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UTILIZING PSEUDOINVERSE MATRIX TO SOLVE AN NONLINEAR BOUNDARY VALUE PROBLEM

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Abstract. *In this paper, by using the concepts of pseudoinverse matrix combining with coincidence degree theory due to Mawhin and constructing suitable operators, we study the existence of solutions for a nonlinear higher-order boundary value problem at resonance case in* $\mathbb{R}^n$ *. An illustrative example is presented at the end of the paper to illustrate the validity of our results*

Keywords: *pseudoinverse matrix; Mawhin's coincidence degree theory; nonlinear boundary value problem.*

1. INTRODUCTION

The study of different kinds of matrices (for example, matrix pencils, generalized inverse, or Moore- Penrose inverse matrix (pseudoinverse)) are by now a very thoughtful subject, with the notion of pseudospectrum and pseudoinverse playing a key role in the theory (for example, perturbation theory, generalized eigenvalue problems theory), see [1-7].

The pseudospectrum and pseudoinverse (generalized inverse) of bounded linear operators or bounded nonlinear operators on a Banach space can be split into subsets in many different ways, depending on the purpose one has in mind. We refer to [8-10].

The concept of a generalized inverse seems to have been first mentioned in print in 1903 by Fredholm [11], where a particular generalized inverse (called by him "pseudoinverse") of an integral operator, he also studied the generalized inverse of differential linear operator. Generalized inverses of differential operators, already implicit in Hilbert's discussion in 1904 of generalized Green functions, [12], were consequently studied by numerous authors. For a history of this subject see the excellent survey by Reid [13].

Generalized inverses of differential and integral operators thus antedated the generalized inverses of matrices, whose existence was first noted by E. H. Moore, who defined a unique inverse (called by him the "general reciprocal") for every finite matrix (square or rectangular). In 1955 Penrose [14] sharpened and extended Bjerhammar's results on linear systems, and showed that Moore's inverse, for a given matrix M, is the unique matrix A^+ called Moore-Penrose pseudoinverse matrix of A, meaning the matrix satisfying

$(i) A^+AA^+ = A^+$,

$(ii) AA^+A = A$,

$(iii) AA^+$ is a orthogonal projection on ImA,

$(iv) I - A^+A$ is orthogonal projection on $\ker A$.

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In this paper, we use the concepts of pseudoinverse matrix (Moore- Penrose inverse matrix) combinig with Mawhin's coincidence degree theory [15] to solving the following nonlinear boundary value problem

$$u^{(n)}(t) = f\left(t, u(t), u'(t), u''(t), \dots, u^{(n-1)}(t)\right), t \in (0,1), \tag{1.1}$$

$$\alpha_i u^{(i-1)}(0) + \beta_i u^{(i-1)}(1) = \gamma_i \int_0^1 u^{(i-1)}(s)ds\,,\ i = 1,2, \dots, n, \tag{1.2}$$

where $f: [0,1] \times \mathbb{R}^{n-1} \to \mathbb{R}$ is a continuous function, and α_i, β_i and γ_i, $i = 1,2, \dots, n$ are real constants. This work concerns a kind of higher-order differential equation which can be written in abstract form as $Lu = Nu$uNu, where L is a linear Fredholm operator of index zero, and N is a nonlinear operator. It is well known that if the kernel of the linear part contains only zero, the corresponding boundary value problem is called non-resonant. In this case, $L$is invertible, the equation can be reduced to a fixed point problem for the $L^{-1}N$ operator. Otherwise, if L is a non-invertible, i.e., $\ker L \geq 1$. We establish the existence results for the boundary problem at resonance when dimKerL enabled to take value arbitrarily. However, it is seem s that the construction a such projections is difficult when $\dim\ker L$ is large. This lead to use the generalized inverse (Moore-Penrose pseudoinverse matrix).

The theory of the boundary value problems with integral boundary conditions arises in different areas of applied mathematics and physics. For example, heat conduction, chimical engineering, underground watetr flow, thermo-elasticity and plasma physics can be reduced to the nonlocal problems with integral boundary conditions. Recently, several authors have studied nonlocal boundary value problems at resonance, see [5, 16-18] and the reference therein. In the papers mentioned above, the coincidence degree theory of Mawhin was applied to establish existence theorems. The reason is that resonant problems are rather complicated due to the non-invertibility of $L$and when the dimension of KerL is large. Non-invertibility leads to the difficulty of constructing a suitable continuous projection on a complement of ImL.

The rest of this articles is organized as follow. In Section 2, we provied some results regarding Mawhin's coincidence degree theory and several important lemmas which are motivation for obtaning our main results. In Section 3, we state and prove the main theorem of our problem and we give an example to illustrate our results.

2. PRELIMINARIES

We first recall some notation and an abstract existence results of coincidence degree theory due to Mawhin [4, 10, 11]. Let X, Z be two real Banach spaces.

Definition 2.1 *A linear operator $L: domL \subset X \to Z$ is called to be a Fredholm operator provided that*

$(i) KerL$ is finite dimensional,

$(ii) ImL$ is closed and has finite codimension.

In addition the Fredholm index of L is defined by the integer number

$$indL = dimKerL - codimImL.$$

From Definition 2.1, it follows that if L is a Fredholm operator, then there exist continuous projections $P: X \to X$ and $Q: Z \to Z$ such that

$$ImP = KerL, KerQ = ImL \; and X = KerL \oplus KerP, Z = ImL \oplus ImQ.$$

Further, the restriction of L on $domL \cap KerP$ that is $L_P: domL \cap KerP \to ImL$ is invertible. We denote by K_P the inverse of L_P and by $K_{P,Q} = K_P(I - Q)$ the generalized inverse of L. Moreover, if $indL = 0$, that is ImQ and $KerL$ are isomorphic, for isomorphism $J: ImQ \to KerL$, the operator $JQ + K_{P,Q}: Z \to domL$ is isomorphic and for all $u \in domL$ we have

$$\left(JQ + K_{P,Q}\right)^{-1} u = (L + J^{-1}P)u$$

for every isomorphism $J^{-1}: KerL \to ImQ$.

