ORIGINAL PAPER

k-BALANCING NUMBERS AND NEW GENERATING FUNCTIONS WITH SOME SPECIAL NUMBERS AND POLYNOMIALS

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Abstract. *In this paper, we consider some operators including symmetric functions. From those operators, we obtain the generating functions of* k *-balancing numbers with some special numbers and Chebyshev polynomials.*

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1. INTRODUCTION

Balancing numbers are the solutions of the Diophantine equation

$$1+2+3+...+(n+1)=(n+1)+(n+2)+...+(n+r),$$

where r is a balancer corresponds to a Balancing number n (see [1, 2]). Balancing numbers satisfy the recurrence relation

$$B_{n+1}=6B_n-B_{n-1},\ n\geq 1,$$

with initial values $B_0=0$ and $B_1=1$.

Balancing numbers are generalized in many ways, see examples [1, 3, 4]. One of the generalized balancing numbers called k -balancing numbers depending on one real parameter k , was recently introduced by Ray in [5]. The n^{th} k -balancing numbers $B_{k,n}$ are recursively defined by [6]

$$B_{k,0}=0,\ B_{k,1}=1 \text{ and } B_{k,n+1}=6kB_{k,n}-B_{k,n-1},\ k\geq 1. \tag{1.1}$$

The first few k -balancing numbers are

$$B_{k,0}=0,\ B_{k,1}=1, B_{k,2}=6k,\ B_{k,3}=36k^2-1, B_{k,4}=21k^3-12k,\ B_{k,5}=129k^4-108k^2+1.$$

Notice that, for $k=1$ in we obtain the sequence of balancing numbers. We, Also, have a second-order difference equation with an auxiliary equation $\alpha^2=6k\alpha-1,$ whose roots are $\alpha_1=3k+\sqrt{9k^2-1}$ and $\alpha_2=3k-\sqrt{9k^2-1}.$

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The author [2] gave some important identities for k -balancing numbers.

Lemma 1.1. For any integer $n \geq 1$, we have

$$\alpha_1^{n+2} = 6k\alpha_1^{n+1} - \alpha_1^n \text{ and } \alpha_2^{n+2} = 6k\alpha_2^{n+1} - \alpha_2^n.$$

Proof: Since α_1 and α_2 are roots of the equation $\alpha_1^2 = 6k\alpha_1 - 1$ and $\alpha_2^2 = 6k\alpha_2 - 1$. The desired results are obtained by multiplying α_1^n and α_2^n to both the equations respectively.

Lemma1.2. (Binet's formula). The closed form n^{th} k -balancing number is

$$B_{k,n} = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2},$$

where $\alpha_1 = 3k + \sqrt{9k^2 - 1}$ and $\alpha_2 = 3k - \sqrt{9k^2 - 1}$.

Proof: By induction, clearly, the result is true for $n = 0$ and $n = 1$. Assume that it is true for all i such that $0 \leq i \leq m+1$ for some positive integer m. Now by (1.1), we obtain:

$$\begin{aligned} B_{k,n+2} &= 6kB_{k,n+1} - B_{k,n} \\ &= 6k\frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2} - \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} \\ &= \frac{\alpha_1^n\left(6k\alpha_1 - 1\right) - \alpha_2^n\left(6k\alpha_2 - 1\right)}{\alpha_1 - \alpha_2} \\ &= \frac{\alpha_1^{n+2} - \alpha_2^{n+2}}{\alpha_1 - \alpha_2}, \end{aligned}$$

hence the desired result.

Let's recall the following theorem (Favard [7]).

Theorem 1.1. Let $\{P_n\}_{n\geq 0}$ be a monic polynomial sequence. Then $\{P_n\}_{n\geq 0}$ is orthogonal if and only if there exist two sequences of complex numbers $\{B_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$, such that $\gamma_n \neq 0$, $n \geq 1$ and satisfies the three-term recurrence relation

$$\begin{cases} P_0(x) = 1, \;\; P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \;\; n \geq 0. \end{cases}$$

Next, we recall some properties of the monic orthogonal Chebyshev polynomials that we will need in the sequel. Let $T_n(x)$ (resp. $U_n(x)$) be the monic Chebyshev polynomial of the first (resp. second) kind. These polynomials can be given by

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x),$$

with initial values $T_0(x) = 1$ and $T_1(x) = x$.

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x),$$

with $U_0(x) = 1$ and $U_1(x) = 2x.$

The aim of this paper is to determine some new generating functions related to k -balancing. The concept of generating functions has been studied by many authors in the literature (see examples [8-12]).

The further contents of this paper are as follows. Section S2 gives some definitions that we will need in the sequel. In section S3, we give the Theorems. As an illustration, we give, in S4 the applications of the obtained results. In S5, we finish the document with a conclusion.

2. DEFINITIONS AND PROPERTIES

In this section, we introduce a new symmetric function and we give some properties. We, also, give some more useful definitions from the literature which are used in the subsequent sections (for more details see [13, 14]).

