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ORIGINAL PAPER

MODULES WITH THE PROPERTY Rad_g

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Abstract. *In this article, we introduce modules with the property Rad_g and provide various properties of this module. Firstly, we prove that every semisimple module has the property Rad_g. We also indicate that the class of modules with the property Rad_g is closed under finite direct sum and direct summands. Moreover, we define the concepts of g – (radical) cover. We show that factor modules of a module with the property Rad_g have the property Rad_g under special condition. Additionally, we characterize semisimple commutative rings via modules with the property Rad_g.*

Keywords: *g-radical supplement; extension; SSI-ring; semisimple commutative ring.*

1. INTRODUCTION AND PRELIMINARIES

In this article, modules with the property Rad_g are defined as a new generalization of $(GE)-$modules which was studied in [1] and some simple features of them are investigated. While preparing this article in addition to the article [2], the articles [3-4] connected with modules possessing a supplement and a Rad-supplement in each cofinite extension are examined. During the article, R will be an associative ring with identity and all modules are unital left $R-$ modules unless otherwise specified. Let X be an $R-$ module. $U \subseteq X$ implies that U is a submodule of X or X is an extension of U. Remind that a submodule $U \subseteq X$ is named *small*, shown by $U = X$, if $U + T \neq X$ for all proper submodules T of X and that $T \subseteq X$, is named *essential* in X, shown by $T \trianglelefteq X$, if $T \cap S \neq 0$ for each non-zero submodule $S \subseteq X$. $Rad(X)$ will indicate Jacobson radical of X.

A module X is named *supplemented*, if each submodule T of X has a supplement in X, i.e. a submodule S of X is minimal with respect to $T + S = X$. S is a supplement of T in X if and only if $T + S = X$ and $T \cap S = S$ [5]. Moreover, the submodule T is said to possess *ample supplements* in X if each submodule L of X includes a supplement of T in X. The module X is named *amply supplemented* if each submodule of X has ample supplements in X [5].

A new notion of small submodules are studied in [6-7]. A submodule T of X is said to be a *generalized small submodule* of X if for each $S \trianglelefteq X$ with $X = T + S$ implies that $S = X$, which is indicated $T =_g X$ (in [7], it is named an $e-$*small submodule* of X and written by $T =_e X$). If T is both essential and maximal submodule of X, then T is called a *generalized maximal submodule* of X. $Rad_g(X)$ indicates the generalized radical of X that is defined as the intersection of all maximal essential submodules of X (in [7], it is indicated

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$Rad_e(X)$). If X possess no maximal essential submodules, then we denoted by $Rad_g(X) = X$. According to [8], if $X / Rad_g(X)$ is semisimple, then X is called a *$g-$semilocal module*. Following [9], if X has a large proper essential submodule that contains all essential submodules of X or X has no proper essential submodules, then X is called a *generalized local module*. Let S, T be submodules of X. Then T is called a *$g-$supplement* of S in X, if $X = T + S$ and $T \cap S =_g T$. A module X is named *$g-$supplemented* if each submodule of X has a $g-$supplement in X. A submodule S of X has *ample $g-$*supplements in X, if for each $T \subseteq X$ such that $X = T + S$ and S has a $g-$supplement T_1 in X with $T_1 \subseteq T$. A module X is named *amply $g-$supplemented* if each submodule of X has ample $g-$supplements in X. For more detailed discussion about, we refer to [6-7]. Similarly, generalized radical supplemented (briefly, $g-$radical supplemented) modules are defined in [9]. Following [9], for a module X and submodules T, S of X, if $X = T + S$ and $T \cap S = Rad_g(T)$, then T is named a *generalized radical supplement* (briefly, $g-$radical supplement) of S in X. If each submodule of X has a generalized radical supplement in X, then X is named a *generalized radical supplemented* (briefly, $g-$radical supplemented) module. For the characterization of generalized radical supplemented modules we refer to [9].

Modules which has a supplement (ample supplements) in each module in where it is included as a submodule are named modules with the property (E)((EE)) in [10]. This kind of modules are also named supplementing modules in [11]. In a recent paper [3], modules which possess a supplement in each cofinite extension have been investigated and these modules are named modules with the property (CE)((CEE)). Following this study, a generalization of these modules has been examined in [4]. Another variant of them are examined in [12]. In addition, (ample) Rad-supplementing are defined in [2]. On the other hand, another adaptation of these modules introduced in [1] and the authors named these modules as $(GE)-$module. X is named a *$(GE)-$module* if it has a $g-$supplement in each extension N. X is named an *$(GEE)-$module* if it has ample $g-$supplements in each extension N.

