ORIGINAL PAPER

STRONG CONSISTENCY RATE OF THE CONDITIONAL QUANTILE ESTIMATOR WITH DATA MISSING AT RANDOM

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Abstract. *In the case of response missing at random conditioned by a functional explanatory variable, this research investigates non parametric estimate of the conditional quantile using the kernel approach. Under an α-mixing assumption and properties of the small ball, we establish the uniform almost complete convergence (with rate) of the proposed estimator.*

Keywords: *conditional quantile; FDA; kernel estimator; missing at random; strong mixing.*

1. INTRODUCTION

One of the most encountered problems in non-parametric statistics is the question of forecasting. In some situations regression is the main tool adapted to answer this question. However, in other situations, such as the case where the conditional density is asymmetrical or multimodale, this tool is inadequate. Therefore, the conditional quantile better predicts the impact of the variable of interest Y on the explanatory variable X. The study [1] was the first to treat the conditional quantile estimate in the case of Markov processes. Stone established the convergence in probability of an estimator of the empirical conditional distribution function [2]. The uniform convergence of the kernel estimator of the conditional quantile and the asymptotic normality was obtained [3]. Another way to estimate conditional quantiles is the robust statistic; let us quote in this context, [4-5] for the asymptotic normality and quadratic mean convergence, by considering independent identically distributed (i.i.d.), α-mixing or β-mixing observations.

In the context of functional data, the first results were introduced by [6] where they studied the almost complete convergence of a kernel estimator of the conditional quantile in the case of i.i.d. This case has been generalized [7] to the α-mixing case and they applied their result to the climatic El Nino phenomenon. The articles [8, 9] studied the asymptotic normality of this estimator in both cases (i.i.d. or strong mixing conditions). The spline function method was introduced [10] for the conditional quantile estimator. In [11] it is considered the convergence in L^p-norm of nonparametric quantile regression under the mixing hypothesis.

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The aim of this paper is to contribute to the research on the functional non parametric model by studying the estimation of the conditional quantile when the response variable is missing at random conditioned by a functional explanatory variable and under α-mixing condition. Note that missing data at random are a kind of incomplete data, however, data are often incompletely observed and part of the responses are missing at random (MAR) in many cases, such as sampling survey, pharmaceutical tracing or reliability. The literature concerning MAR in the case of functional data is still limited, see among others [12-14].

The organization of the paper is as follow: the estimation model of the conditional distribution function and conditional quantile is presented in Section 2. Section 3 describes the assumptions necessary to demonstrate the main results found in Section 4. Finally, the proofs of the auxiliary results are given in section Appendix.

2. MODEL AND ESTIMATOR

Let $(X_i, Y_i)_{i=1,\ldots,n}$ be the statistical sample of pairs which are identically distributed like (X, Y), but dependent. Where Y is valued in $\mathbb{R}$ and X is valued in some infinite dimensional semi-metric vector space $(\mathcal{F}, d)$. Let x be fixed in $\mathcal{F}$, the conditional distribution function $F^x(y)$ of Y given $X = x$ is defined as follows:

$$F^x(y) = \mathbb{P}(Y \leq y | X = x).$$

The conditional quantile, of the order $\alpha \in (0,1)$, is defined by:

$$t_\alpha(x) = \inf\{y \in R: F^x(y) \geq \alpha\}.$$

$\forall x \in \mathcal{F}$, $F^x(y)$ admits a unique conditional quantile. Let $\alpha \in (0,1)$, the α th conditional quantile, denoted $t_\alpha(x)$, satisfies the following equation:

$$F^x(t_\alpha(x)) = \alpha. \tag{1}$$

In the case of complete data, the estimator of the conditional distribution function $F^x(y)$ of Y given $X = x$ is defined as follows:

$$\tilde{F}^x(y) = \frac{\sum_{i=1}^{n} K(h_K^{-1} d(x, X_i)) H(h_H^{-1}(y - Y_i))}{\sum_{i=1}^{n} K(h_K^{-1} d(x, X_i))}, \qquad \forall y \in \mathbb{R}. \tag{2}$$

where K is the kernel, H is a distribution function and $h_K = h_{K,n}$ (resp. $h_H = h_{H,n}$) is a sequence of positive real numbers such that $\lim_{n\to+\infty} h_K = \lim_{n\to+\infty} h_H = 0$.