We shall handle functions on different sets of indeterminates (called alphabets, though we shall mostly use commutative indeterminates for the moment). A symmetric function of an alphabet A is a function of the letters which is invariant under the permutation of the letters of A. Taking an extra indeterminate z , one has two fundamental series

$$\lambda_z(A) = \Pi_{a\in A}(1+za),\ \sigma_z(A) = \frac{1}{\Pi_{a\in A}(1-za)}.$$

The expansion of which gives the elementary symmetric functions $\Lambda_n(A)$ and the complete functions $S_n(A)$:

$$\lambda_z(A) = \sum_{n=0}^{+\infty}\Lambda_n(A)z^n,\ \sigma_z(A) = \sum_{n=0}^{+\infty}S_n(A)z^n.$$

Let us now start with the following definition.

Definition 2.1. Let A and B be any two alphabets, then we give $S_n(A-B)$ by the following form

$$\frac{\Pi_{b\in B}(1-zb)}{\Pi_{a\in A}(1-za)} = \sum_{n=0}^{+\infty} S_n(A-B)z^n = \sigma_z(A-B), \tag{2.1}$$

with the condition $S_n(A-B) = 0$ for $n < 0$.

Corollary 2.1. Taking $A = 0$ in (2.1) gives

$$\Pi_{b\in B}(1+zb) = \sum_{n=0}^{+\infty} S_n(-B)z = \lambda_z\left(-B\right).$$

Further, in the case $A = 0$ or $B = 0$, we have

$$\sum_{n=0}^{+\infty}S_n(A-B)z^n = \sigma_z(A)\times\lambda_z(-B).$$

Thus

$$S_n(A-B)=\sum_{k=0}^{n}S_{n-k}(A)S_k(-B) \text{ (see } [15]).$$

Definition 2.2. [13] Let g be any function on $\mathbb{R}^n$, then we consider the divided difference operator as the following form

$$\partial_{x_i x_{i+1}}(g)=\frac{g(x_1,\cdots,x_i,x_{i+1},\cdots x_n)-g(x_1,\cdots x_{i-1},x_{i+1},x_i,x_{i+2}\cdots x_n)}{x_i-x_{i+1}},$$

Definition 2.3. [14] Given an alphabet $E=\{e_1,e_2\}$, the symmetrizing operator $\delta^k_{e_1e_2}$ is defined by

$$\delta^k_{e_1e_2}(e_1^n)=\frac{e_1^{k+n}-e_2^{k+n}}{e_1-e_2}=S_{k+n-1}(e_1+e_2), \text{ for all } k\in N \text{ and } n\in N.$$

3. THEOREMS

In this section, we provide some theorems by using the symmetrizing operator. We now begin with the following theorem.

Theorem 3.1. Let the alphabets $A=\{a_1,a_2\}, B=\{b_1,b_2\}$ and $C=\{c_1,c_2\}$ be given, we have

$$\sum_{n=0}^{\infty}h_n(a_1,a_2)h_n(b_1,b_2)h_n(c_1,c_2)z^n=\frac{b_1b_2}{c_1-c_2}\times\frac{\Pi_1}{\Pi_2},$$

where

$$\Pi_1=\prod_{a\in A}(1-ab_1c_1z)\prod_{a\in A}(1-ab_2c_1z)\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)h_{n-2}(b_1,b_2)c_2^{n+1}z^n$$
$$-\prod_{a\in A}(1-ab_1c_2z)\prod_{a\in A}(1-ab_2c_2z)\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)h_{n-2}(b_1,b_2)c_1^{n+1}z^n.$$

$$\Pi_2=\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)b_1^nc_1^nz^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)b_2^nc_1^nz^n\right)$$
$$\times\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)b_1^nc_2^nz^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)b_2^nc_2^nz^n\right).$$

Proof: Let $\sum_{n=0}^{\infty}h_n(a_1,a_2)b_1^nc_1^nz^n$ and $\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)b_1^nc_1^nz^n$ be two sequences such that

$$\left(\sum_{n=0}^{\infty}h_n(a_1,a_2)b_1^nc_1^nz^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)b_1^nc_1^nz^n\right)=1$$