By taking all studies on these into consideration, we describe and examine modules which has the property (ample) Rad_g. An $R-$module X has the property Rad_g if it has a $g-$radical supplement in each module where it is included as a submodule. Similar to this definition, a module X has *the property ample Rad_g* if it has ample $g-$radical supplements in each module where it is included as a submodule where a submodule S has ample $g-$radical supplements in each $T \subseteq X$, with $X = S + T$, there is a $g-$radical supplement T_1 of S with $T_1 \subseteq T$. In this paper, some properties of these modules are examined. It is shown that $(GE)-$modules and $g-$radical modules have the property Rad_g. Usually, modules with the property Rad_g need not be a $(GE)-$module. An example will be given to show that under this fact. It is proved that each direct summand of a module with the property Rad_g has also the property Rad_g. Nevertheless, it is proved that modules with the property Rad_g are preserved under extensions. Following, it is given that an immediate theorem such that each module with composition series is module with the property Rad_g. On the other hand, it is pointed that if each module with the property Rad_g over the ring R is injective, then the ring R is a left $SSI-$ring. As a result of this theorem, semisimple commutative rings are characterized using modules with the property Rad_g. It is also evidenced that for a module X, the necessary and sufficient condition of having with the property ample Rad_g is that each submodule of X has the property Rad_g. In addition, it is proved that each module with the property ample Rad_g has the property Rad_g, and it is obtained that each module with the property ample Rad_g is a $g-$radical supplemented.

2. MAIN RESULTS

Definition 2.1. An $R-$module X is called having the property Rad_g, if X has a $g-$radical supplement in every extension, and X is called having the property ample Rad_g if X has an amply $g-$radical supplement in every extension.

Proposition 2.2. Let N be a $(GE)-$module. Then N has the property Rad_g. In addition, the same property is provided for $g-$radical module.

Proof: Assume that X is a module and $X \subseteq N$. If X is a $(GE)-$module, then there exists a submodule S of N where S is a $g-$supplement of X in N. In other words, we have $X \cap S =_g S$. By [9], $X \cap S \subseteq Rad_g(S)$. Since X has a $g-$radical supplement S in N, we can obtain that X is a module with the property Rad_g. If $Rad_g(X) = X$, then N is a $g-$radical supplement of X in N.

The following results are obtained from preceding Proposition 2.2 and Lemma 1 in [1].

Corollary 2.3. If X is a semisimple module, then X has the property Rad_g.

The next example shows that modules with the property Rad_g are a proper generalization of the class of $(GE)-$modules.

Example 2.4. (See [13] Example 6.2) Let F be a field and $I = (a_1^2, a_2^2 - a_1, a_3^2 - a_2, ..)$ be an ideal generated by a_1^2 and $a_{n+1}^2 - a_n$ for each $n \in \mathrm{N}$ in the polynomial ring $F[a_1, a_2, \ldots]$ with countably many indeterminates $a_n, n \in N$. Then the quotient ring $R = F[a_1, a_2, \ldots]/I$ is a local ring with the unique maximal ideal $Rad(R) = Rad(R)^2$. Let $X = Rad(R)^N$. Then R is a generalized local ring. Since $Rad(X) = X$ and $Rad(X) \subseteq Rad_g(X)$, then $Rad_g(X) = X$. So X is a module with the property Rad_g by Proposition 2.2. Suppose that X is a $(GE)-$module. Since R is $g-$semilocal, $Rad_g(R)$ is a $t-$nilpotent. But it is a contradiction. Thus X is not a $(GE)-$module.

Proposition 2.5. If X is a module with the property Rad_g, then each direct summand of the module has the property Rad_g.