Hence, following Mawhin's equivalent theorem, $u \in \overline{\Omega}$ is solution to equation $Lu = Nu$ if and only if it is fixed point of Mawhin's operator

$$\Phi = P + \left(JQ + K_{P,Q}\right)N$$

where Ω is an given open bounded subset of Xsuch that $domL \cap \Omega \neq \phi$.

Definition 2.2 *Let L be a Fredholm operator of index zero. The operator $N : X \to Z$ is said to be $L - compact$ in $\overline{\Omega}$ provided that*

(i) the map $QN: \overline{\Omega} \to Z$ is cotinuous $QN\left(\overline{\Omega}\right)$ is bounded in Z,

(ii) the map $K_{P,Q}N: \overline{\Omega} \to X$ is completely continuous.

In addition, we say that N is L −completely continuous if it is L −compact on every bounded set in X.

We will formulate the boundary value problem $(1.1) - (1.2)$ as $Lu = Nu$ where L and N are appropriate operators. To obtain our existence results we use the following fixed point theorem of Mawhin.

Theorem 2.3 *(See* [4]*) Let L be a Fredholm operator of index zero and N be $L - compact$ on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

$(i) Lu \neq \lambda Nu$ for every $(u, \lambda) \in [(domL \backslash KerL) \cap \partial\Omega] \times (0,1)$.

$(ii) Nu \notin ImL$ every $u \in KerL \cap \partial\Omega$KerL$\partial\Omega$.

$(iii) deg(QN|_{KerL}, \Omega \cap KerL, 0) \neq 0$,

where $Q: Z \to Z$ is a projection as above with $ImL = KerQ$.

Then the abstract equation $Lu = Nu$ has at least one solution in $domL \cap \overline{\Omega}$.

Next, to obtain the solvability of problem $(1.1) - (1.2)$ by using Theorem 2.3, we renew the space $X = C^{n-1}[0,1]$ endowed with the norm

$$\|u\| = max\left\{\left\|u^{(i)}\right\|_\infty : i = 1,2, \dots, n\right\},$$

where $\|\cdot\|_\infty$ stands for the sup-norm and $Z = L^1(0,1)$ equipped with the Lebesgue norm denoted by $\|\cdot\|_1$.

Define the linear operator $L : domL \subset X \to Z$ by

$$Lu = u^{(n)}, \quad u \in domL$$

where

$$domL = \left\{ u \in AC^n[0,1] : \ \alpha_i u^{(i-1)}(0) + \beta_i u^{(i-1)}(1) = \gamma_i \int_0^1 u^{(i-1)}(t)dt \, , i = 1,2, \dots, n \right\},$$

and

$$u^{(n)}(t) = f\left(t, u(t), u'(t), u''(t), \dots, u^{(n-1)}(t)\right), \quad t \in (0,1).$$

Let $N : X \to Z$ be the operator defined by

$$Nu(t) = f\left(t, u(t), u'(t), u''(t), \dots, u^{(n-1)}(t)\right), \quad t \in (0,1). \tag{2.1}$$

Then the boundary value problem $(1.1) - (1.2)$ can be written as

$$Lu = Nu. \tag{2.2}$$

Now, we show that

$$ImL = \{z \in Z : \ \phi(z) \in ImA\}.$$

Indeed, for $z \in ImL$, there exists $u \in domL$ such that $u^{(n)}(t) = z(t)$, and then

$$u(t) = c_1 + c_2 t + \frac{c_3}{2!}t^2 + \dots + \frac{c_n}{(n-1)!}t^{n-1} + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} z(s)ds,$$

where $c_i = u^{(i-1)}(0), \ \ i = 1,2, \dots, n.$

Using the boundary conditions (1.2), we have

$$A[c_1 \ c_2 \ \dots \ c_n]^T = D[I^{n+1}z(1) \ I^n z(1) \ \dots \ Iz(1)]^T, \tag{2.3}$$

where $A = (a_{ij})$ is squar matrix of order n with

$$a_{1j} = \begin{cases} \alpha_1 + \beta_1 - \gamma_1, & if \quad j = 1, \\ \dfrac{\beta_1}{(j-2)!} - \dfrac{\gamma_1}{j!}, & if \quad 2 \le j \le n, \end{cases}$$

and

$$\begin{cases} 0, & if \ \ j = 1,2, \dots, i-1, \\ \alpha_i + \beta_i - \gamma_i, & if \ \ j = i, \qquad i = 2,3, \dots, n, \\ \dfrac{\beta_i}{(j-2)!} - \dfrac{\gamma_i}{!}, & if \ \ j = i+1, i+2, \dots, n, \end{cases}$$

and D is $n \times (n+1)$ matrix defined by

$$D = \begin{bmatrix} \gamma_1 & -\beta_1 & 0 & 0 & \cdots & & \cdots & 0 \\ 0 & \gamma_2 & -\beta_2 & 0 & & & & \vdots \\ 0 & 0 & \gamma_3 & -\beta_3 & & & & \\ \vdots & 0 & 0 & \gamma_4 & -\beta_4 & & & \\ \vdots & \vdots & 0 & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & & 0 & \gamma_n & -\beta_n \end{bmatrix},$$

and

$$I^k z(1) = \frac{1}{(k-1)!}\int_0^1 (1-s)^{k-1} y(s)ds \, , k = 1, 2, \dots, n.$$

It is clear that

$$KerL = \left\{u(t) = c_1 + c_2 t + \cdots + \frac{c_n}{(n-1)!} t^{n-1} : (c_1, c_2, \dots, c_n) \in KerA\right\} \cong KerA$$

Now, we consider the function $\phi : Z \to \mathbb{R}^n$ by

$$\phi(z) = D[I^{n+1}z(1)\ I^n z(1)\ \dots\ Iz(1)]^T \tag{2.4}$$

where D defined above.