On one hand, since $g(b_1,c_1)=\sum_{n=0}^{\infty}h_n(a_1,a_2)b_1^nc_1^nz^n$, we have

$$\delta_{c_1c_2}\delta_{b_1b_2}g\left(b_1,c_1\right)=\delta_{c_1c_2}\left(\frac{\sum\limits_{n=0}^{\infty}h_n\left(a_1,a_2\right)b_1^{n+1}c_1^nz^n-\sum\limits_{n=0}^{\infty}h_n\left(a_1,a_2\right)b_2^{n+1}c_1^nz^n}{b_1-b_2}\right)$$

$$=\delta_{c_1c_2}\sum_{n=0}^{\infty}h_n\left(a_1,a_2\right)\frac{b_1^{n+1}-b_2^{n+1}}{b_1-b_2}c_1^nz^n$$

$$=\delta_{c_1c_2}\sum_{n=0}^{\infty}h_n\left(a_1,a_2\right)h_n\left(b_1,b_2\right)c_1^nz^n$$

$$=\frac{\sum\limits_{n=0}^{\infty}h_n\left(a_1,a_2\right)h_n\left(b_1,b_2\right)c_1^{n+1}z^n-\sum\limits_{n=0}^{\infty}h_n\left(a_1,a_2\right)h_n\left(b_1,b_2\right)c_2^{n+1}z^n}{c_1-c_2}$$

$$=\sum_{n=0}^{\infty}h_n\left(a_1,a_2\right)h_n\left(b_1,b_2\right)h_n\left(c_1,c_2\right)z^n$$

On the other hand, from $g(b_1,c_1)=\dfrac{1}{\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2)b_1^nc_1^nz^n}$, we get

$$\delta_{c_1c_2}\delta_{b_1b_2}g\left(b_1,c_1\right)=\delta_{c_1c_2}\delta_{b_1b_2}\left(\frac{1}{\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2)b_1^nc_1^nz^n}\right)$$

$$=\delta_{c_1c_2}\frac{b_1b_2\left(\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2)b_2^{n-1}c_1^nz^n-\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2)b_1^{n-1}c_1^nz^n\right)}{\left(b_1-b_2\right)\left(\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2)b_1^nc_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2)b_2^nc_1^nz^n\right)}.$$

Equivalently

$$\delta_{c_1c_2}\delta_{b_1b_2}g\left(b_1,c_1\right)=\delta_{c_1c_2}\left(\frac{-b_1b_2\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2)h_{n-2}(b_1,b_2)c_1^nz^n}{\left(\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2)b_1^nc_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2)b_2^nc_1^nz^n\right)}\right)$$

$$=\frac{b_1b_2}{c_1-c_2}\left(\frac{\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2)h_{n-2}(b_1,b_2)c_2^{n+1}z^n}{\left(\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2)b_1^nc_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2)b_2^nc_1^nz^n\right)}\right.$$
$$\left.-\frac{\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2)h_{n-2}(b_1,b_2)c_1^{n+1}z^n}{\left(\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2)b_1^nc_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}(-1)^ne_n(a_1,a_2)b_2^nc_1^nz^n\right)}\right).$$

Using the fact that $\sum\limits_{n=0}^{\infty}\left(-1\right)^ne_n\left(a_1,a_2\right)b_1^nc_1^nz^n=\prod\limits_{a\in A}\left(1-ab_1c_1z\right)$, then

$$\delta_{c_1c_2}\delta_{b_1b_2}g\left(b_1,c_1\right)=\frac{b_1b_2}{c_1-c_2}\frac{\begin{pmatrix}\prod_{a\in A}\left(1-ab_1c_1z\right)\prod_{a\in A}\left(1-ab_2c_1z\right)\sum_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)h_{n-2}\left(b_1,b_2\right)c_2^{n+1}z^n\\-\prod_{a\in A}\left(1-ab_1c_2z\right)\prod_{a\in A}\left(1-ab_2c_2z\right)\sum_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)h_{n-2}\left(b_1,b_2\right)c_1^{n+1}z^n\end{pmatrix}}{\begin{array}{c}\left(\sum_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_1^nc_1^nz^n\right)\left(\sum_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_2^nc_1^nz^n\right)\\ \times\left(\sum_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_1^nc_2^nz^n\right)\left(\sum_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_2^nc_2^nz^n\right)\end{array}}.$$

This completes the proof.

Theorem 3.2. Given three alphabets $A=\{a_1,a_2\},B=\{b_1,b_2\}$ and $C=\{c_1,c_2\}$ we have

$$\sum_{n=0}^{\infty}h_n\left(a_1,a_2\right)h_{n-1}\left(b_1,b_2\right)h_{n-1}\left(c_1,c_2\right)z^n=\frac{\begin{pmatrix}\prod_{a\in A}\left(1-ab_1c_1z\right)\prod_{a\in A}\left(1-ab_2c_1z\right)\times\sum_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)h_{n-1}\left(b_1,b_2\right)c_2^nz^n\\-\prod_{a\in A}\left(1-ab_1c_2z\right)\prod_{a\in A}\left(1-ab_2c_2z\right)\times\sum_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)h_{n-1}\left(b_1,b_2\right)c_1^nz^n\end{pmatrix}}{\left(c_1-c_2\right)\begin{pmatrix}\left(\sum_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_1^nc_1^nz^n\right)\left(\sum_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_2^nc_1^nz^n\right)\\ \left(\sum_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_1^nc_2^nz^n\right)\left(\sum_{n=0}^{\infty}\left(-1\right)^n e_n\left(a_1,a_2\right)b_2^nc_2^nz^n\right)\end{pmatrix}}$$

