ORIGINAL PAPER

NEW CHARACTERIZATIONS FOR SPHERICAL INDICATRICES OF INVOLUTES OF A SPACELIKE CURVE WITH A TIMELIKE BINORMAL IN MINKOWSKI 3-SPACE

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Abstract. *In this paper, we study the spherical indicatrices of involutes of a spacelike curve with spacelike binormal. Then we give some important relationships between arc lengths and geodesic curvatures of the spherical indicatrices of involute-evolute curve couple in Minkowski 3-space. Also, we give some important results about curve couple.*

Keywords: *involute-evolute curve couple; spherical indicatrix, geodesic curvature; Minkowski space.*

1. INTRODUCTION

One of the topics of interest to researchers is the involute of a curve in classical differential geometry. The notion of involute was firstly discovered by Huygens in the 17th century while trying to develop a more accurate clock. Involute of a curve is a well-known concept in the theory of curves, see for detailed information [1, 2].

In [3], authors expressed the relations between Frenet frames of the involute-evolute curve couple (β, α) depending on the curvatures of the evolute curve α. Bilici and Çalışkan [4] found the relations between the Frenet frames of involute-evolute curve couple by using the Lorentzian timelike angle θ between the binormal vector $\boldsymbol{B}$ and the Darbux vector $\boldsymbol{W}$ of the spacelike evolute curve α. This angular approach was the first step in making some geometric calculations and obtaining some important results. Recently Bilici and Çalışkan [5, 6] have computed arc lengths and geodesic curvatures of spherical indicatrices of the involutes of a given timelike curve and spacelike curve with a spacelike binormal in Minkowski 3-space.

In this study, Frenet-Serret frames of the curves α and β are $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ and $\{\boldsymbol{T}^*, \boldsymbol{N}^*, \boldsymbol{B}^*\}$, respectively. More specifically, the causal characteristics of the Frenet frames of the curves α and β are $\{\boldsymbol{T}$ spacelike, $\boldsymbol{N}$ spacelike, $\boldsymbol{B}$ timelike$\}$ and $\{\boldsymbol{T}^*$ spacelike, $\boldsymbol{N}^*$ spacelike (timelike), $\boldsymbol{B}^*$ timelike (spacelike)$\}$. Then we carry tangents of the spacelike involute with a timelike (or spacelike) binormal to the center of the pseudosphere $\boldsymbol{S}_1^2$ and we obtain a spacelike curve $(\boldsymbol{T}^*)$ with equation $\beta_{\boldsymbol{T}^*} = \boldsymbol{T}^*$ on the pseudohyperbolic space $\boldsymbol{H}_0^2$. This curve is called the first spherical indicatrix or tangent indicatrix of the involute curve. Similarly one consider the principal normal indicatrix $(\boldsymbol{N}^*)$ and the binormal indicatrix $(\boldsymbol{B}^*)$ on the $\boldsymbol{H}_0^2$ or $\boldsymbol{S}_1^2$. Then we give some important relationships between arc lengths and geodesic curvatures of the base curve and its involutes on $\boldsymbol{E}_1^3$, $\boldsymbol{S}_1^2$, $\boldsymbol{H}_0^2$. Additionally, some important results concerning curve couple (β, α) are given.

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2. MATERIALS AND METHODS

In this section we give some basic consepts and definitions to understand the main subject of the study. Minkowski 3-space $\boldsymbol{E}_1^3$ is the real vector space $\boldsymbol{E}^3$ with the standart metric given by

$$g(\ ,\ ) = -dx^2 + dy^2 + dz^2,$$

where $X = (x, y, z)$ is a rectangular coordinate system of $\boldsymbol{E}^3$. The vector $\boldsymbol{A} \in \boldsymbol{E}^3$ is called spacelike, timelike or lightlike if $g(\boldsymbol{A}, \boldsymbol{A}) > 0$, $g(\boldsymbol{A}, \boldsymbol{A}) < 0$, $g(\boldsymbol{A}, \boldsymbol{A}) = 0$, respectively and the norm of the vector $\boldsymbol{A} \in \boldsymbol{E}_1^3$ is given by $\|\boldsymbol{A}\| = \sqrt{|g(\boldsymbol{A}, \boldsymbol{A})|}$. For any $\boldsymbol{A} = (a_1, a_2, a_3)$, $\boldsymbol{B} =$ *b1,b2,b3∈E13*, we define the Lorentzian vector product as follows:

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{bmatrix} -e_1 & -e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = (a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_1b_2 - a_2b_1), [7].$$

Let's denote by $\{\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}(s)\}$ the moving Frenet frame along the curve α with curvature $\boldsymbol{\kappa}$ and torsion $\boldsymbol{\tau}$. If the curve α is non-unit speed curve in space $\boldsymbol{E}_1^3$, then

$$\boldsymbol{\kappa} = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3},\ \boldsymbol{\tau} = \frac{det(\alpha' \times \alpha'' \times \alpha''')}{\|\alpha' \times \alpha''\|^2}.$$

From [4, 8], depending on the causal character of the curve α, we have the following Frenet formulae and Darboux vectors:

Case 1) If α is a unit speed spacelike space curve with a timelike binormal then we can write the following expressions.

$$\boldsymbol{T} \times \boldsymbol{N} = \boldsymbol{B},\ \boldsymbol{N} \times \boldsymbol{B} = -\boldsymbol{T},\ \boldsymbol{B} \times \boldsymbol{T} = -\boldsymbol{N}$$

$$\begin{cases} \boldsymbol{T}' = \boldsymbol{\kappa N}, \quad \boldsymbol{N}' = -\boldsymbol{\kappa T} + \boldsymbol{\tau B}, \quad \boldsymbol{B}' = \boldsymbol{\tau N}, \\ g(\boldsymbol{B}, \boldsymbol{B}) = -1, g(\boldsymbol{T}, \boldsymbol{T}) = g(\boldsymbol{N}, \boldsymbol{N}) = 1, g(\boldsymbol{T}, \boldsymbol{N}) = g(\boldsymbol{T}, \boldsymbol{B}) = g(\boldsymbol{N}, \boldsymbol{B}) = 0. \end{cases}$$

and the Darboux vector is defined by

$$\boldsymbol{W} = \boldsymbol{\tau T} - \boldsymbol{\kappa B}.$$

There are two cases corresponding to the causal characteristic of Darboux vector $\boldsymbol{W}$.
i. If $|\boldsymbol{\tau}| > |\boldsymbol{\kappa}|$, then $\boldsymbol{W}$ is a spacelike vector then we can write

$$\begin{cases} \boldsymbol{\kappa} = \|\boldsymbol{W}\| \boldsymbol{sinh\theta} \\ \boldsymbol{\tau} = \|\boldsymbol{W}\| \boldsymbol{cosh\theta} \end{cases}$$

and the unit vector $\boldsymbol{C}$ of direction $\boldsymbol{W}$ is

$$\boldsymbol{C} = cosh\theta \boldsymbol{T} - sinhh\theta \boldsymbol{B},$$

where θ is the Lorentzian timelike angle between $-\boldsymbol{B}$ and unit timelike vector $\boldsymbol{C}'$ that Lorentz orthogonal to the normalisation of the Darboux vector $\boldsymbol{C}$ (see Fig. 1).

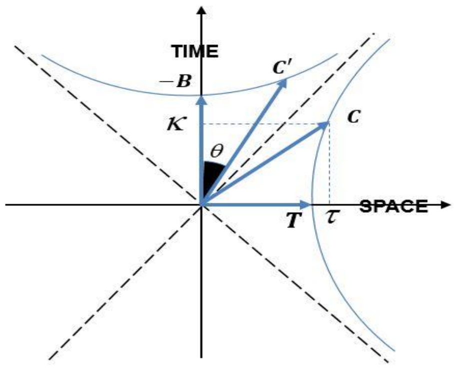

Figure 1. Lorentzian timelike angle θ.

ii. If $|\boldsymbol{\tau}| < |\boldsymbol{\kappa}|$, then $\boldsymbol{W}$ is a timelike vector. In this situation, we have

$$\begin{cases} \boldsymbol{\kappa} = \|\boldsymbol{W}\| cosh\theta \\ \boldsymbol{\tau} = \|\boldsymbol{W}\| sinh\theta \end{cases}$$

and the unit vector $\boldsymbol{C}$ of direction $\boldsymbol{W}$ is

$$\boldsymbol{C} = sinh\theta\boldsymbol{T} - cosh\theta\boldsymbol{B}.$$

Case 2) If α is a unit speed spacelike space curve with a spacelike binormal then we can write the following expressions.