Then, it naturally follows an estimator of conditional quantile $\widetilde{t_\alpha}$ defined as follows:

$$\widetilde{t_\alpha}(x) = \inf\{y \in \mathbb{R}: \tilde{F}^x(y) \geq \alpha\},$$

which satisfies

$$\tilde{F}^x(\widetilde{t_\alpha}(x)) = \alpha. \tag{3}$$

In the case of missing response, we consider a random sample of incomplete data $\{(X_i, Y_i, \delta_i)_{i=1,\ldots,n}\}$, when all the X_i are observed and $\delta_i = 1$ if Y_i is observed, and $\delta_i = 0$ otherwise. The MAR assumption implies that δ and Y are conditionally independent given X.

Such that

$$\mathbb{P}(\delta = 1|X = x, Y = y) = \mathbb{P}(\delta = 1|X = x) = p(x).$$

This assumption was introduced by [15]. The new functional estimator of $F^x(y)$ adapted to MAR response can be defined as:

$$\widehat{F}^x(y) = \frac{\sum_{i=1}^n \delta_i K(h_K^{-1} d(x,X_i)) H(h_H^{-1}(y-Y_i))}{\sum_{i=1}^n \delta_i K(h_K^{-1} d(x,X_i))}, \qquad \forall y \in \mathbb{R}. \tag{4}$$

In order to simplify the notations, we pose:

$$K_i(x) = K\left(h_K^{-1} d(x, X_i)\right) \, and \, H_i(y) = H(h_H^{-1}(y - Y_i)).$$

We can write

$$\widehat{F}^x(y) = \frac{\widehat{F}_N^x(y)}{\widehat{F}_D^x}$$

where

$$\widehat{F}_N^x(y) = \frac{1}{n\mathbb{E}K_1(x)} \sum_{i=1}^n \delta_i K_i(x) H_i(y) \tag{5}$$

and

$$\widehat{F}_D^x = \frac{1}{n\mathbb{E}K_1(x)} \sum_{i=1}^n \delta_i K_i(x). \tag{6}$$

Then, the new estimator of the conditional quantile can be defined as

$$\widehat{t_\alpha}(x) = \inf\{y \in \mathbb{R}: \widehat{F}^x(y) \geq \alpha\},$$

which satisfies

$$\widehat{F}^x(\widehat{t_\alpha}(x)) = \alpha. \tag{7}$$

We introduce the following notations in order to define the property of the strong mixture. Denote by $\mathcal{F}_1^k(Z)$ the σ-algebra generated by $(X_1, Y_1), \dots, (X_k, Y_k)$ and $\mathcal{F}_{k+n}^{\infty}$ generated by (X_{k+n}, Y_{k+n}).... Now, for any $n \geq 1$, we have

$$\alpha(n) = \sup_{k \geq 1} \{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|, A \in \mathcal{F}_1^k B \in \mathcal{F}_{k+n}^{\infty}\}. \tag{8}$$

The process $(X_n, Y_n)_{n \geq 1}$ is said to be strongly mixing if

$$\lim_{n \to \infty} \alpha(n) = 0. \tag{9}$$

3. HYPOTHESES

To formulate our assumptions for the almost complete convergence of $\widehat{F}^x(.)$, we consider the following hypotheses. Let N_x the fixed neighborhood of x and let $B(x, h) = \{x' \in \mathcal{F}/d(x, x') < h\}$ be the ball of center x and radius h.

(H1) $\forall \; h > 0, \; \mathbb{P}(X \in B(x, h)) = \phi_x(h) > 0.$

(H2) $(X_i, Y_i)_{i \in \mathbb{N}}$ is an α-mixing sequence whose coefficients of mixture satisfy:

$$\exists a > 0, \exists c > 0 : \forall n \in \mathbb{N}, \alpha(n) \leq c n^{-a}$$

(H3) $\sup_{i \neq j} \mathbb{P}\left((X_i, X_j) \in B(x,h) \times B(x,h)\right) = O\left(\frac{\left(n^{-1}\phi_x(h)\right)^{\frac{(a+1)}{a}}}{n^{\frac{1}{a}}}\right)$.