$$=\frac{N_1}{D},$$

where

$$\begin{aligned}N_1 = {} & (a_1+a_2)z - a_1a_2(b_1+b_2)(c_1+c_2)z^2 + c_1c_2b_1b_2(a_1+a_2)(2a_1a_2-(a_1+a_2)^2)z^3\\ & + a_1a_2b_1b_2c_1c_2(a_1+a_2)^2(b_1+b_2)(c_1+c_2)z^4\\ & - a_1^2a_2^2b_1b_2c_1c_2(a_1+a_2)(b_1b_2(c_1+c_2)^2+c_1c_2(b_1+b_2)^2-b_1b_2c_1c_2)z^5\\ & + a_1^3a_2^3b_1^2b_2^2c_1^2c_2^2(c_1+c_2)(b_1+b_2)z^6.\end{aligned}$$

$$\begin{aligned}D = {} & 1-(a_1+a_2)(b_1+b_2)(c_1+c_2)z + b_1b_2(a_1+a_2)^2(c_1+c_2)^2\\ & + ((b_1+b_2)^2-2b_1b_2)((a_1+a_2)^2c_1c_2-2a_1a_2c_1c_2+a_1a_2(c_1+c_2)^2)z^2\\ & - (a_1+a_2)(b_1+b_2)(c_1+c_2)(b_1b_2c_1c_2(a_1+a_2)^2+b_1b_2a_1a_2(c_1+c_2)^2\\ & + a_1a_2c_1c_2(b_1+b_2)^2-5a_1a_2b_1b_2c_1c_2)z^3 + a_1^2a_2^2c_1^2c_2^2(b_1+b_2)^4\\ & + c_1^2c_2^2b_1^2b_2^2(a_1+a_2)^4 + a_1^2a_2^2b_1^2b_2^2(c_1+c_2)^4\\ & - a_1a_2b_1b_2c_1c_2(4b_1b_2c_1c_2(a_1+a_2)^2+4a_1a_2c_1c_2(b_1+b_2)^2\\ & + 4a_1a_2b_1b_2(c_1+c_2)^2-(a_1+a_2)^2(b_1+b_2)^2(c_1+c_2)^2)\\ & + 6a_1^2a_2^2b_1^2b_2^2c_1^2c_2^2)z^4\\ & - a_1a_2b_1b_2c_1c_2(a_1+a_2)(b_1+b_2)(c_1+c_2)(a_1a_2c_1c_2(b_1+b_2)^2\\ & + b_1b_2c_1c_2(a_1+a_2)^2+b_1b_2a_1a_2(c_1+c_2)^2-5a_1a_2b_1b_2c_1c_2)z^5\\ & + (a_1^2a_2^2b_1^3b_2^3c_1^2c_2^2(a_1+a_2)^2(c_1+c_2)^2+a_1^2a_2^2b_1^2b_2^2c_1^2c_2^2((b_1+b_2)^2\\ & - 2b_1b_2)(c_1c_2(a_1+a_2)^2-2a_1a_2c_1c_2+a_1a_2(c_1+c_2)^2))z^6\\ & - a_1^3a_2^3b_1^3b_2^3c_1^3c_2^3(a_1+a_2)(c_1+c_2)(b_1+b_2)z^7 + a_1^4a_2^4b_1^4b_2^4c_1^4c_2^4z^8\end{aligned}$$

Proof: By using the divided difference operator $\partial_{c_1,c_2}\partial_{b_1,b_2}$, the proof is similar to the proof of Theorem 3.1.

4. APPLICATION OF THEOREMS

We now consider the previous theorems in order to derive new generating functions of k -Balancing numbers.

Theorem 4.1. [16] Given three alphabets $A=\{a_1,a_2\}, B=\{b_1,b_2\}$ and $C=\{c_1,c_2\}$ we have

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,a_2)\,h_{n-1}(b_1,b_2)\,h_{n-1}(c_1,c_2)\,z^n=\frac{N_2}{D},$$

where

$$N_2=b_1b_2-\left(\prod_{a\in A}(1-ab_1c_1z)\prod_{a\in A}(1-ab_2c_1z)\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)\,h_{n-2}(b_1,b_2)\,c_2^{n+1}z^{n+1}\right)$$
$$-b_1b_2\left(\prod_{a\in A}(1-ab_1c_2z)\prod_{a\in A}(1-ab_2c_2z)\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2)\,h_{n-2}(b_1,b_2)\,c_1^{n+1}z^{n+1}\right)$$
$$=z-\left(a_1a_2c_1c_2(b_1+b_2)^2+a_1a_2b_1b_2(c_1+c_2)^2+b_1b_2c_1c_2(a_1+a_2)^2-3a_1a_2b_1b_2c_1c_2\right)z^3$$
$$+2a_1a_2b_1b_2c_1c_2(a_1+a_2)(b_1+b_2)(c_1+c_2)z^4-(a_1^2a_2^2b_1b_2c_1^2c_2^2(b_1+b_2)^2+$$
$$+a_1^2a_2^2b_1^2b_2^2c_1c_2(c_1+c_2)^2+a_1a_2b_1^2b_2^2c_1^2c_2^2(a_1+a_2)^2-3a_1^2a_2^2b_1^2b_2^2c_1^2c_2^2)z^5+a_1^3a_2^3b_1^3b_2^3c_1^3c_2^3z^7.$$