Proof: Assume that K is a direct summand of X and $K \subseteq S$. We can find a submodule L of X where $X = K \oplus L$. By the assumption, X has a $g-$supplement in the module $L \oplus S$

including X, that is, there exists a submodule T of $L \oplus S$ such that $(L \oplus K) + T = L \oplus S$ and $(L \oplus K) \cap T \subseteq Rad_g(T)$. Now, let $\pi : L \oplus S \to S$ be a canonical projection onto S. Then

$$K + \pi(T) = \pi(L \oplus K) + \pi(T) = \pi\big((L \oplus K) + T\big) = \pi(L \oplus S) = S,$$

and

$$K \cap \pi(T) = \pi\big((L \oplus K) \cap T\big) \subseteq \pi\big(Rad_g(T)\big) \subseteq Rad_g\big(\pi(T)\big).$$

Hence $\pi(T)$ is a $g-$supplement of K in S, as desired.

Proposition 2.6. Let X be a module. Then X has a $g-$radical supplement in every essential extension if and only if X has a $g-$radical supplement in its injective envelope $E(X)$.

Proof: ($\Rightarrow$) This side is clear.
($\Leftarrow$) Suppose that $X \subseteq S$ with $X \trianglelefteq S$. Let $\alpha_1 : X \to S$ and $\beta_1 : X \to E(X)$ be inclusion maps. Then we obtain the commutative diagram with γ necessarily monic:

$$\begin{tikzcd} X \arrow[r, "\alpha_1"] \arrow[d, "\beta_1"'] & S \arrow[dl, "\gamma"] \\ E(X) & \end{tikzcd}$$

By using the assumption, X has a $g-$radical supplement in $E(X)$, say V, that is, $X + V = E(X)$ and $X \cap V \subseteq Rad_g(V)$. Since $X \subseteq \gamma(S)$, we have

$$\gamma(S) = \gamma(S) \cap E(X) = \gamma(S) \cap (X + V) = X + \big(\gamma(S) \cap V\big).$$

For every $s \in S$, we can write $\gamma(s) = m + \gamma(s_1) = \gamma(m + s_1)$ where $m \in X$ and $\gamma(s_1) \in \gamma(S) \cap V$. So, $s = m + s_1 \in X + \gamma^{-1}(V)$ since γ is monic and thus $X + \gamma^{-1}(V) = S$. Therefore $X \cap \gamma^{-1}(V) = \gamma^{-1}(X \cap V) \subseteq \gamma^{-1}\big(Rad_g(V)\big) \subseteq Rad_g\big(\gamma^{-1}(V)\big)$because $\gamma^{-1}(X) = X$. Thus $\gamma^{-1}(V)$ is a $g-$radical supplement of X in S.

The following lemma is a key as a notion of this kind of modules.

Lemma 2.7. If X is a non-zero non-simple $R-$module, $\alpha : X \to N$ is an epimorphism and $\ker(\alpha) \subseteq Rad_g(X)$, then $Rad_g(N) = \alpha\big(Rad_g(X)\big)$. In particular $Rad_g\big(X / Rad_g(X)\big) = 0$.

Proof: It is clear by [14, 8.17(2)] and [6, Lemma 1.1(3)].

If T and X are modules, then we call an epimorphism $\alpha : T \to X$ is a $g-$*(radical) cover* in case $\ker(\alpha) =_g T$ ($\ker(\alpha) \subseteq Rad_g(T)$), then every $g-$cover is a $g-$radical cover.

Proposition 2.8. If both $\alpha : T \to X$ and $\beta : X \to S$ are $g-$(radical) covers, then $\beta \circ \alpha : T \to S$ is a $g-$(radical) cover for non-zero non-simple $R-$module T.

Proof: Let α and β be $g-$radical covers. If we show that $\ker(\beta\circ\alpha)\subseteq Rad_g(T)$, then we will have completed the first part of the proof. Suppose that $a\in\ker(\beta\circ\alpha)$. Then $(\beta\circ\alpha)(a)=0$ and $\alpha(a)\subseteq\ker(\beta)\subseteq Rad_g(X)$. Since $\ker(\alpha)\subseteq Rad_g(T)$, it follows from Lemma 2.7 that $\alpha(Rad_g(T))=Rad_g(X)$. Hence $\alpha(a)=\alpha(a')$ for some $a'\in Rad_g(T)$ and so $a-a'\in\ker(\alpha)\subseteq Rad_g(T)$. Finally, we have $a\in Rad_g(T)$. Thus $\ker(\beta\circ\alpha)\subseteq Rad_g(T)$ and similar proof can be done for $g-$covers.

Proposition 2.9. Let X be a module and U be a submodule of X. If U and X/U are modules with the property *Rad*$_g$, then so is X.