This follows,

$$ImL \subset \{z \in Z : \phi(z) \in ImA\}.$$

Conversely, if $z \in L^1[0,1]$ and holds $\phi(z) \in ImA$, then there exists $c = (c_1, c_2, \dots, c_n) \in \mathbb{R}^n$ such that

$$A[c_1\ c_2\ \dots\ c_n]^T = \phi(z).$$

Then by setting

$$u(t) = c_1 + c_2 t + \frac{c_3}{2!} t^2 + \cdots + \frac{c_n}{(n-1)!} t^{n-1} + \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} z(s) ds,$$

it calculates strainghforwardly that $u \in domL$ and $Lu = z \in ImL$. Thus

$$ImL \supset \{z \in Z : \phi(z) \in ImA\}.$$

Therefore

$$ImL = \{z \in Z : \phi(z) \in ImA\}.$$

From (2.3) and (2.4) we deduce that we deduce the following lemma.

Lemma 2.4 *Let $\phi : Z \to \mathbb{R}^n$ be a linear operator defined by* (2.4). *then the following statement hold.*

$(i) |\phi(z)|_{\mathbb{R}^n} \leq \|D\|_* \|z\|_1$ for all $z \in Z$.

$(ii)\ Im\phi = ImD$.

where $|.|_{\mathbb{R}^n}$ and $\|.\|_*$ are the max norms on $\mathbb{R}^n$ and $M_{nx(n+1)}(\mathbb{R})$ respectively.

Proof: Setting the operator $T : Z \to \mathbb{R}^{n+1}$ by

$$Tz = [I^{n+1}z(1)\ I^n z(1)\ \dots\ Iz(1)]^T, \qquad z \in Z.$$

We derive that $\phi = D \circ T$. Thanks to the linearity of T, we obtain the linearity of ϕ. Furthermore, for $k \geq 1, k \in \mathbb{N}$, we have

$$|I^k z(1)| \leq \frac{1}{(k-1)!} \int_0^1 (1-s)^{k-1} |z(s)| ds \leq \frac{1}{(k-1)!} \|z\|_1.$$

It follows that

$$\boldsymbol{|\phi(z)|_{\mathbb{R}^n} \leq \|D\|_* max\left\{\left|I^k z(1)\right| : k = 1, 2, \dots, n+1\right\} \leq \|D\|_* \|z\|_1},$$

for all $z \in Z$.

To prove (ii) it is suffieces to show that the operator T is surjective. In fact, it is obviously that $ImT \subset \mathbb{R}^{n+1}$.

Coversely, for $(\zeta_1, \zeta_2, \dots, \zeta_{n+1}) \in \mathbb{R}^{n+1}$, we shall show that there exists $z \in Z$ to be form of

$$z(t) = c_1 + c_2(1-t) + \cdots + c_{n+1}(1-t)^n,$$

which satisfies $Tz = \zeta$. Notice that for $k = 1,2, \dots, n+1$, we have

$$I^k z(1) = \frac{1}{(k-1)!}\int_0^1 (1-s)^{k-1}[c_1 + c_2(1-s) + \cdots + c_{n+1}(1-s)^n]ds,$$

i.e,

$$I^k z(1) = \frac{1}{(k-1)!}\begin{bmatrix}\frac{1}{k} & \frac{1}{k+1} & \cdots & \frac{1}{k+n}\end{bmatrix}[c_1 + c_2 + \cdots + c_{n+1}].$$

Hence, one has

$$Tz = C[c_1 + c_2 + \cdots + c_{n+1}]^T,$$

where C denotes the following squar matrix of order $n+1$,

$$C = \begin{bmatrix} \frac{1}{n!\,(n+1)} & \frac{1}{n!\,(n+2)} & \cdots & \frac{1}{n!\,(2n+1)} \\ \frac{1}{(n-1)!\,n} & \frac{1}{(n-1)!} & \cdots & \frac{1}{(n-1)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{1} & \frac{1}{2} & \cdots & \frac{1}{n+1} \end{bmatrix}.$$

On the other hand, it is easy to see that C is invertible matrix and denote its inverse as C^{-1}. Now, we let $c = (c_1, c_2, \dots, c_{n+1}) = C^{-1} \cdot \zeta$ and $z(t) = c_1 + c_2(1-t) + \cdots + c_{n+1}(1-t)^n$, then $Tz = \zeta$. The proof is completed.

In the following we will present a general way to construct the projectors Q, P and pseudo-inverse $K_{P,Q}$. For this aim we denote the orthogonal complement of $ImA \cap ImB$ in $Im\phi = ImD$ by

$$\Sigma = \{\omega_k : k = 1,2, \dots, m\},$$

for some $1 \le m \le n$. Then we could represent the rang of L as follows

$$ImL = \{z \in Z : \langle \phi(z), \omega_k \rangle = 0, k = 1,2, \dots, m\}, \tag{2.5}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$.

On the other hand, for each $\omega_k \in Im\phi = ImD$, there exists $\zeta_k = (\zeta_1^k, \zeta_2^k, \dots, \zeta_{n+1}^k) \in \mathbb{R}^{n+1}$ to be a solution of the system linear equation $DC\zeta = \omega_k$.