Case 1. Replacing a_2,b_2,c_2, respectively, by $-a_2,-b_2,-c_2$, and taking $a_1a_2=b_1b_2=c_1c_2=-1$ and $a_1-a_2=b_1-b_2=c_1-c_2=6k$ in Theorem 4.1, we have the following result.

Theorem 4.2. For $n\in\mathrm{N}$, the generating function of the cubes of k -balancing numbers is given by

$$\sum_{n=0}^{\infty} B_{k,n}^3 z^n=\frac{N_{B_{k,n}B_{k,n}B_{k,n}}}{D_{B_{k,n}B_{k,n}B_{k,n}}},$$

with

$$N_{B_{k,n}B_{k,n}B_{k,n}}=z-\left(108k^2-3\right)z^3+432k^3z^4-\left(108k^2-3\right)z^5+z^7,$$

$$D_{B_{k,n}B_{k,n}B_{k,n}}=1-216k^3z-4\left(270k^2+648k^4+1\right)z^2$$
$$-216k^3\left(108k^2-5\right)z^3+6\left(648k^4+7776k^6-72k^2+1\right)z^4$$
$$+216k^3\left(108k^2-5\right)z^5-4\left(972k^4-54k^2+1\right)z^6-216k^3z^7+z^8.$$

In the following, we will present some cases.

Case 2. By replacing a_2 by $(-a_2)$, b_2 by $(-b_2)$ and c_2 by $(-c_2)$, taking $a_1a_2=b_1b_2=-1$, $c_1c_2=1$, $a_1-a_2=b_1-b_2=6k$ and $c_1-c_2=k$ in Theorem 4.1, we obtain the following theorem.

Theorem 4.2. For $n\in\mathrm{N}$, a new generating function of the product for squares of k - balancing numbers and k -Fibonacci numbers is given by

$$\sum_{n=0}^{\infty} B_{k,n}^2 F_{k,n}\ z^n = \frac{N_{B_{k,n}^2 F_{k,n}}}{D_{B_{k,n}^2 F_{k,n}}},$$

with

$$N_{B_{k,n}^2 F_{k,n}} = z - \left(71k^2+3\right)z^3 - 72k^3 z^4 - \left(71k^2-3\right)z^5 - z^7,$$

$$\begin{aligned} D_{B_{k,n}^2 F_{k,n}} &= 1 - 36k^3 z + \left(36k^4 - \left(36k^2-2\right)\left(35k^2-2\right)\right)z^2 \\ &+ 36k^3\left(71k^2-5\right)z^3 + \left(2593k^4 - 292k^2 + 1296k^6 + 6\right)z^4 \\ &- 36k^3\left(71k^2-5\right)z^5 + \left(36k^4 + \left(36k^2-2\right)\left(37k^2-2\right)\right)z^6 + 36k^3 z^7 + z^8. \end{aligned}$$

Case 3. By replacing a_2 by $(-a_2)$, b_2 by $(-b_2)$ and c_2 by $(-c_2)$, taking $a_1 a_2 = b_1 b_2 = -1$, $c_1 c_2 = k$, $a_1 - a_2 = b_1 - b_2 = 6k$ and $c_1 - c_2 = 2$ in Theorem 4.1, we have the following theorem.

Theorem 4.3. For $n \in \mathrm{N}$, a new generating function of the product for squares of k -Balancing numbers and k -Pell numbers is given by

$$\sum_{n=0}^{\infty} B_{k,n}^2 P_{k,n}\ z^n = \frac{N_{B_{k,n}^2 P_{k,n}}}{D_{B_{k,n}^2 P_{k,n}}},$$

with

$$N_{B_{k,n}^2 P_{k,n}} = z + \left(72k^3 - 3k - 4\right)z^3 - 144k^3 z^4 - \left(72k^3 - 3k - 4\right)kz^5 - k^3 z^7,$$

$$\begin{aligned} D_{B_{k,n}^2 P_{k,n}} &= 1 - 72k^2 z + 4\left(k - 36k^3 + 324k^5 + 2\right)z^2 \\ &- 72k^2\left(5k - 72k^3 + 4\right)z^3 + \left(2592k^6 - 288k^4 - 5184k^5 + 6k^2 + 16k + 16\right)z^4 \\ &+ 72k^3\left(5k - 72k^3 + 4\right)z^5 - 4k^2\left(k - 72k^2 - 36k^3 + 324k^5 + 2\right)z^6 + 72k^5 z^7 + k^4 z^8. \end{aligned}$$

Case 4. By replacing a_2 by $(-a_2)$, b_2 by $(-b_2)$ and c_2 by $(-c_2)$, taking $a_1 a_2 = b_1 b_2 = -1$, $c_1 c_2 = 2$, $a_1 - a_2 = b_1 - b_2 = 6k$ and $c_1 - c_2 = k$ in theorem 4.1, we have the following theorem.