(H4) The conditional distribution function $F^x(y)$ is differentiable, continuous and it has a first derivative uniformly bounded, denoted $f^x(y)$ and satisfies: $\forall (y_1, y_2) \in \mathbb{R} \times \mathbb{R}$, $\forall (x_1, x_2) \in N_{x_1} \times N_{x_2}$, there exist some constants $C, b_1, b_2 > 0$, such that, for $j = 0,1$, we have

$$|F^{x_1^{(j)}}(y_1) - F^{x_2^{(j)}}(y_2)| \leq C(d(x_1, x_2)^{b_1} + |y_1 - y_2|^{b_2}).$$

(H5) K is a function with support (0,1) such that $0 < C_1 \mathbb{I}_{[0,1]} < K(t) < C_2 \mathbb{I}_{[0,1]} < \infty$,where $\mathbb{I}_A$ is the indicator function and $\mathbb{E}K_1(x)$ imply that $0 < C_1 \phi_x(h_K) < \mathbb{E}K_1(x) < C_2 \phi_x(h_K)$.

(H6) $\forall (y_1, y_2) \in \mathbb{R}^2$, $|H(y_1) - H(y_2)| \leq C|y_1 - y_2|$ and $\int |t|^{\beta_2} H^{(1)}(t) dt < \infty$.

(H7) $p(x)$ is continuous in a neighbourhood of x, such that $0 < p(x) < 1$.

(H8) $\lim_{n \to +\infty} n^b h_H = \infty$, $\forall\ b > 0$ and $\lim_{n \to +\infty} \frac{\log n}{n\phi(h_K)} = 0$.

4. MAIN RESULTS

Proposition 1. Under assumptions $(H1) - (H8)$, and if

$$\exists \eta > 0, C n^{\frac{3-a}{a+1}+\eta} \leq \phi_x(h_K) \leq C' n^{\frac{1}{1-a}} \tag{10}$$

holds with $a > \frac{5+\sqrt{17}}{2}$, we get

$$\sup_{y \in S} |\widehat{F}^x(y) - F^x(y)| = O_{a.s}\left(h_K^{b_1} + h_H^{b_2}\right) + O_{a.s}\left(\left(\frac{\log n}{n\phi(h_K)}\right)^{\frac{1}{2}}\right).$$

Theorem 1. Under hypotheses $(H1)$-$(H8)$, we have

$$|\widehat{t_\alpha}(x) - t_\alpha(x)| = O_{a.s}\left(h_K^{b_1} + h_H^{b_2}\right) + O_{a.s}\left(\left(\frac{\log n}{n\phi(h_K)}\right)^{\frac{1}{2}}\right).$$

Proof of Proposition 1: The proof of proposition 1 is based on the following decomposition and lemmas below:

$$\widehat{F}^x(y) - F^x(y) = \frac{1}{\widehat{F}_D^x}\left\{\left(\widehat{F}_N^x(y) - \mathbb{E}\widehat{F}_N^x(y)\right) - \left(F^x(y) - \mathbb{E}\widehat{F}_N^x(y)\right)\right\} + \frac{F^x(y)}{\widehat{F}_D^x}\left\{\widehat{F}_D^x - \mathbb{E}\widehat{F}_D^x\right\} \tag{11}$$

Lemma 1. Under conditions of proposition 1, we have

$$\sup_{y\in S}|F^x(y)-\mathbb{E}[\hat{F}_N^x(y)]| = O\left(h_K^{b_1}\right)+O\left(h_H^{b_2}\right), \quad a.co.$$

Lemma 2. Under assumptions $(H1)-(H8)$, we get

$$\sup_{y\in S}\left|\hat{F}_N^x(y)-\mathbb{E}\hat{F}_N^x(y)\right| = O\left(\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right), a.co. \tag{12}$$