$$\boldsymbol{T} \times \boldsymbol{N} = -\boldsymbol{B},\ \boldsymbol{N} \times \boldsymbol{B} = -\boldsymbol{T},\ \boldsymbol{B} \times \boldsymbol{T} = \boldsymbol{N}$$

$$\begin{cases} \boldsymbol{T}' = \boldsymbol{\kappa N}, \qquad \boldsymbol{N}' = \boldsymbol{\kappa T} + \boldsymbol{\tau B}, \qquad \boldsymbol{B}' = \boldsymbol{\tau N}, \\ g(\boldsymbol{N},\boldsymbol{N}) = -\boldsymbol{1}, g(\boldsymbol{T},\boldsymbol{T}) = g(\boldsymbol{B},\boldsymbol{B}) = 1, g(\boldsymbol{T},\boldsymbol{N}) = g(\boldsymbol{T},\boldsymbol{B}) = g(\boldsymbol{N},\boldsymbol{B}) = 0. \end{cases}$$

and the Darboux vector is defined by

$$\boldsymbol{W} = -\boldsymbol{\tau T} + \boldsymbol{\kappa B}.$$

In this situation, $\boldsymbol{W}$ is a spacelike vector then we get

$$\begin{cases} \boldsymbol{\kappa} = \|\boldsymbol{W}\| cos\theta \\ \boldsymbol{\tau} = \|\boldsymbol{W}\| sin\theta \end{cases}$$

and the unit vector $\boldsymbol{C}$ of direction $\boldsymbol{W}$ is

$$\boldsymbol{C} = -sin\theta\boldsymbol{T} + cos\theta\boldsymbol{B}.$$

Remark 1. From Case 1) and Case 2), we says that θ is a constant angle then α is a general helix.

For the arc length of the spherical indicatrix of $(\boldsymbol{C})$ we get

$$s_{\boldsymbol{C}} = \int_0^s |\theta'| ds.$$

After some calculations, we have for the arc lenghts of the spherical indicatrices $(\boldsymbol{T}), (\boldsymbol{N}), (\boldsymbol{B})$ measured from the points corresponding to $s = 0$

$$s_{\boldsymbol{T}} = \int_0^s |\boldsymbol{\kappa}| ds, \quad s_{\boldsymbol{N}} = \int_0^s \|\boldsymbol{W}\| ds, \quad s_{\boldsymbol{B}} = \int_0^s |\boldsymbol{\tau}| ds$$

for their geodesic curvatures with respest to $\boldsymbol{E}_1^3$

$$\boldsymbol{k_T} = \begin{cases} \dfrac{1}{sinh\theta}, \boldsymbol{W}\ spacelike \\ \dfrac{1}{cosh\theta}, \boldsymbol{W}\ timelike \end{cases}$$

$$\boldsymbol{k_N} = \begin{cases} \dfrac{1}{\|\boldsymbol{W}\|}\sqrt{|\theta'^2 + \|\boldsymbol{W}\|^2|}, \boldsymbol{W}\ spacelike \\ \dfrac{1}{\|\boldsymbol{W}\|}\sqrt{|-\theta'^2 + \|\boldsymbol{W}\|^2|}, \boldsymbol{W}\ timelike \end{cases}$$

$$\boldsymbol{k_B} = \begin{cases} \dfrac{1}{cosh\theta}, \boldsymbol{W}\ spacelike \\ \dfrac{1}{sinh\theta}, \boldsymbol{W}\ timelike \end{cases}$$

and for their geodesic curvatures with respest to $\boldsymbol{S}_1^2$ or $\boldsymbol{H}_0^2$

$$\boldsymbol{g_T} = \left\|\overline{\boldsymbol{\nabla}}_{\boldsymbol{t_T}}\boldsymbol{t_T}\right\| = \begin{cases} coth\theta, \boldsymbol{W}\ spacelike \\ tanh\theta, \boldsymbol{W}\ timelike \end{cases},$$

$$\boldsymbol{g_N} = \left\|\overline{\boldsymbol{\nabla}}_{\boldsymbol{t_N}}\boldsymbol{t_N}\right\| = \frac{\theta'}{\|\boldsymbol{W}\|}$$

$$\boldsymbol{g_B} = \left\|\overline{\overline{\boldsymbol{\nabla}}}_{\boldsymbol{t_B}}\boldsymbol{t_B}\right\| = \begin{cases} tanh\theta, \boldsymbol{W}\ spacelike \\ coth\theta, \boldsymbol{W}\ timelike \end{cases}, \text{[9]}.$$