Theorem 4.4. For $n \in \mathrm{N}$, a new generating function of the product for squares of k -Balancing numbers and k -Jacobsthal numbers is given by

$$\sum_{n=0}^{\infty} B_{k,n}^2 J_{k,n}\ z^n = \frac{N_{B_{k,n}^2 J_{k,n}}}{D_{B_{k,n}^2 J_{k,n}}},$$

with

$$N_{B_{k,n}^2 J_{k,n}} = z - \left(6 - 143k^2\right)z^3 - 144k^3 z^4 - \left(286k^2 - 12\right)z^5 - 8z^7,$$

$$\begin{aligned} D_{B_{k,n}^2 J_{k,n}} &= 1 - 36k^3 z - 2\left(20k^2 - 1\right)\left(63k^2 - 4\right)z^2 + 36k^2\left(143k^2 - 10\right)z^3 \\ &- \left(1144k^2 - 10369k^4 + 2592k^6 - 24\right)z^4 + 72k^3\left(143k^2 - 10\right)z^5 \\ &- 8\left(1278k^4 - 161k^2 + 4\right)z^6 + 288k^3 z^7 + 16z^8. \end{aligned}$$

Case 5. By replacing a_2 by $(-a_2)$, b_2 by $(-b_2)$ and c_2 by $(-c_2)$, taking $a_1a_2 = b_1b_2 = -1$, $c_1c_2 = -2$, $a_1 - a_2 = b_1 - b_2 = 6k$ and $c_1 - c_2 = 3k$ in theorem 4.1, we have the following theorem.

Theorem 4.5. For $n \in \mathrm{N}$, a new generating function of the product for squares of k-Balancing numbers and k-Mersenne numbers is given by

$$\sum_{n=0}^{\infty} B_{k,n}^2 M_{k,n}\ z^n = \frac{N_{B_{k,n}^2 M_{k,n}}}{D_{B_{k,n}^2 M_{k,n}}},$$

with

$$N_{B_{k,n}^2 M_{k,n}} = z - \left(135k^2 - 6\right)z^3 - 432k^3 z^4 - \left(306k^2 - 12\right)z^5 + 8z^7,$$

$$\begin{aligned} D_{B_{k,n}^2 M_{k,n}} &= 1 - 108k^3 z + \left(324k^4 + \left(36k^2 - 2\right)\left(81k^2 - 4\right)\right)z^2 - 108k^3\left(153k^2 - 10\right)z^3 \\ &+ \left(10449k^4 - 2\left(612k^2 - 11664k^6\right) + 24\right)z^4 + 216k^3\left(-135k^2 + 10\right)z^5 \\ &+ 8\left(1620k^4 - 153k^2 + 4\right)z^6 - 864k^3 z^7 + 16z^8. \end{aligned}$$

Case 6. By replacing a_2 by $(-a_2)$, b_2 by $(-b_2)$ and c_2 by $(-c_2)$, taking $a_1a_2 = -1$, $b_1b_2 = c_1c_2 = 1$, $a_1 - a_2 = 6k$, $b_1 - b_2 = c_1 - c_2 = k$ in theorem 4.1, we have the following theorem.

Theorem 4.6. For $n \in \mathrm{N}$, a new generating function of the product of k-Balancing numbers and squares of k-Fibonacci numbers is given by

$$\sum_{n=0}^{\infty} B_{k,n} F_{k,n}^2\ z^n = \frac{N_{B_{k,n} F_{k,n}^2}}{D_{B_{k,n} F_{k,n}^2}},$$

with

$$N_{B_{k,n} F_{k,n}^2} = z + \left(34k^2 - 3\right)z^3 + 12k^3 z^4 - \left(34k^2 - 3\right)z^5 + z^7,$$

$$\begin{aligned} D_{B_{k,n} F_{k,n}^2} &= 1 - 6k^3 z + \left(-36k^4 + \left(k^2 + 2\right)\left(35k^2 + 2\right)\right)z^2 - 6k^3\left(34k^2 - 5\right)z^3 \\ &+ \left(1298k^4 - \left(136k^2 - 36k^6\right) + 6\right)z^4 - 6k^3\left(34k^2 - 5\right)z^5 + \left(36k^4 + \left(k^2 + 2\right)\left(35k^2 - 2\right)\right)z^6 - 6k^3 z^7 + z^8. \end{aligned}$$

Case 7. By replacing a_2 by $(-a_2)$, b_2 by $(-b_2)$ and c_2 by $(-c_2)$, taking $a_1a_2 = -1$, $b_1b_2 = c_1c_2 = k$, $a_1 - a_2 = 6k$, $b_1 - b_2 = c_1 - c_2 = 2$ in Theorem 4.1, we have the following theorem.