Lemma 3. Assume that hypotheses $(H1)-(H8)$ and condition (10) are satisfied, we have

$$|\hat{F}_D^x-\mathbb{E}\hat{F}_D^x| = O\left(\left(\frac{\log n}{n\phi_x(h_K)}\right)^{\frac{1}{2}}\right), \quad a.co.$$

and

$$\sum_n \mathbb{P}\left(\hat{F}_D^x<1/2\right)<\infty.$$

Lemma 4. Under the conditions of Theorem 1, we have

$$\widehat{t_\alpha}(x)\longrightarrow t_\alpha(x). a.co. \tag{13}$$

4. APPENDIX

Proof of Lemma 1: We have

$$\begin{aligned}
|\mathbb{E}[\hat{F}_N^x(y)]-F^x(y)| &= \mathbb{E}\left[\frac{1}{n\mathbb{E}K_1}\sum_{i=1}^n \delta_i K_i H\left(\frac{y-Y_i}{h_H}\right)\right]-F^x(y)\\
&= \frac{1}{n\mathbb{E}K_1}\sum_{i=1}^n \mathbb{E}\left[\delta_i K_i H\left(\frac{y-Y_i}{h_H}\right)\right]-F^x(y)\\
&= \frac{1}{n\,\mathbb{E}K_1}\sum_{i=1}^n \mathbb{E}\left(\mathbb{E}\left[\delta_i K_i H\left(\frac{y-Y_i}{h_H}\right)|X_i\right]\right)-F^x(y)\\
&= \frac{1}{\mathbb{E}K_1}\mathbb{E}\left(p(x)K_1\mathbb{E}\left[H\left(\frac{y-Y_i}{h_H}\right)\right]\right)-F^x(y).
\end{aligned}$$

After integration by parts and a variables change, we obtain

$$\begin{aligned}
\mathbb{E}(H_1(h_H^{-1}(y-Y_i))/X) &= \int_{\mathbb{R}} H\left(\frac{y-u}{h_H}\right)f^X(u)du\\
&= \int_{\mathbb{R}} H^{(1)}(t)F^X(y-h_Ht)dt.
\end{aligned}$$

Thus, under conditions $(H4),(H6)$ and $(H7)$, we have

$$\begin{aligned}
|\mathbb{E}[\hat{F}_N^x(y)]-F^x(y)| &= \frac{1}{\mathbb{E}K_1}\mathbb{E}\left\{p(x)K_1\int_{\mathbb{R}} H^{(1)}(t)|F^X(y-h_Ht)-F^x(y)|dt\right\}\\
&\le C(h_K^{b_1}+h_H^{b_2})(p(x)+o(1))\\
&= O_{a.co}(h_K^{b_1}+h_H^{b_2}).
\end{aligned}$$

Finally, the proof is achieved.

Proof of Lemma 2: The idea is to cover the compact $S = \{t_\alpha(x) - \epsilon, t_\alpha(x) + \epsilon\}$ by intervals of S_k, we can write that $S \subset \cup_{k=1}^{s_n} S_k$, where $S_k = (t_k - l_n, t_k + l_n)$, with $l_n = n^{-\beta-\frac{1}{2}}$. We put $m_y = arg\min_{k\in\{1,\dots,s_n\}} |y - m_k|$ and we show that:

$$\sup_{y\in S}\left|\hat{F}_N^x(y) - \mathbb{E}F_N^x(y)\right| \leq \underbrace{\sup_{y\in S}\left|\hat{F}_N^x(y) - \hat{F}_N^x(m_y)\right|}_{T_1} + \underbrace{\sup_{y\in S}\left|\hat{F}_N^x(m_y) - \mathbb{E}\hat{F}_N^x(m_y)\right|}_{T_2}$$
$$+ \underbrace{\sup_{y\in S}\left|\mathbb{E}\hat{F}_N^x(m_y) - \mathbb{E}F_N^x(y)\right|}_{T_3}.$$

Concerning T_1 The hypothesis $(H6)$ lead

$$\begin{aligned} T_1 &\leq \frac{1}{n\mathbb{E}K_1(x)} \sum_{i=1}^n \delta_i K_i(x) |H_i(y) - H_i(t_y)| \\ &\leq \frac{1}{n\mathbb{E}K_1(x)} \sup_{y\in S} |y - t_y| \sum_{i=1}^n \delta_i \frac{K_i(x)}{h_H} \\ &\leq \hat{F}_D^x \sup_{y\in S} \frac{|y - t_y|}{h_H} \\ &\leq A \frac{l_n}{h_H}. \end{aligned}$$

We take $l_n = n^{-\beta-\frac{1}{2}}$ to show that

$$l_n/h_H = o\left(\sqrt{\log n (n\phi_x(h_K))^{-1}}\right),$$

finally, we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left(T_1 > \epsilon\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right) < \infty. \tag{14}$$