Note that $\overline{\boldsymbol{\nabla}}$ and $\overline{\overline{\boldsymbol{\nabla}}}$ are Levi-Civita connections on $\boldsymbol{S}_1^2$ and $\boldsymbol{H}_0^2$, respectively. Then Gauss equations are given by the followings:

$$\boldsymbol{\nabla}_X Y = \overline{\boldsymbol{\nabla}}_X Y + \varepsilon g(\boldsymbol{S}(X), Y)\boldsymbol{\xi}, \quad \boldsymbol{\nabla}_X Y = \overline{\overline{\boldsymbol{\nabla}}}_X Y + \varepsilon g(\boldsymbol{S}(X), Y)\boldsymbol{\xi}, \varepsilon = g(\boldsymbol{\xi}, \boldsymbol{\xi}),$$

where $\boldsymbol{\xi}$ is a unit normal vector field and S is the shape operator of $\boldsymbol{S}_1^2$ (or $\boldsymbol{H}_0^2$).

The unit pseudosphere and pseudohyperbolic space of radius 1and center 0 in $\boldsymbol{E}_1^3$ are given by

$$\boldsymbol{S}_1^2 = \left\{X = \left(x_1, x_{2,} x_3\right) \in \boldsymbol{E}_1^3 : g(X, X) = 1\right\}$$

and

$$\boldsymbol{H}_0^2 = \left\{X = \left(x_1, x_{2,} x_3\right) \in \boldsymbol{E}_1^3 : g(X, X) = -1\right\}$$

respectively, [10].

Definition 1. Let $\alpha = \alpha(s), \beta = \beta(s) \in \boldsymbol{E}_1^3$ be two curves. Let Frenet frames of α and β be $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ and $\{\boldsymbol{T}^*, \boldsymbol{N}^*, \boldsymbol{B}^*\}$, respectively. β is called the involute of α (α is called the evolute of β) if

$$g(\boldsymbol{T}, \boldsymbol{T}^*) = 0.$$

It's noticed that α is a spacelike curve with a timelike binormal then its involute curve β must be a spacelike curve with a spacelike or timelike binormal.

Lemma 1. Let (β, α) be involute-evolute curve couple. The relations between the Frenet vectors of the curve couple as follow:
i. If $\boldsymbol{W}$ is a spacelike vector $(|\boldsymbol{\kappa}| > |\boldsymbol{\tau}|)$, then

$$\begin{bmatrix} \boldsymbol{T}^* \\ \boldsymbol{N}^* \\ \boldsymbol{B}^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ sinh\theta & 0 & -cosh\theta \\ -cosh\theta & 0 & sinh\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{bmatrix}.$$

ii. If $\boldsymbol{W}$ is a timelike vector $(|\boldsymbol{\kappa}| < |\boldsymbol{\tau}|)$, then

$$\begin{bmatrix} \boldsymbol{T}^* \\ \boldsymbol{N}^* \\ \boldsymbol{B}^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -cosh\theta & 0 & sinh\theta \\ -sinh\theta & 0 & cosh\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{bmatrix}, \text{[4]}.$$