Put

$$z_k = \zeta_1^k + \zeta_2^k(1-t) + \cdots + \zeta_{n+1}^k(1-t)^n,$$

then we obtain $z_k \in Z$ and $\phi(z_k) = DC\zeta_k = \omega_k$. Moreover, thanks to the linearity of the operator ϕ and the independence of system vector $\Sigma = \{\omega_k : k = 1,2, \dots, m\}$, we deduce that $\{z_k : k = 1,2, \dots, m\}$ is an independent system in Z.

Now, we state and prove three important lemmas.

Lemma 2.5 *Suppose that $ImA + ImD = \mathbb{R}^n$. Then the operator $L : domL \subset X \to Z$ is a Fredholm and has index zero.*

Proof: Since ϕ is continuous and ImA is closed in $\mathbb{R}^n$, it is clear that $ImL = \phi^{-1}(ImA)$ is closed in Z. Further, we have $dimkerL = dimKerA \leq n < \infty$. Hence it remains to prove that

$$\boldsymbol{dimKerA - codimImL = 0}.$$

To do this, we formulate the continuous operator $Q : Z \to Z$ defined as, for $z \in Z$,

$$Qz(t) = \sum_{k=1}^{m} \langle \phi(z), \omega_k \rangle z_k(t). \tag{2.6}$$

Since $\phi(z_k) = \omega_k$ and $\{\omega_k : k = 1,2, \dots, m\}$ being an orthonormal basis we deduce that

$$\langle \boldsymbol{\phi(Q(z), \omega\_k)} \rangle = \langle \boldsymbol{\phi(z), \omega\_k} \rangle$$

for all $k = 1,2, \dots, m$. This implies that Q is idempotent and therefore Q is projector. Next, utilizing $\{z_k : k = 1,2, \dots, m\}$ an independent system of Z, we argue that

$$\begin{aligned} \boldsymbol{z \in KerQ} &\Leftrightarrow \sum_{k=1}^{m} \langle \boldsymbol{\phi(z), \omega_k} \rangle \boldsymbol{z_k(t) = 0}, \\ &\Leftrightarrow \langle \boldsymbol{\phi(z), \omega_k} \rangle = \boldsymbol{0}, \quad \forall \boldsymbol{k = 1, 2, \dots, m}, \\ &\Leftrightarrow \boldsymbol{z \in ImL}. \end{aligned}$$

Hence $KerQ = ImL$. On the other hand,

$$\begin{aligned} \boldsymbol{codimImL} &= \boldsymbol{dimImQ}, \\ &= \boldsymbol{dimImD - dim(ImA \cap ImD)}, \\ &= \boldsymbol{dim(ImA + ImD) - dimImA}, \\ &= \boldsymbol{dim}(\mathbb{R}^{\boldsymbol{n}}) - \boldsymbol{dimImA}, \\ &= \boldsymbol{dimKerA = dimKerL}, \end{aligned}$$

where we use the hypothesis $ImA + ImD = \mathbb{R}^n$. The proof is complete.

Let $P : X \to X$ be the operator defined by

$$\boldsymbol{Pu(t)} = \left[\boldsymbol{1\ t\ \dots\ \frac{t^{n-1}}{(n-1)!}}\right] \boldsymbol{(I_n - A^+A)} \left[\boldsymbol{u(0)\ u'(0)\ \dots\ u^{(n-1)}(0)}\right]^{\boldsymbol{T}},$$

where A^+ is Moore-Penrose pseudo-inverse of A and I_n denotes the square matrix of n order.

Since $P_A = (I_n - A^+A)$ is an orthogonal projector onto $KerA$, it is not difficult to see that P is projector onto $KerL$ and

$$KerP = \left\{u \in X : \left[u(0)\, u'(0)\ \dots\ u^{(n-1)}(0)\right]^T = A^+A\left[u(0), u'(0), \dots\ u^{(n-1)}(0)\right]^T\right\}.$$

Lemma 2.6 *Let $K_P : ImL \to domL \cap Kerp$ be a linear operator defined by*

$$(\boldsymbol{K_P z})(\boldsymbol{t}) = \left[\boldsymbol{1\ t\ \dots\ \frac{t^{n-1}}{(n-1)!}}\right]\boldsymbol{A^+\phi(z) + I^n z(t)},$$

for $z \in ImL$. Then K_P is a pseudo-inverse of L, wich means that

$$\boldsymbol{K_P = (L|_{domL \cap KerP})^{-1}}.$$

Moreover, we have the following estimate

$$\|\boldsymbol{K_P z}\| \leq (\boldsymbol{1 + n}\|\boldsymbol{A^+}\|_*\|\boldsymbol{D}\|_*)\|\boldsymbol{z}\|_{\boldsymbol{1}},$$

for every $z \in ImL$.

Proof: For each $z \in ImL$, it is not difficult to see that $K_P z \in AC^n[0,1]$ and

$$\left[K_P z(0)\ (K_P z)'(0)\ \dots\ (K_P z)^{(n-1)}(0)\right]^T = A^+\phi(z).$$