Concerning T_3, using analogous arguments as for T_1, we have

$$\begin{aligned} T_3 &\leq \frac{1}{n\mathbb{E}\Delta_1(x)} \sum_{i=1}^n \mathbb{E}[\delta_i \Delta_i(x) |H_i(y) - H_i(t_y)| |\mathrm{X}_i] \\ &\leq A\frac{l_n}{h_H} \longrightarrow 0 \quad a.co. \quad as \quad n \to \infty. \end{aligned}$$

So, for n large enough, we can found $\epsilon > 0$ such that

$$\sum_{n=1}^{\infty} \mathbb{P}\left(T_3 > \epsilon\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right) < \infty. \tag{15}$$

However, for T_2, we have

$$\mathbb{P}\left(\sup_{y\in S}\left|\hat{F}_N^x(m_y) - \mathbb{E}\hat{F}_N^x(m_y)\right| > \epsilon\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right) \leq$$
$$s_n \max_{k\in\{1,\dots,s_n\}} \mathbb{P}\left(\left|\hat{F}_N^x(m_y) - \mathbb{E}\hat{F}_N^x(m_y)\right| > \epsilon\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right).$$

On the other hand, we have

$$\left|\widehat{F}_N^x(m_y) - \mathbb{E}\widehat{F}_N^x(m_y)\right| = \frac{1}{n\,\mathbb{E}K_1}\sum_{i=1}^{n} \underbrace{\delta_i K_i H_i(m_y) - \mathbb{E}\big(\delta_i K_i H_i(m_y)\big)}_{\Delta_i},$$

So, to apply Fuck-Nagaev's exponential inequality (see [16]), we calculate

$$\begin{aligned} s_n'^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} \left|Cov(\Delta_i,\Delta_j)\right|. \\ &= \sum_{i\neq j} Cov(\Delta_i,\Delta_j) + \sum_{i=1}^{n} \mathrm{Var}(\Delta_i) \\ &= s_n^{2*} + n\mathrm{Var}(\Delta_1). \end{aligned} \tag{16}$$

Firstly, by the same technique used in [17], we define the sets

$$E_1 = \{(i,j) \;\; such\ that \;\; 1 \le i-j \le u_n\},$$

and

$$E_2 = \{(i,j) \;\; such\ that \;\; u_n + 1 \le i-j \le n-1\}.$$

where u_n is an arbitrary sequence of positive integer. Let $J_{1,n}$ and $J_{2,n}$ be the sums of covariances over E_1 and E_2 respectively. Then

$$J_{1,n} = \sum_{E_1} Cov(\Delta_i,\Delta_j) \le$$

$$\sum_{E_1} \left|\mathbb{E}[\delta_i\delta_j K_i(x)K_j(x)H_i(m_y)H_j(m_y)] - \mathbb{E}[\delta_i K_i(x)H_i(m_y)]\mathbb{E}[\delta_j K_j(x)H_j(m_y)]\right|.$$

Under $(H1), (H3), (H5), (H7)$ and because H is a cumulative kernel $\big(H_i(m_y) \le 1\big)$, we obtain

$$J_{1,n} \le Cnu_n\phi_x(h)\left(\left(\frac{\phi_x(h)}{n}\right)^{1/a} + \phi_x(h)\right).$$

For E_2, the covariance can be controlled by means of the usual Davydov's covariance inequality for mixing processes (see [16])and condition $(H2)$. This inequality leads to:

$$\forall i \neq j, |Cov(\Delta_i,\Delta_j)| \le C(|i-j|)^{-a}.$$

Thus,

$$J_{2,n} = \sum_{E_2} |Cov(\Delta_i,\Delta_j)| \le n^2 u_n^{-a}.$$

By choosing $u_n = \left(\frac{\phi_x(h_k)}{n}\right)^{-1/a}$ and under condition (10), we get

$$s_n^{2*} = J_{1,n} + J_{2,n} = O(n\phi_x(h_K)). \tag{17}$$

Concerning the variance term, we deduce from $(H1)$ that

$$\mathrm{Var}(\Delta_1(x)) = O(\phi_x(h_K)). \tag{18}$$

Finally, from (16), (17) and (18) we obtain

$$s_n'^2 = O(n\phi_x(h_K)).$$

Now, we can apply the inequality of Fuck-Nagaev:

$$\mathbb{P}\left(\left|\hat{F}_N^x(m_y)-\mathbb{E}\hat{F}_N^x(m_y)\right|>\epsilon_0\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right)$$
$$\leq 4\left(1+\frac{\epsilon_0^2 n^2\mathbb{E}^2K_1\frac{\log n}{n\phi_x(h_K)}}{16rs_n'^2}\right)^{\frac{-r}{2}}+2ncr^{-1}\left(\frac{8r}{\epsilon_0 n\mathbb{E}K_1\sqrt{\frac{\log n}{n\phi_x(h_K)}}}\right)^{a+1}$$

We put $r=C(\log n)^2$,

$$\mathbb{P}\left(\left|\hat{F}_N^x(m_y)-\ \mathbb{E}\hat{F}_N^x(m_y)\right|>\epsilon_0\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right),$$
$$\leq An^{\frac{-\epsilon_0^2}{32}}+A(\log n)^{2a-\frac{a+1}{2}}n^{1-\frac{(a+1)}{2}}n^{\frac{-(a+1)}{2}(\frac{3-a+\eta}{a+1})}.$$

Therefore, for ϵ_0 large enough and $\nu>0$, we have

$$\mathbb{P}\left(\sup_{y\in S}\left|\hat{F}_N^x(m_y)-\mathbb{E}\hat{F}_N^x(m_y)\right|>\epsilon\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right)$$
$$\leq Al_n^{-1}\left(n^{\frac{-\varepsilon^2}{32}}+n^{-1-\frac{a+1}{2}\eta}\right)$$
$$\leq An^{-1-\nu}.$$

Finally, the proof is achieved.

Proof of Lemma 3: For the demonstration of the first part of this lemma we use the same arguments as the previous lemma, the only change is for Δ_i, where:

$$\hat{F}_D^x-\mathbb{E}\hat{F}_D^x=\frac{1}{n\,\mathbb{E}K_1}\sum_{i=1}^n\Lambda_i.$$

with $\Lambda_i(x)=\delta_iK_i-\mathbb{E}(\delta_iK_i)$.

All the calculus previously made with the variables $\Delta_i(x)$ remain valid with the variables $\Lambda_i(x)$ and we obtain:

$$\mathbb{P}\left(\left|\hat{F}_D^x-\mathbb{E}\hat{F}_D^x\right|>\epsilon\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right)$$
$$\leq 4\left(1+\frac{\epsilon_0^2n^2\ \ \mathbb{E}^2K_1\frac{\log n}{n\phi_x(h_K)}}{16rs_n'^2}\right)^{\frac{-r}{2}}+2ncr^{-1}\left(\frac{8r}{\epsilon_0 n\ \ \mathbb{E}\,K_1\sqrt{\frac{\log n}{n\phi_x(h_K)}}}\right)^{a+1}$$
$$\leq An^{\frac{-\epsilon^2}{32}}+A(\log n)^{2a-\frac{a+1}{2}}n^{1-\frac{(a+1)}{2}}n^{\frac{-(a+1)}{2}(\frac{3+a}{a+1}+\eta)}$$
$$\leq An^{\frac{-\epsilon^2}{32}}+An^{2a-\frac{a+1}{2}}n^{1-\frac{(a+1)}{2}}n^{\frac{-(a+1)}{2}(\frac{3+a}{a+1}+\eta)}$$
$$\leq An^{\frac{-\epsilon^2}{32}}+An^{-1-(\frac{1-a}{2}+\frac{a+1}{2}\eta)}$$

For ϵ large enough and $\nu > 0$, we leads to,

$$\mathbb{P}\left(\left|\widehat{F}_D^x - \mathbb{E}\widehat{F}_D^x\right| > \varepsilon\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right) \leq A' n^{-1-\nu}. \tag{19}$$

Finally,

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left|\widehat{F}_D^x - \mathbb{E}\widehat{F}_D^x\right| > \varepsilon\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right) \leq \sum_{n=1}^{\infty} A' n^{-1-\nu} < \infty.$$