Theorem 4.7. For $n \in \mathrm{N}$, a new generating function of the product of k-balancing numbers and squares of k-Pell numbers is given by

$$\sum_{n=0}^{\infty} B_{k,n} P_{k,n}^2\ z^n = \frac{N_{B_{k,n} P_{k,n}^2}}{D_{B_{k,n} P_{k,n}^2}},$$

with

$$N_{B_{k,n} P_{k,n}^2} = z - \left(-8k - 3k^2 + 36k^4\right)z^3 - 48k^3 z^4 - \left(36k^6 - 8k^3 - 3k^4\right)z^5 + k^6 z^7,$$

$$\begin{aligned}D_{B_{k,n}P_{k,n}^2} &= 1-24kz+\left(144k^3+(4+2k)\left(36k^3-4-2k\right)\right)z^2 \\ &-24k^2\left(36k^3-5k-8\right)z^3+\left(32k^2+1296k^8-k^2\left(-32k-576k^2+144k^4\right)+6k^4\right)z^4 \\ &+24k^3\left(8k+5k^2+36k^4\right)z^5-\left(144k^7+k^4(4+2k)\left(36k^3-2k-4\right)\right)z^6-24k^7z^7+k^8z^8.\end{aligned}$$

Case 8. By replacing a_2 by $(-a_2)$, b_2 by $(-b_2)$ and c_2 by $(-c_2)$, taking $a_1a_2=-1$, $b_1b_2=c_1c_2=2$, $a_1-a_2=6k$, $b_1-b_2=c_1-c_2=k$ in Theorem 4.1, we have the following theorem.

Theorem 4.8. For $n\in\mathbb{N}$, a new generating function of the product of k -Balancing numbers and squares of k -Jacobsthal numbers is given by

$$\sum_{n=0}^{\infty}B_{k,n}J_{k,n}^2\ z^n=\frac{N_{B_{k,n}J_{k,n}^2}}{D_{B_{k,n}J_{k,n}^2}},$$

with

$$N_{B_{k,n}J_{k,n}^2}=z-\left(140k^2-12\right)z^3+48k^3z^4-\left(560k^2-48\right)z^5+64z^7,$$

$$\begin{aligned}D_{B_{k,n}J_{k,n}^2} &= 1-6k^3z+\left(-72k^4+\left(k^4+4\right)\left(-71k^2+4\right)\right)z^2-6k^3\left(140k^2-20\right)z^3 \\ &+\left(20744k^4-4\left(560k^2-36k^6\right)+96\right)z^4-24k^3\left(140k^2-20\right)z^5 \\ &-\left(1152k^4+16\left(k^2+4\right)\left(71k^2-4\right)\right)z^6-384k^3z^7+256z^8.\end{aligned}$$

Case 9. By replacing a_2 by $(-a_2)$, b_2 by $(-b_2)$ and c_2 by $(-c_2)$, taking $a_1a_2=-1$, $b_1b_2=c_1c_2=-2$, $a_1-a_2=6k$, $b_1-b_2=c_1-c_2=3k$ in Theorem 4.1, we have the following theorem.

Theorem 4.9. For $n\in\mathbb{N}$, a new generating function of the product of k -balancing numbers and squares of k -Mersenne numbers is given by

$$\sum_{n=0}^{\infty}B_{k,n}M_{k,n}^2\ z^n=\frac{N_{B_{k,n}M_{k,n}^2}}{D_{B_{k,n}M_{k,n}^2}},$$

with

$$N_{B_{k,n}M_{k,n}^2}=z-\left(180k^2-12\right)z^3+432k^3z^4-\left(720k^2-48\right)z^5+64z^7,$$

$$\begin{aligned}D_{B_{k,n}M_{k,n}^2} &= 1-54k^3z+\left(-648k^4+\left(9k^2-4\right)\left(4-81k^2\right)\right)z^2-54k^3\left(180k^2-20\right)z^3 \\ &+\left(2138k^4-2880k^2+11664k^6+96\right)z^4-216k^3\left(180k^2-20\right)z^5 \\ &+\left(10368k^4+16\left(9k^2-4\right)\left(81k^2-4\right)\right)z^6-3456k^3z^7+256z^8.\end{aligned}$$

Case 10. By replacing a_1 by $(2a_1)$, a_2 by $(-2a_2)$, b_2 by $(-b_2)$ and c_2 by $(-c_2)$, taking $a_1-a_2=x$, $b_1-b_2=c_1-c_2=6k$, $a_1a_2=-\frac{1}{4}$, $b_1b_2=c_1c_2=-1$ in theorem 3.2, we have the following theorem.