3. RESULTS AND DISCUSSION

In this section, we compute the arc-lengths of the spherical indicatrix curves $(\boldsymbol{T}^*), (\boldsymbol{N}^*), (\boldsymbol{B}^*)$ and then we calculate the geodesic curvatures of these curves in $\boldsymbol{E}_1^3$ and $\boldsymbol{S}_1^2$ or $\boldsymbol{H}_0^2$. Firstly, for the arc-length $s_{\boldsymbol{T}^*}$ of tagent indicatrix $(\boldsymbol{T}^*)$ of the involute curve β, we can write

$$s_{\boldsymbol{T}^*} = \int_0^{s^*} \left\| \frac{d\boldsymbol{T}^*}{ds^*} \right\| ds^*,$$

$$s_{\boldsymbol{T}^*} = \int_0^{s} \sqrt{|\boldsymbol{\kappa}^2 - \boldsymbol{\tau}^2|} ds,$$

$$s_{\boldsymbol{T}^*} = \int_0^{s} \|\boldsymbol{W}\| ds$$

If the arc length for the principal normal indicatrix $(\boldsymbol{N}^*)$ is $s_{\boldsymbol{N}^*}$ it is

$$s_{\boldsymbol{N}^*} = \int_0^{s^*} \left\| \frac{d\boldsymbol{N}^*}{ds^*} \right\| ds^*,$$

$$s_{\boldsymbol{N}^*} = \int_0^{s} \sqrt{\left|\theta'^2 + \|\boldsymbol{W}\|^2\right|} ds.$$

For timelike $\boldsymbol{W}$, from the Lemma1. ii. we have

$$s_{\boldsymbol{N}^*} = \int_0^{s} \sqrt{\left|-\theta'^2 + \|\boldsymbol{W}\|^2\right|} ds$$

Similarly, the arc length $s_{\boldsymbol{B}^*}$ of the binormal indicatrix $(\boldsymbol{B}^*)$ can be obtained as

$$s_{\boldsymbol{B}^*} = \int_0^{s} |\theta'| ds.$$

If $\boldsymbol{W}$ is a timelike vector, then we have the same result. Thus we can give the following corollaries:

Corollary 3.1. For the arc lenght of the tagent indicatrix $(\boldsymbol{T}^*)$ of the involute of a timelike curve, it is obvious that

$$s_{\boldsymbol{T}^*} = s_{\boldsymbol{N}}.$$

Corollary 3.2. If the evolute curve α is a helix then for the arc-length of the principal normal indicatrix $(\boldsymbol{N}^*)$, we can write

$$s_{\boldsymbol{N}^*} = s_{\boldsymbol{N}}.$$

Corollary 3.3. For the arc lenght of the binormal indicatrix $(\boldsymbol{B}^*)$ of the involute of a timelike curve, we have

$$s_{\boldsymbol{B}^*} = s_{\boldsymbol{C}}.$$

Now let us compute the geodesic curvatures of the spherical indicatrices $(\boldsymbol{T}^*), (\boldsymbol{N}^*), (\boldsymbol{B}^*)$ with respect to $\boldsymbol{E}_1^3$. For the geodesic curvature $k_{\boldsymbol{T}^*}$ of the tangent indicatrix $(\boldsymbol{T}^*)$ of the curve β, we can write

$$\boldsymbol{k}_{\boldsymbol{T}^*} = \left\|\boldsymbol{\nabla}_{\boldsymbol{t}_{\boldsymbol{T}^*}} \boldsymbol{t}_{\boldsymbol{T}^*}\right\| \tag{1}$$

Differentiating the curve $\beta_{\boldsymbol{T}^*}(s_{\boldsymbol{T}^*}) = \boldsymbol{T}^*(s)$ with the respect to $s_{\boldsymbol{T}^*}$ and normalizing, we obtain

$$\boldsymbol{t}_{\boldsymbol{T}^*} = -sinh\theta\boldsymbol{T} + cosh\theta\boldsymbol{B}. \tag{2}$$

By taking derivative of the last equation we have

$$\boldsymbol{\nabla}_{\boldsymbol{t}_{\boldsymbol{T}^*}} \boldsymbol{t}_{\boldsymbol{T}^*} = (-\theta' cosh\theta\boldsymbol{T} + \|\boldsymbol{W}\|\boldsymbol{N} - \theta' sinh\theta\boldsymbol{B})\frac{1}{\|\boldsymbol{W}\|} \tag{3}$$

By substituting (3) into the Eq. (1) we get

$$\boldsymbol{k}_{\boldsymbol{T}^*} = \frac{1}{\|\boldsymbol{W}\|}\sqrt{\left|\theta'^2 + \|\boldsymbol{W}\|^2\right|}. \tag{4}$$