Concerning the second part, we have

$$\{|\widehat{F}_D^x| \leq 1/2\} \subseteq \{|\widehat{F}_D^x - p(x)| > 1/2\}$$

then

$$\begin{aligned} \mathbb{P}\{|\widehat{F}_D^x| \leq 1/2\} &\leq \mathbb{P}\{|\widehat{F}_D^x - p(x)| > 1/2\} \\ &\leq \mathbb{P}\{|\widehat{F}_D^x - \mathbb{E}\widehat{F}_D^x| > 1/2\}, (\text{because}\, \mathbb{E}\widehat{F}_D^x = p(x)). \end{aligned}$$

We come to show that

$$\sum_{n=1}^{\infty} \mathbb{P}(\widehat{F}_D^x < 1/2) < \infty.$$

Proof of Lemma 4: $F(y|x)$ is a continuous and strictly increasing function that admits a unique quantile of order α, So, we have:

$$\forall \epsilon > 0 \ \ \exists \ \delta(\epsilon) > 0, \ \forall y \in \mathbb{R}, |t_\alpha(x) - y| \geq \epsilon \Rightarrow |F^x(t_\alpha(x)) - F^x(y)| \geq \delta(\epsilon).$$

Under equations (1) and (7), we obtain

$$\begin{aligned} \exists \ \delta(\epsilon) > 0, \mathbb{P}(|\hat{t}_\alpha(x) - t_\alpha(x)| > \epsilon) &\leq \mathbb{P}(|\widehat{F}^x(\hat{t}_\alpha(x)) - \widehat{F}^x(t_\alpha(x))| > \delta(\epsilon)) \\ &= \mathbb{P}(|F^x(t_\alpha(x)) - \widehat{F}^x(t_\alpha(x))| > \delta(\epsilon)) \\ &\leq \sup_{y\in S}|\widehat{F}^x(y) - F^x(y)|. \end{aligned}$$

Since $F^x(y)$ is continuously differentiable, we have

$$\sum_n \mathbb{P}(|\hat{t}_\alpha(x) - t_\alpha(x)| > \epsilon) \leq \sum_n \mathbb{P}\left(\sup_{y\in S}|\widehat{F}^x(y) - F^x(y)| > \delta(\epsilon)\right) < \infty.$$

which complete the proof.

Proof of Theorem 1: Writing a Taylor expansion of order one of the function $\widehat{F}^x$ at point $\widehat{t_\alpha}(x)$ leads to the existence of some $t_\alpha{}^*(x)$ betweens $\widehat{t_\alpha}(x)$ and $t_\alpha(x)$ and taking into account Lemma 4 such that

$$\begin{aligned} \widehat{F}^x(\widehat{t_\alpha}(x)) - \widehat{F}^x(t_\alpha(x)) &= (\widehat{t_\alpha}(x) - t_\alpha(x))\widehat{F}^{x^{(1)}}(t_\alpha^*(x)) \\ |\widehat{t_\alpha}(x) - t_\alpha(x)| \quad &= \frac{1}{\widehat{F}^{x^{(1)}}(t_\alpha^*(x))}[|\widehat{F}^x(t_\alpha(x)) - F^x(t_\alpha(x))|]. \end{aligned}$$

Then,

$$|\widehat{t_\alpha}(x) - t_\alpha(x)|\widehat{F}^{x^{(1)}}(t_\alpha^*(x)) = O_{a.co.}\left(\left|\widehat{F}^x(t_\alpha(x)) - F^x(t_\alpha(x))\right|\right)$$

The result provided by Lemma 4 insures that $t_\alpha{}^*(x)$ tends to $t_\alpha(x)$ as it follows that

$$|\widehat{t_\alpha}(x) - t_\alpha(x)| = O_{a.co.}\left(\left|\hat{F}^x(t_\alpha(x)) - F^x(t_\alpha(x))\right|\right).$$

So, the result of the Theorem is an easy consequence of Proposition 1.

5. CONCLUSION

In this paper, we have established under certain mild conditions the almost complete convergence of the conditional quantile estimator. In the case of missing data at random and α-mixing structure our estimator has good asymptotic properties. The k-NN method is a smoothing method that includes an adaptive estimator. A very important feature of this approach is that it allows building a neighborhood that adapts to the local structure of the data. It is also interesting to investigate the asymptotic properties of the k-NN estimator of the conditional quantile function.

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