Proof: Suppose that T is any extension of X. By the assumption, there exists a $g-$radical supplement S/U of X/U in T/U and $g-$radical supplement V of U in S. Beside this $T=S+X=(V+U)+X=V+X$. Then we can obtain epimorphisms $\alpha_1:V\to S/U$ and $\beta_1:S/U\to T/X$ such that $\ker(\alpha_1)=V\cap U\subseteq Rad_g(V)$ and $\ker(\beta_1)=(S/U)\cap(X/U)$ $\subseteq Rad_g(S/U)$. Thus $\beta_1\circ\alpha_1:V\to T/X$ is an epimorphism with $V\cap X=\ker(\beta_1\circ\alpha_1)\subseteq$ $Rad_g(V)$ by Proposition 2.8.

Corollary 2.10. If X_1 and X_2 are modules with the property *Rad*$_g$, then so is $X_1\oplus X_2$.

Proof: If we take the short exact sequence $0\to X_1\to X_1\oplus X_2\to X_2\to 0$ into consideration, then the proof is clear from Proposition 2.5.

Recall from [5] that a normal series $0=X_0\subseteq X_1\subseteq X_2\subseteq\ldots\subseteq X_k=X$ is called a *composition series* of X if all factors X_i/X_{i-1} are simple modules.

Theorem 2.11. If X is a module with a composition series, then it has the property *Rad*$_g$.

Proof: Let $0=X_0\subseteq X_1\subseteq X_2\subseteq\ldots\subseteq X_n=X$ be a composition series of a module X. If $n=1$, then $X=X_1$ is (semi)simple, and so X is module with the property Rad_g by Corollary 2.3. Assume that this is true for each $k\le n-1$. Then X_{n-1} is a module with the property *Rad*$_g$. Since X_n/X_{n-1} is a (semi)simple module, X_n/X_{n-1} is a module with the property *Rad*$_g$. By Proposition 2.9 that $X=X_n$ is a module with the property *Rad*$_g$. The result follows from by induction.

Proposition 2.12. Let $T\subseteq X\subseteq S$ be modules with S/T injective. If X is a module with the property *Rad*$_g$, then so is X/T.

Proof: Suppose that N is an extension of X/T. Since S/T is injective, the next commutative diagram with exact rows can be obtained by [2, Lemma 2.16].

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r, "inc"] \arrow[d, "id"] & X \arrow[r, "\delta"] \arrow[d, "\gamma"] & X/T \arrow[r] \arrow[d, "inc"', "\alpha"] & 0 \arrow[d] \\
0 \arrow[r] & T \arrow[r] & L \arrow[r, "\beta"] & N \arrow[r] & 0
\end{tikzcd}$$

While γ is monic and X is a module with the property Rad_g, $Im(\gamma)$ has a $g-$radical supplement U in L. So $Im(\gamma)+U=L$ and $Im(\gamma)\cap U\subseteq Rad_g(U)$. Then we have

$$N=\beta(L)=\beta(\gamma(X))+\beta(U)=(\alpha\delta)(X)+\beta(U)=(X/T)+\beta(U)$$

and

$$(X/T)\cap\beta(U)=(\alpha\delta)(X)\cap\beta(U)=\beta[\gamma(B)\cap U]\subseteq\beta(Rad_g(U))\subseteq Rad_g(\beta(U)).$$

Recall from [15, Theorem 5.4] that a ring R is left hereditary if and only if each factor module of an injective $R-$module is injective, the result which we will give now is directly derived from the Proposition 2.12 if we choose N as an injective envelope $E(X)$ of X.

Corollary 2.13. Let R be a left hereditary ring. Then every factor module of an $R-$module with the property Rad_g is a module with the property Rad_g.

Recall from [16] that a ring R is a $SSI-$ring if and ony if every semisimple left $R-$module is injective.

Proposition 2.14. Let R be a ring with the property Rad_g – as an $R-$module is injective. Then R is a left $SSI-$ring.

Proof: Suppose that X is a semisimple $R-$module. Then X is a module with the property Rad_g by Corollary 2.2. By the hypothesis X is injective. Thus R is a left $SSI-$ring.

Using Proposition 2.14, we can characterize semisimple commutative rings via modules with the property Rad_g.

Corollary 2.15. Let R be a commutative ring. Then R is semisimple if and only if every left module with the property Rad_g is injective over the ring ${}_RR$.