It is strainghforward to verify that $K_P z \in domL \cap KerP$. Hence $K_P z$ is well defined. On the other hand, it is clear that $LK_P z(t) = z(t)$ for all $t \in [0,1]$ and $z \in ImL$. Moreover, $u \in domL \cap KerP$., we have $u \in domL$ which implies

$$A\left[u(0)\, u'(0)\ \dots\ u^{(n-1)}(0)\right]^T = \phi(Lu).$$

It follows

$$(K_P Lu)(t) = \left[1\ \ t\ \dots\ \frac{t^{n-1}}{(n-1)!}\right]A^+\phi(Lu) + I^n Lu(t),$$

$$= u(t) - \left[1\ \ t\ \dots\ \frac{t^{n-1}}{(n-1)!}\right](I_n - A^+A)\left[u(0)\, u'(0)\ \dots\ u^{(n-1)}(0)\right]^T,$$

$$= u(t) - Pu(t) = u(t),$$

where we use the fact that $u \in KerP$ in the last inequality. Hence $(K_P Lu)(t) = u(t)$ for all $t \in [0,1]$ and for every $u \in domL \cap KerP$. Thus, $K_P = (L|_{domL \cap KerP})^{-1}$. Furthermore, by the definition of the pseudo-inverse K_P of L, we get

$$(K_P Lu)^{(i)}(t) = \left[0\ \dots\ 1\ t\ \dots\ \frac{t^{n-1-i}}{(n-1)!}\right]A^+\phi(Lu) + I^{n-i}Lu(t), \tag{2.7}$$

where 1 is $(i+1)^{th}$ position, for $i = 0,1,\dots,(n-1)$. It follows from (2.7) and Lemma 2.4 that

$$\left|(K_P z)^{(i)}(t)\right| \leq (n-i)\|A^+\|_*|\phi(z)|_{\mathbb{R}^n} + \frac{1}{(n-i-1)!}\int_0^1 (1-s)^{n-1-i}|z(s)|ds,$$

$$\leq (1 + (n-i)\|A^+\|_*\|D\|_*)\|z\|_1,$$

for $i = 0,1,\dots,(n-1)$, which implies

$$\|K_P z\| = max\left\{\left\|(K_P z)^{(i)}\right\|_\infty : i = 0,1,2,\dots,n-1\right\},$$

$$\leq (1 + n\|A^+\|_*\|D\|_*)\|z\|_1.$$

The proof is complete.

Lemma 2.7 *The operator N defined by* (2.1) *is $L-$ completely continuous on X.*

Proof: Let Ω be an open bounded set in X. Put $R = sup\{\|u\| : u \in \Omega\}$. Then it is clearly seen that $QN : \overline{\Omega} \to Z$ is continuous QN is continuous by using the Lebesgue's dominanted convergence theorem and $QN(\overline{\Omega})$ is bounded because is N is continuous mapping and takes bounded sets into bounded sets. Now, we shall prove that $K_{P,Q}N$ is completely continuous on $\overline{\Omega}$. In fact, since $K_{P,Q}N$ is composition of the continuous operators N, Q and K_P, so $K_{P,Q}N$ is also continuous operator. In addition, by the deffinition of the operator $K_{P,Q}$, we have

$$\left(K_{P,Q}Nu\right)^{(i)}(t) = (K_P(I-Q)Nu)^{(i)}(t),$$

$$= \left[0 \ \dots \ 1 \ t \ \dots \ \frac{t^{n-1-i}}{(n-1)!}\right] A^+\phi\big((I-Q)Nu\big) + I^{n-i}(I-Q)Nu(t), \qquad (2.8)$$

where 1 is $(i+1)^{th}$ position, for $i = 0,1,\dots,(n-1)$ and $u \in \overline{\Omega}$.

Set $z(t) = (I-Q)Nu(t)$ for $t \in [0,1]$, then there exists a positive constant M such that $\|z\|_1 \leq M$ for all $u \in \overline{\Omega}$. This implies that $K_{P,Q}N\ (\overline{\Omega})$ is bounded by using Lemma 2.6.

On the other hand, since $\{t^i : t \in [0,1]\}$ and $\{I^{n-i}z(t)\}$, for each $i = 0,1,\dots,(n-1)$ are equicontinuous families, it follows from (2.8) that $\{(K_P z)^{(i)}(t) : t \in [0,1]\}$ are equicontinuous. Hence, one obtains $\left(K_{P,Q}Nu\right)^{(i)}(\overline{\Omega})$ is relatively compact, for $i = 0,1,\dots,(n-1)$ due to Arzela-Ascoli theorem. Thus, $K_{P,Q}N\ (\overline{\Omega})$ is relatively compact in X. The proof is complete.

3. MAIN RESULTS

In this section, we state our results on the existence of a solution for $(1.1)-(1.2)$. For this purpose we will assume that

$$ImA + ImD = \mathbb{R}^n,$$

and the following conditions hold.

(H_1) There exist the positve functions $a_0, a_1, \dots, a_n \in Z$ with $C\sum_{i=0}^{n-1}\|a_i\|_1 < 1$ such that

$$|f(t,u_0,u_1,\dots,u_{n-1})| \leq \sum_{i=0}^{n-1} a_i(t)|u_i| + a_n(t),$$

for all $t \in [0,1]$ and $u, v \in \mathbb{R}^n$, where $C = 1 + n\|I_n - A^+A\|_* + n\|A^+\|_*\|D\|_*$.

(H_2) There exists a constant $\Lambda_1 > 0$, such that for $u \in domL$, if $max\{|u^{(i)}(t)| : i = 0,1,\dots,n-1\} > \Lambda_1$ for all $t \in [0,1]$, then

$$\boldsymbol{\phi(Nu) \notin ImA}.$$

(H_3) There exists a constant $\Lambda_2 > 0$ such that for any $\{c_i\}_{i=1}^m \subset \mathbb{R}$ with $\sum_{i=1}^m |c_i| > \Lambda_2$, then either

$$c_i \left\langle \phi \circ N\left(\sum_{j=0}^m c_j u_j\right), \omega_j \right\rangle < 0, \tag{3.1}$$

or

$$c_i \left\langle \phi \circ N\left(\sum_{j=1}^m c_j u_j\right), \omega_i \right\rangle > 0, \tag{3.2}$$

for all $i = 0,1,\dots,m$ where $\{u_j : j = 1,2,\dots,m\}$ is basis of $KerL$ and $\langle\cdot,\cdot\rangle$ stand for the scalar product in $\mathbb{R}^n$.

