho}]\right||f'''(tx+(1-t)a)|$$
$$+\int_0^1|t^{\lambda+2}\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}t^{\rho}]-t^2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]||f'''(tx+(1-t)b)|dt\le$$
$$\le\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(x-a)^{\rho k}}{\Gamma(\rho k+\lambda+3)}\int_0^1\left(t^2-t^{\rho k+\lambda+2}\right)|f'''(tx+(1-t)a)|dt+$$
$$+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-x)^{\rho k}}{\Gamma(\rho k+\lambda+3)}\int_0^1\left(t^2-t^{\rho k+\lambda+2}\right)|f'''(tx+(1-t)b)|dt.$$

By power-mean inequality and s-convexity of $|f'''|^q$ in the second sense, we get the following inequality:

$$
\begin{aligned}
&|I_1 - I_3 - (I_2 - I_4)| \\
&\le \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (x-a)^{\rho k}}{\Gamma(\rho k+\lambda+3)} \left(\int_0^1 \left(t^2 - t^{\rho k+\lambda+2}\right) dt\right)^{1-\frac{1}{q}} \left(\int_0^1 \left(t^2 - t^{\rho k+\lambda+2}\right) |f'''(tx+(1-t)a)|^q \, dt\right)^{\frac{1}{q}} \\
&+ \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-x)^{\rho k}}{\Gamma(\rho k+\lambda+3)} \left(\int_0^1 \left(t^2 - t^{\rho k+\lambda+2}\right) dt\right)^{1-\frac{1}{q}} \left(\int_0^1 \left(t^2 - t^{\rho k+\lambda+2}\right) |f'''(tx+(1-t)b)|^q \, dt\right)^{\frac{1}{q}} \\
&\le \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (x-a)^{\rho k}}{\Gamma(\rho k+\lambda+3)} \left(\frac{1}{3} - \frac{1}{\rho k+\lambda+3}\right)^{1-\frac{1}{q}} \left(\int_0^1 \left(t^2 - t^{\rho k+\lambda+2}\right) [t^s |f'''(x)|^q + (1-t)^s |f'''(a)|^q] \, dt\right)^{\frac{1}{q}} \\
&+ \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-x)^{\rho k}}{\Gamma(\rho k+\lambda+3)} \left(\frac{1}{3} - \frac{1}{\rho k+\lambda+3}\right)^{1-\frac{1}{q}} \left(\int_0^1 \left(t^2 - t^{\rho k+\lambda+2}\right) [t^s |f'''(x)|^q + (1-t)^s |f'''(b)|^q] \, dt\right)^{\frac{1}{q}}
\end{aligned}
$$

and by calculus the proof is completed.

Corollary 3.8. If in previous theorem we take $s = 1$, then we have,

$$
\begin{aligned}
&\Bigg| f'(x) \left\{ \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(x-a)^{\rho}] - 2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}]}{(x-a)^2} - \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(b-x)^{\rho}] - 2\mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}]}{(x-b)^2} \right\} \\
&+ \frac{2[f(x)-f(a)]}{(x-a)^3} \mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(x-a)^{\rho}] - \frac{2[f(x)-f(b)]}{(x-b)^3} \mathcal{F}^{\sigma}_{\rho,\lambda+3}[w(b-x)^{\rho}] \\
&- f(x) \left\{ \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]}{(x-a)^3} - \frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]}{(x-b)^3} \right\} + \frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x^-;w} f\right)(a)}{(x-a)^{\lambda+3}} \\
&+ \frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x^+;w} f\right)(b)}{(b-x)^{\lambda+3}} \Bigg| \le \mathcal{F}^{\sigma_{3,1}}_{\rho,\lambda+3}[w(x-a)^{\rho}] + \mathcal{F}^{\sigma_{4,1}}_{\rho,\lambda+3}[w(b-x)^{\rho}],
\end{aligned}
$$

where

$$\sigma_{3,1}(k) = \sigma(k) \left(\frac{\rho k+\lambda}{3(\rho k+\lambda+3)}\right)^{1-\frac{1}{q}} \left[|f'''(x)|^q \frac{\rho k+\lambda}{4(\rho k+\lambda+4)} + |f'''(a)|^q (B(3,2) - B(\rho k+\lambda+3,2))\right]^{\frac{1}{q}}$$

$$\sigma_{4,1}(k) = \sigma(k) \left(\frac{\rho k+\lambda}{3(\rho k+\lambda+3)}\right)^{1-\frac{1}{q}} \left[|f'''(x)|^q \frac{\rho k+\lambda}{4(\rho k+\lambda+4)} + |f'''(b)|^q (B(3,2) - B(\rho k+\lambda+3,2))\right]^{\frac{1}{q}}$$

$$\rho, \lambda > 0, w \in \boldsymbol{R}.$$

Corollary 3.9. If in previous theorem we take $x = \frac{a+b}{2}$, then we have,

$$\left|\frac{16}{(b-a)^3}\left\{f\left(\frac{a+b}{2}\right)\left[2\mathcal{F}^{\sigma}_{\rho,\lambda+3}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]-\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]\right]-[f(a)+f(b)]\mathcal{F}^{\sigma}_{\rho,\lambda+3}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]\right\}+\frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\frac{a+b}{2}^{-};w}f\right)(a)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,\frac{a+b}{2}^{+};w}f\right)(b)}{\left(\frac{b-a}{2}\right)^{\lambda+3}}\right| \le \mathcal{F}^{\sigma_{3,s}}_{\rho,\lambda+3}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]+\mathcal{F}^{\sigma_{4,s}}_{\rho,\lambda+3}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right] \text{ where}$$