Proof: $(\Rightarrow)$Since R is a semisimple commutative ring, then every $R-$module is injective.
$(\Leftarrow)$ R is a left $SSI-$ring by Proposition 2.14. Then R is semisimple by corollary of Proposition 1 in [16].

Proposition 2.16. Let X be a module. Then X has the property ample Rad_g if and only if each submodule of it is a module with the property Rad_g.

Proof: $(\Leftarrow)$ Assume that X is a module and $X\subseteq N$. Assume that for a submodule L of N, $L+X=N$. According to assumption, the submodule $L\cap X$ of X has a $g-$radical supplement U in L, that is $(L\cap X)+U=L$ and $(L\cap X)\cap U\subseteq Rad_g(U)$. Then we have

$$N=X+L=X+(L\cap X)+U=X+U$$

and

$$X \cap U = X \cap (U \cap L) = (L \cap X) \cap U \subseteq Rad_g(U).$$

Hence U is a $g-$radical supplement of X in N such that $U \subseteq L$.

$(\Rightarrow)$ Assume that K is a submodule of X and L is any submodule containing K. Then we have the pushout for inclusion homomorphisms $i_1 : K \to L$ and $i_2 : K \to X$.

$$\begin{tikzcd} X \arrow[r, "\theta_1"] & F \\ K \arrow[u, "i_2"] \arrow[r, "i_1"'] & L \arrow[u, "\gamma_1"'] \end{tikzcd}$$

In the diagram above, θ_1 and γ_1 are also monomorphisms by the properties of pushout. Let $X' = Im(\theta_1)$ and $L' = Im(\gamma_1)$. Then $F = X' + L'$ by the properties of pushout. So by the assumption, $X' \cong X$ has a $g-$radical supplement W in F such that $W \subseteq L'$. Hence $X' + W = F$ and $X' \cap W \subseteq Rad_g(W)$.

Since

$$L' = L' \cap F = L' \cap (X' + W) = (L' \cap X') + W$$

and

$$(X' \cap L') \cap W = X' \cap W \subseteq Rad_g(W)$$

therefore, W is a $g-$radical supplement of $X' \cap L'$ in L'. Then we obtain an isomorphism $\tilde{\gamma} : L \to L'$ defined as $\tilde{\gamma}(a) = \gamma_1(a)$ for all $a \in L$, because $\gamma_1 : L \to F$ is a monomorphism with $L' = Im(\gamma_1)$. Since W is a $g-$radical supplement of $X' \cap L'$ in L', we have $\tilde{\gamma}^{-1}(W)$ is a $g-$radical supplement of $\tilde{\gamma}^{-1}(X' \cap L')$ in $\tilde{\gamma}^{-1}(L')$ by this isomorphism. The result follows from $\tilde{\gamma}^{-1}(W) = \gamma_1^{-1}(W)$, $\tilde{\gamma}^{-1}(L') = L$ and $\tilde{\gamma}^{-1}(X' \cap L') = K$.

As a consequence of Proposition 2.16, we have the next result.

Corollary 2.17. Let X be a module with the property ample Rad_g. Then X has the property Rad_g. In addition, X is a $g-$radical supplemented module.

Using Theorem 21 [1], we can easily obtain the following proposition.

Proposition 2.18. Let R be a local Dedekind domain and X be an $R-$module. Then X is a module with the property Rad_g if and only if X is $Rad-$supplementing.

Corollary 2.19. Let X be a module with the property ample Rad_g over a local Dedekind domain. Then X is ample $Rad-$ supplementing.

Proof: Suppose that K is any submodule of X. By Proposition 2.16, K is a module with the property Rad_g. It follows from Proposition 2.18 that K is $Rad-$ supplementing. Thus X is ample $Rad-$ supplementing by Proposition 3.1 [2].

CONCLUSION

In [10], Zöschinger characterized two properties in module categories as the property (E) and the property (EE), and it was determined the modules with the property (E) over non-local Dedekind ring *R*. Then it was defined (amply) Rad-supplementing modules as a proper generalization of property ((EE)) (E) in [2]. Based on these two studies, we examined the important algebraic properties that the module with the property (ample) Rad_g provide by deriving from the notion g – supplement in [7]. So, modules with g – radical supplement in each extension are defined on the basis of existing concepts in the literature and some properties of these modules are examined.

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