Theorem 3.1 *Assume that the assumptions* $(H_1),(H_2),(H_3),(3.1)$ *and* (3.2) *hold. Then the problem* $(1.1)-(1.2)$ *has at least one solution in* X.

To prove the above theorem, we need the following lemmas.

Lemma 3.2 *Let* $\Omega_1 = \{u \in domL\backslash KerL: Lu = \lambda Nu, \quad \lambda \in (0,1]\}$. *Then* Ω_1 *is buonded subset in* X.

Proof: Suppose that $u \in \Omega_1$, and $Lu = \lambda Nu$ for $\lambda \in (0,1]$. then it is clear that $Nu \in ImL = KerQ$, which also implies $QNu \in ImA$ by caractirization of ImL. On the other hand, we have

$$QNu(t) = \sum_{k=1}^m \langle \phi(z), \omega_k\rangle z_k(t) = 0, \qquad \forall t \in [0,1].$$

Thanks to the linearly independent property of $\{z_k : k = 1,2,\dots,m\}$, we derive that $\langle Q(Nu), \omega_k\rangle = 0$ for all $k = 1,2,\dots,m$. Therefore, we possess $\phi(Nu) \in ImA \cap ImD$ which implies $\phi(Nu) \in ImA$. By utilizing the assumption (H_2) there exits $t_0 \in [0,1]$ such that

$$max\{|u^{(i)}(t_0)| : i = 0,1,\dots,(n-1)\} \le \Lambda_1.$$

It follows from the identities

$$u^{(i)}(t) = u^{(i)}(t_0) + u^{(i+1)}(t_0)(t-t_0) + \cdots + \frac{1}{(n-i-1)!}\int_{t_0}^{\mathrm{t}} (\mathrm{t}-s)^{n-1-i} u^{(n)}(s)ds,$$

for all $i = 1,\dots,n-1$, and

$$u(t) = u(t_0) + u'(t_0)(t-t_0) + \cdots + \frac{1}{(n-1)!}\int_{t_0}^{\mathrm{t}} (\mathrm{t}-s)^{n-1-i} u^{(n)}(s)ds,$$

that

$$|u^{(i)}(0)| \le (n-i)\Lambda_1 + \|u^{(n)}\|_1, \quad i = 1,\dots,n-1,$$

and

$$|u(0)| \leq n\Lambda_1 + \left\|u^{(n)}\right\|_1.$$

Therefore, we get

$$max\left\{\left|u^{(i)}(0)\right| : \ i = 0,1,\dots,(n-1)\right\} \leq n\Lambda_1 + \left\|u^{(n)}\right\|_1,$$

$$\leq \boldsymbol{n\Lambda_1 + \|Nu\|_1}.$$

It follows from the definition of the projector P and the inequality above that

$$\boldsymbol{\|Pu\| = n\|I_n - A^+A\|_* max\left\{\left|u^{(i)}(0)\right| : \ i = 0,1,\dots,n-1\right\}},$$

$$\leq n\|I_n - A^+A\|_*(n\Lambda_1 + \|Nu\|_1). \tag{3.3}$$

On the other hand, since $(I-P)u \in domL \cap KerP$ and using Lemma 2.5, we achieve

$$\|(I-P)u\| = \|K_PL(I-P)u\| = \|K_PLu\|, \tag{3.4}$$

$$\leq (1 + n\|A^+\|_*\|D\|_*)\|Nu\|_1. \tag{3.5}$$

Combining $(3.3) - (3.5)$, we obtain

$$\|u\| = \|Pu + (I-P)u\|,$$

$$\leq \|Pu\| + \|(I-P)u\|,$$

$$\leq \|I_n - A^+A\|_* n^2\Lambda_1 + C\|Nu\|_1,$$

where $C = 1 + n\|I_n - A^+A\|_* + n\|A^+\|_*\|D\|_*$. Exploiting the assumptions of nonlinear term (H_1) and the definition of the operator N, we get

$$\|Nu\|_1 \leq \int_0^1 \left|f\left(s, u(s), u'(s), \dots, u^{(n-1)}(s)\right)\right| ds,$$

$$\leq \sum_{i=1}^{n-1} \|a_i\|_1 \left\|u^{(i)}\right\|_\infty + \|a_n\|_1,$$

$$\leq \left(\sum_{i=1}^{n-1} \|a_i\|_1\right)\|u\| + \|a_n\|_1. \tag{3.6}$$

It follows from (3.5) and (3.6), $C\sum_{i=0}^{n-1}\|a_i\|_1 < 1$ that

$$\|u\| \leq \frac{\|I_n - A^+A\|_* n^2\Lambda_1 + C\|a_n\|_1}{1 - C\sum_{i=1}^{n-1}\|a_i\|_1}.$$

Thus, Ω_1 is bounded in X.

Lemma 3.3 *The set $\Omega_2 = \{u \in KerL : \ Nu \in ImL\}$ is a bounded subset in X.*

Proof: Let $u \in \Omega_2$. Assume that

$$u(t) = c_0 + c_1 t + \frac{c_3}{2!} t^2 + \cdots + c_{n-1} \frac{t^{n-1}}{(n-1)!},$$

for all $t \in [0,1]$, where $(c_0, c_1, \dots, c_{n-1}) \in KerA$. It follows from $Nu \in ImL$ that $QNu \in ImA$. From condition (H_2), by the same arguments as in Lemma 3.2, we arrive at

$$max\{|u^{(i)}(t_0)| : i = 0,1, \dots, (n-1)\} \leq \Lambda_1,$$

for some $t_0 \in [0,1]$. As a result, for each $i = 0,1, \dots, n-1$, c_i is bounded in $\mathbb{R}$, that is Ω_2 is bounded in X. The proof is complete.