From $\boldsymbol{k}_{\boldsymbol{T}} = \frac{1}{sinh\theta}$ we have $\theta' = -\frac{\boldsymbol{k}_{\boldsymbol{T}}'}{\boldsymbol{k}_{\boldsymbol{T}}\sqrt{1+\boldsymbol{k}_{\boldsymbol{T}}^2}}$. If we set θ' in the Eq. (4) then we have

$$\boldsymbol{k}_{\boldsymbol{T}^*} = \frac{1}{\|\boldsymbol{W}\|}\sqrt{\left|1 + \frac{\boldsymbol{k}_{\boldsymbol{T}}'^2}{\|\boldsymbol{W}\|^2\, \boldsymbol{k}_{\boldsymbol{T}}^2(1+\boldsymbol{k}_{\boldsymbol{T}}^2)}\right|}. \tag{5}$$

If $\boldsymbol{W}$ is a timelike vector, then the similar results in the Eqs. (4), (5) can be easily obtained as

$$\boldsymbol{k}_{\boldsymbol{T}^*} = \frac{1}{\|\boldsymbol{W}\|}\sqrt{\left|-\theta'^2 + \|\boldsymbol{W}\|^2\right|}, \tag{6}$$

$$\boldsymbol{k}_{\boldsymbol{T}^*} = \frac{1}{\|\boldsymbol{W}\|}\sqrt{\left|1 - \frac{\boldsymbol{k}_{\boldsymbol{T}}'^2}{\|\boldsymbol{W}\|^2\, \boldsymbol{k}_{\boldsymbol{T}}^2(1-\boldsymbol{k}_{\boldsymbol{T}}^2)}\right|}. \tag{7}$$

Using the Remark 1, we can give the following result:

Corollary 3.4. If the evolute curve α is a helix then we have for the geodesic curvature of the tangent indicatrix $(\boldsymbol{T}^*)$ of the involute curve β

$$\boldsymbol{k}_{\boldsymbol{T}^*} = 1.$$

Similarly, by differentiating the curve $\beta_{\boldsymbol{N}^*(s_{\boldsymbol{N}^*})} = \boldsymbol{N}^*(s)$ with the respect to $s_{\boldsymbol{N}^*}$ and by normalizing we obtain

$$\boldsymbol{t}_{\boldsymbol{N}^*} = \sigma cosh\theta \boldsymbol{T} - \frac{\boldsymbol{1}}{\boldsymbol{k_N}} \boldsymbol{N} + \sigma sinh\theta, \tag{8}$$

where $\sigma = \frac{\gamma_N}{\boldsymbol{k_N}}$. By taking derivative of the last equation and using the definition of geodesic curvature, we have

$$\nabla_{\boldsymbol{t}_{\boldsymbol{N}^*}} \boldsymbol{t}_{\boldsymbol{N}^*} = \Big\{\Big(\sigma' cosh\theta + \sigma\theta' sinh\theta - \frac{\kappa}{\boldsymbol{k_N}}\Big)\boldsymbol{T} + \Big(-\frac{\boldsymbol{k_N}'}{\boldsymbol{k_N}^2}\Big)\boldsymbol{N} + \Big(-\sigma' sinh\theta - \sigma\theta' cosh\theta + \frac{\tau}{\boldsymbol{k_N}}\Big)\boldsymbol{B}\Big\}\frac{\boldsymbol{1}}{\|\boldsymbol{W}\|\boldsymbol{k_N}} \tag{9}$$

$$\boldsymbol{k}_{\boldsymbol{N}^*} = \frac{1}{\|\boldsymbol{W}\|\boldsymbol{k_N}}\sqrt{\left|\sigma'^2 - \Big(\sigma\theta' + \frac{\|\boldsymbol{W}\|}{\boldsymbol{k_N}}\Big)^2 + \frac{\boldsymbol{k_N}'^2}{\boldsymbol{k_N}^4}\right|} \tag{10}$$

In the case of $\boldsymbol{W}$ is a timelike vector, similar result can be easily obtained as follow in following same procedure.