Theorem 4.10. For $n \in \mathrm{N}$, a new generating function of the product of Chebyshev polynomials of the second kind and squares of k -balancing numbers is given by

$$\sum_{n=0}^{\infty} U_n(x) B_{k,n}^2 \ z^n = \frac{N_{U_n(x)B_{k,n}^2}}{D_{U_n(x)B_{k,n}^2}},$$

with

$$N_{U_n(x)B_{k,n}^2} = 2xz - 36k^2z^2 - 2x(2+4x^2)z^3 + 144k^2x^2z^4 - 2x(72k^2+1)z^5 + 36k^2z^6,$$

$$\begin{aligned} D_{U_n(x)B_{k,n}^2} &= 1 - 72k^2xz + (144k^2x^2 + (36k^2-2)(4x^2+36k^2-2))z^2 \\ &- 72k^2x(4x^2+72k^2-5)z^3 + (2592k^4+16x^4+16x^2-288k^2+5184k^4x^2+6)z^4 \\ &- 72k^2x(72k^2+4x^2-5)z^5 + \left(144k^2x^2 - \left(36k^2-2\right)(4x^2+36k^2-2)\right)z^6 - 72k^2xz^7 + z^8. \end{aligned}$$

Theorem 4.11. For $n \in \mathrm{N}$, a new generating function of the product of Chebyshev polynomials of the first kind and squares of k -balancing numbers is given by

$$\sum_{n=0}^{\infty} T_n(x) B_{k,n}^2 \ z^n = \frac{N_{T_n(x)B_{k,n}^2}}{D_{T_n(x)B_{k,n}^2}},$$

with

$$N_{T_n(x)B_{k,n}^2} = xz - 36k^2z^2 + x\left(72k^2-4x^2-7\right)z^3 + x\left(-72k^2+4x^2-5\right)z^5 + 36k^2z^6 - xz^7,$$

and

$$D_{T_n(x)B_{k,n}^2} = D_{U_n(x)B_{k,n}^2}.$$

Proof: We have

$$\begin{aligned} \sum_{n=0}^{+\infty} T_n\left(x\right) B_{k,n}^2 \ z^n &= \sum_{n=0}^{+\infty} \left[h_n\left(2a_1 + \left[-2a_2\right]\right) - xh_{n-1}\left(2a_1 + \left[-2a_2\right]\right)\right] B_{k,n}^2 z^n \\ &= \sum_{n=0}^{+\infty} h_n\left(2a_1 + \left[-2a_2\right]\right) B_{k,n}^2 z^n - x\sum_{n=0}^{+\infty} h_{n-1}\left(2a_1 + \left[-2a_2\right]\right) B_{k,n}^2 z^n \\ &= \sum_{n=0}^{+\infty} U_n(x) B_{k,n}^2 z^n - x\sum_{n=0}^{+\infty} h_{n-1}\left(2a_1 + \left[-2a_2\right]\right) B_{k,n}^2 z^n. \end{aligned}$$

Replacing a_1 by $\left(2a_1\right)$, a_2 by $\left(-2a_2\right)$, b_2 by $\left(-b_2\right)$ and c_2 by $\left(-c_2\right)$, taking $a_1 - a_2 = x$, $b_1 - b_2 = c_1 - c_2 = 6k$, $a_1a_2 = -\dfrac{1}{4}$, $b_1b_2 = c_1c_2 = -1$ in theorem 4.1 , we have

$$\sum_{n=0}^{\infty} T_n(x) B_{k,n}^2 \ z^n = \frac{N_{U_n(x)B_{k,n}^2} - xz + x\left(72k^2+4x^2-3\right)z^3 - 144k^2x^2z^4 + x\left(72k^2+4x^2-3\right)z^5 - xz^7}{D_{T_n(x)B_{k,n}^2}},$$

with $D_{T_n(x)B_{k,n}^2} = D_{U_n(x)B_{k,n}^2}$. Then, we obtain

$$\sum_{n=0}^{\infty} T_n(x) B_{k,n}^2 \, z^n = \frac{N_{U_n(x)B_{k,n}^2}}{D_{U_n(x)B_{k,n}^2}} - x \times \frac{z - \left(72k^2 + 4x^2 - 3\right)z^3 + 144k^2 x z^4 - \left(72k^2 + 4x^2 - 3\right)z^5 + z^7}{D_{U_n(x)B_{k,n}^2}}$$

$$= \frac{N_{T_n(x)B_{k,n}^2}}{D_{T_n(x)B_{k,n}^2}},$$

with

$$N_{T_n(x)B_{k,n}^2} = xz - 36k^2 z^2 + x\left(72k^2 - 4x^2 - 7\right)z^3 + x\left(-72k^2 + 4x^2 - 5\right)z^5 + 36k^2 z^6 - xz^7.$$

5. CONCLUSION

In this paper, we have derived new theorems in order to determine generating functions of the product of k -balancing numbers and some numbers and polynomials. The derived theorems and corollaries are based on symmetric functions and products of these numbers and polynomials.

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