$$\sigma_{3,s}(k)=\sigma(k)\left(\frac{\rho k+\lambda}{3(\rho k+\lambda+3)}\right)^{1-\frac{1}{q}}\left[\left|f'''\left(\frac{a+b}{2}\right)\right|^{q}\frac{\rho k+\lambda}{(s+3)(\rho k+\lambda+s+3)}+|f'''(a)|^{q}(B(3,s+1)-B(\rho k+\lambda+3,s+1))\right]^{\frac{1}{q}}$$

$$\sigma_{4,s}(k)=\sigma(k)\left(\frac{\rho k+\lambda}{3(\rho k+\lambda+3)}\right)^{1-\frac{1}{q}}\left[\left|f'''\left(\frac{a+b}{2}\right)\right|^{q}\frac{\rho k+\lambda}{(s+3)(\rho k+\lambda+s+3)}+|f'''(b)|^{q}(B(3,s+1)-B(\rho k+\lambda+3,s+1))\right]^{\frac{1}{q}}$$

$$\rho,\lambda>0, w\in\boldsymbol{R}, s\in(0,1].$$

4. CONCLUSIONS

In this paper a new integral identity is established for three times differentiable mapping involving fractional integral operators in order to present new versions of Hermite-Hadamard type inequalities, involving the Riemann-Liouville fractional integral, for functions whose absolute values of third derivatives are s-convex in the second sense. Then several consequences in some special cases like when $s=1$ or when $x=\frac{a+b}{2}$ are presented. The particular case when $\lambda=\alpha,\ \sigma(0)=1, w=0$ or some combinations of these three special cases can be also analyzed as new consequences and applications. In addition, new versions of Hermite-Hadamard type inequalities, involving the Riemann-Liouville fractional integral, for functions whose absolute values of third derivatives are s-convex in the first sense could be established.

REFERENCES

[1] Hadamard, J., *J. Math. Pures Appl*., **58**, 171, 1893.

[2] Fejèr L., *Naturwise. Anz Ungar. Akad., Wiss*, **24**, 369, 1906.

[3] Dragomir, S.S., Fitzpatrick, S., *Demonstratio Math.* , **32**(4), 687, 1999.

[4] Dragomir, S.S., Pearce, C.E.M., *Selected Topics on Hermite-Hadamard Inequalities and Applications*, Science Direct Working Paper No S1574-0358(04)70845-X, 2003, available online: https://ssrn.com/abstract=3158351.

[5] Dahmani Z., *Int. J. Nonlinear Sci*., **9**(4), 493, 2010.

[6] Gorenflo, R., Mainardi, F., Fractional calculus: Integral and differential equations of fractional order. In Carpinteri, A., Mainardi, F. (Eds.), *Fractals and Fractional Calculus in Continuum Mechanics*, Springer Verlag, Wien and New York, 1997, pp. 223-276.

[7] Set, E., Sarikaya, M.Z., Ozdemir, M.E., Yildrim, H., *J. Appl. Math. Statis. Inform.*, **10**(2), 69, 2014.

[8] Set, E., Iscan, I., Sarikaya, M.Z., Ozdemir, M.E., *Appl. Math. Comput*., **259**, 875, 2015.

[9] Sarikaya, M.Z., Set, E., Yaldiz, H., Basak, N., *Math. Comput. Model*., **57**(9), 2403, 2013.

[10] Ciurdariu, L., *Int. J. of Math. Anal.*, **6**(33), 1623, 2012.
[11] Ciurdariu, L., *Int. J. of Math. Anal*., **6**(48), 2361, 2012.
[12] Tirtirau, L., *Appl. Math. Sci*., **14**(7), 337, 2020.
[13] Ozdemir, M.E., Ekinci, A., Akdemir, A.O., *TWMS J. Pure Appl. Math.*, **2**(10), 212, 2019.
[14] Ekinci, A., Akdemir, A.O., Ozdemir, M.E., *Turk. J. Sci*., **5**(3), 305, 2020.
[15] Kunt, M., Iscan, I., *Turk. J. Inequal*., **2**(1), 1, 2018.
[16] Breckner, W.W., *Pupl. Inst. Math*., **23**, 13, 1978.
[17] Toader, G., *J. Math. Inequal.*, **3**(3), 305, 2009.
[18] Set, E., Dragomir, S.S., Gozpinar, A., *Res. Rep. Coll. RGMIA*, **20**, 14, 2017.
[19] Raina, R.K., *East Asian Math. J*, **21**(2), 191, 2005.
[20] Agarwal, R.P., Luo, M.J., Raina, R.K., *Fasciculi Mathematici*, **204**, 5, 2016.
[21] Barsam, H., Ramezani, S.M., Sayyari, Y., *Afrika Matematika*, **32**, 1355, 2021.