Lemma 3.4. *The sets*

$$\boldsymbol{\Omega_3^- = \{u \in KerL : -\lambda u + (1-\lambda) JQNu = 0, \quad \lambda \in [0,1]\},}$$

and

$$\Omega_3^+ = \{u \in KerL : \lambda u + (1-\lambda) JQNu = 0, \quad \lambda \in [0,1]\},$$

are bounded in X provided that (3.1) and (3.2) of the assumption (H_3) hold, respectively.

Where $J : ImQ \to KerL$ is a linear isomorphism which is defined by

$$J\left(\sum_{i=1}^{m} c_i z_i\right) = \sum_{i=1}^{m} c_i u_i,$$

for $\sum_{i=1}^{m} c_i z_i \in ImQ$.

Proof: Assume that (H_3) and (3.1) holds. Let $u \in \Omega_3^-$, then we might assume that $u = \sum_{i=1}^{m} c_i u_i \in KerL$, where $c_i \in \mathbb{R}, i = 1,2, \dots, m$ and

$$\lambda J^{-1}\left(\sum_{i=1}^{m} c_i u_i\right) = (1-\lambda) QN\left(\sum_{i=1}^{m} c_i u_i\right),$$

for $\lambda \in [0,1]$. It follows from the definitions of the operators J and Q that

$$\lambda \sum_{i=1}^{m} c_i z_i = (1-\lambda) \sum_{i=1}^{m} \left\langle \phi \circ N\left(\sum_{j=1}^{m} c_j u_j\right), \omega_i \right\rangle z_i.$$

This implies

$$\lambda c_i = (1-\lambda) \left\langle \phi \circ N\left(\sum_{j=1}^{m} c_j u_j\right), \omega_i \right\rangle,$$

for all $i = 1,2, \dots, m$. If $\lambda = 1$, then $c_i = 0$ for all $i = 1,2, \dots, m$. In this case by assumption (H_3) and (3.1) we get a contradiction

$$0 \leq \lambda c_i^2 = (1-\lambda)c_i \left\langle \phi \circ N\left(\sum_{j=1}^{m} c_j u_j\right), \omega_i \right\rangle < 0,$$

for some $i = 1,2,\dots,m$. Thus, Ω_3 is bounded in X. If (H_3) and (3.2) holds by using the same arguments as in above we are also able to prove that Ω_3^+ is bounded in X. The proof is complete.

For the Proof of Theorem 3.1 we shall apply Theorem 2.3 and the above Lemmas.

Proof: Proof of Theorem 3.1. Let Ω to be an open bounded subset of X such that $\bigcup_{i=1}^{3} \overline{\Omega}_i \subset \Omega$. It is clear that the operator L is a Fredholm of index zero by Lemma 2.5 and N is $L-$ compact on $\overline{\Omega}$ by Lemma 2.7. Furthermore, the conditions (i) and (ii) of the Theorem 2.3 are fulfilled. By exploiting Lemmas 3.2 and 3.3. So it remains to verify the third condition of Theorem 2.3. For this purpose, we apply the degree property of invariance under a homotopy . Let us define

$$H(u,\lambda) = \pm\lambda Ju + (1-\lambda)QNu = 0, \qquad \lambda \in [0,1],$$

where the isomorphisme $J : ImQ \to KerL$ is defined in Lemma 3.4. According to Lemmas 3.4, we know that $H(u,\lambda) \neq 0$ for every $(u,\lambda) \in (KerL \cap \partial\Omega) \times [0,1]$. Thus, by the invariance under a homotopy property of degree, we have

$$\boldsymbol{deg(JQN|_{KerL}, \Omega \cap KerL, 0) = deg(H(\cdot,0), \Omega \cap KerL, 0)},$$

$$\boldsymbol{= deg(H(\cdot,1), \Omega \cap KerL, 0)},$$

$$\boldsymbol{= deg(\pm I, \Omega \cap KerL, 0) \neq 0}.$$

Then, by Theorem 2.3, $Lu = Nu$ has at least one solution in $domL \cap \overline{\Omega}$, so the boundary value problem $(1.1)-(1.2)$ has at least one solution in X. The proof is complete.

We construct an example to illustrate the applicability of the results presented.

Example 3.1 *Consider the existence of solutions to the following boundary value problem*

$$u''(t) = f\big(t, u(t), u'(t)\big), t \in (0,1), \tag{3.7}$$

associated with the integral boundary condition

$$\begin{cases} \frac{3}{4}u(0) + \frac{1}{2}u(1) = \int_0^1 u(s)ds, \\ \frac{1}{2}u'(0) + \frac{5}{8}u'(1) = \int_0^1 u'(s)ds, \end{cases} \tag{3.8}$$

where $\alpha_1 = \frac{3}{4}$, $\alpha_2 = \frac{1}{2}$, $\beta_1 = \frac{1}{2}$, $\beta_2 = \frac{5}{8}$, $\gamma_1 = \gamma_2 = 1$, and the function $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R})$ is defined by

$$f(t,u) = \big(f_1(t,u), f_2(t,u)\big),$$

where the functions $f_i : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$, $(i = 1,2)$ are

$$f_1(t,u) = \frac{t^5+1}{44}(u_1+u_2) + \sqrt{t+1},$$

$$f_2(t,u) = \frac{t^5+1}{44}(|u_1| + |u_2|) + \frac{\sqrt{t+1}}{2},$$

for all $t \in [0,1]$ and $u \in \mathbb{R}^2$. Set

$$A = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{8} \end{bmatrix},$$

which has $ImA = \langle (4,0),(0,8) \rangle$ and the Moore-Pensore matrix $A^+ = \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}$. Moreover, we could see that the matrix $D = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{5}{8} \end{bmatrix}$ and the operator $\phi(z) : Z \to \mathbb{R}^2$ defined by

$$\phi(z) = D[I^3 z(1) \;\; I^2 z(1) \;\; Iz(1)]^T = \left(I^3 z(1) - \frac{1}{2} I^2 z(1), I^2 z(1) - \frac{5}{8} Iz(1) \right).$$

So the image of L

$$\boldsymbol{ImL} = \{\boldsymbol{z} \in \boldsymbol{L^1(0,1)} : \boldsymbol{\phi(z)} \in \boldsymbol{ImA}\},$$

$$= \left\{ \boldsymbol{z} \in \boldsymbol{L^1(0,1)} : \int_0^1 \left(-\boldsymbol{s^2} + \frac{\boldsymbol{3}}{\boldsymbol{8}} \right) \boldsymbol{z(s)ds}. = \boldsymbol{0} \right\}.$$

It is clearly seen that $ImD = \mathbb{R}^2$ and $ImA \cap ImD = ImA$. Therefore, we have

$$dim(ImA + ImD) = dimImA + dimImD - dim(ImA \cap ImD) = dim\mathbb{R}^2$$

It follows that $ImA + ImD = \mathbb{R}^2$. Hence, according to Lemma 2.4, L is a Fredholm operator of index zero.

Now, taking $\left\{ \omega_1 = \frac{\sqrt{2}}{2}(1,-1) \right\}$ is an orthonormal basis of the orthogonal complement of $ImA \cap ImD = ImA$ in $\mathbb{R}^2$ and setting

$$z_1(t) = \zeta_1 + \zeta_2(1-t) + \zeta_3(1-t)^2,$$

where $\zeta = (\zeta_1, \zeta_2, \zeta_3) = \left(-\frac{3\sqrt{2}}{4}, -15\sqrt{2}, 0 \right) \in \mathbb{R}^3$ is a solution of equation $DC\zeta = \omega_1$. Then one has $\phi(z_1) = \omega_1$.

We can now define the projections $P : X \to X$ and $Q : Z \to Z$ by

$$Pu(t) = [1 \;\; t](I_2 - A^+A)[u(0) \;\; u'(0)]^T = \frac{1}{2}\big(u(0) - u'(0)\big)u_1(t),$$

and

$$Qz(t) = \langle \phi(z), \omega_1 \rangle z_1(t) = \frac{\sqrt{2}}{2} \int_0^1 \left[\left(-s^2 + \frac{3}{8} \right) z(s) ds \right] z_1(t).$$

The pseudoinverse K_P is define by

$$K_P(z)(t) = [1 \;\; t]A^+\phi(z) + \int_0^1 (t-s)z(s)ds,$$

which implies

$$\|K_P z\| \leq (1 + 2\|A^+\|_*\|D\|_*)\|z\|_1 = 25\|z\|_1,$$

for all $z \in Z$.

First, it is not difficult to see that the function f satisfies the Caratheodory condition. Further, we have

$$|f(t,u_0,u_1)| \leq a_0(t)|u_0| + a_1(t)|u_1| + a_2,$$

where

$$a_0(t) = a_1(t) = \frac{t^5+1}{44}, \quad a_2(t) = \sqrt{t+1},$$

are the positive integrable functions on [0,1]. Moreover, some direct calculations give us

$$C = 1 + 2\|I_n - A^+A\|_* + 2\|A^+\|_*\|D\|_* = 25$$

and

$$C(\|a_0\|_\infty + \|a_1\|_\infty) = \tfrac{1}{2} < 1.$$

Hence, the condition (H_1) holds. In order to chek (H_2) we note that $\phi(Nu) \in ImA$ is equivalent to

$$\int_0^1 \left|-s^2 + \frac{3}{8}\right| Nu(s)ds = 0,$$

i.e.

$$\int_0^1 \left|-s^2 + \frac{3}{8}\right| f\big(s,u(s),u'(s)\big)ds = 0.$$

On the other hand, if $|u_1| > 22$, then we have $f(t,u_0,u_1) > 2$ for all $t \in [0,1]$. Hence, taking $M_1 = 22$, we have

$$\int_0^1 \left|-s^2 + \frac{3}{8}\right| f\big(s,u(s),u'(s)\big)ds > 0,$$

provided that $max\{|u^{(i)}(t)| \;:\; i = 0,1\} > M_1$ for all $t \in [0,1]$. This results $\phi(Nu) \notin ImA$.

The condition (H_2) holds.

Finaly to chek (H_3), for $u_1(t) = 1 - t$, we have

$$\langle \phi \circ N(c_1u_1), \omega_1\rangle = \frac{\sqrt{2}}{2}\int_0^1 \left|-s^2 + \frac{3}{8}\right| c_1 f(s, c_1(1-s), -c_1)ds.$$

Similarly, taking $\Lambda_2 = 22$, then we get $c_1 f(t, c_1(1-t), -c_1) > 11 > 0$ if $c_1 > \Lambda_2$ and $c_1 f(t, c_1(1-t), -c_1) < -22 < 0$ if $c_1 < \Lambda_2$.

Therefore, if $|c_1| > \Lambda_2$, then

$$c_1\langle \phi \circ N(c_1u_1), \omega_1\rangle = \frac{\sqrt{2}}{2}\int_0^1 \left|-s^2 + \frac{3}{8}\right| c_1 f(s, c_1(1-s), -c_1)ds > 0$$

or

$$c_1\langle \phi \circ N(c_1u_1), \omega_1\rangle = \frac{\sqrt{2}}{2}\int_0^1 \left|-s^2 + \frac{3}{8}\right| c_1 f(s, c_1(1-s), -c_1)ds < 0.$$

Hence (H_3) holds. Thus, problem $(3.7) - (3.8)$ has at least one solution.

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