$$\boldsymbol{k}_{\boldsymbol{N}^*} = \frac{1}{\|\boldsymbol{W}\|\boldsymbol{k_N}}\sqrt{\left|-\sigma'^2 - \Big(\sigma\theta' - \frac{\|\boldsymbol{W}\|}{\boldsymbol{k_N}}\Big)^2 + \frac{\boldsymbol{k_N}'^2}{\boldsymbol{k_N}^4}\right|} \tag{11}$$

Corollary 3.5. If the evolute curve α is a helix then we have for the geodesic curvature of the principal normal indicatrix $(\boldsymbol{N}^*)$ of the involute curve β

$$\boldsymbol{k}_{\boldsymbol{N}^*} = 1.$$

By differentiating the curve $\beta_{\boldsymbol{B}^*(s_{\boldsymbol{N}^*})} = \boldsymbol{B}^*(s)$ with the respect to $s_{\boldsymbol{B}^*}$ and by normalizing we obtain

$$\boldsymbol{t}_{\boldsymbol{B}^*} = -sinh\theta \boldsymbol{T} + cosh\theta \boldsymbol{B}. \tag{12}$$

By taking derivative of the last equation

$$\nabla_{\boldsymbol{t}_{\boldsymbol{B}^*}} \boldsymbol{t}_{\boldsymbol{B}^*} = -cosh\theta \boldsymbol{T} + \frac{\|\boldsymbol{W}\|}{\theta'}\boldsymbol{N} + sinh\theta \boldsymbol{B} \tag{13}$$

and by taking the norm of the last equation, we obtain

$$\boldsymbol{k}_{\boldsymbol{B}^*} = \sqrt{\left|1 + \Big(\frac{\|\boldsymbol{W}\|}{\theta'}\Big)^2\right|}. \tag{14}$$

From $\boldsymbol{k_B} = \frac{1}{cosh\theta}$ we have $\theta' = -\frac{\boldsymbol{k_B}'}{\boldsymbol{k_B}\sqrt{1+\boldsymbol{k_B}^2}}$. If we set θ' in the Eq. (14) then we have

$$\boldsymbol{k}_{\boldsymbol{B}^*} = \sqrt{\left|1 + \frac{\|\boldsymbol{W}\|^2\, \boldsymbol{k_B}^2(1+\boldsymbol{k_B}^2)}{\boldsymbol{k_B}'^2}\right|} \tag{15}$$

In the case of timelike $\boldsymbol{W}$, similar results can be easily obtained as follow,

$$\boldsymbol{k_{B^*}} = \sqrt{\left|-1+\left(\frac{\|\boldsymbol{W}\|}{\theta'}\right)^2\right|}, \tag{16}$$

$$\boldsymbol{k_{B^*}} = \sqrt{\left|-1+\frac{\|\boldsymbol{W}\|^2\,\boldsymbol{k_B}^2\left(1+\boldsymbol{k_B}^2\right)}{\boldsymbol{k_B}'^2}\right|}. \tag{17}$$

Corollary 3.6. If the evolute curve α is a helix then the geodesic curvature $\boldsymbol{k_{B^*}}$ of the binormal indicatrix $(\boldsymbol{B}^*)$ of the involute curve β is undefined.

Now let us compute the geodesic curvatures of the spherical indicatrices $(\boldsymbol{T}^*), (\boldsymbol{N}^*), (\boldsymbol{B}^*)$ with respect to $\boldsymbol{S}_1^2$ (or $\boldsymbol{H}_0^2$). For the geodesic curvature $\boldsymbol{g_{T^*}}$ of the tangent indicatrix $(\boldsymbol{T}^*)$ of the curve β with respect to $\boldsymbol{S}_1^2$, we can write

$$\boldsymbol{g_{T^*}} = \left\|\overline{\boldsymbol{\nabla}}_{\boldsymbol{t_{T^*}}}\boldsymbol{t_{T^*}}\right\| \tag{18}$$

From the Gauss equation we can write

$$\boldsymbol{\nabla}_{\boldsymbol{t_{T^*}}}\boldsymbol{t_{T^*}} = \overline{\boldsymbol{\nabla}}_{\boldsymbol{t_{T^*}}}\boldsymbol{t_{T^*}} + \varepsilon g(\boldsymbol{S}(\boldsymbol{t_{T^*}}), \boldsymbol{t_{T^*}})\boldsymbol{T}^*, \tag{19}$$

where $\varepsilon = g(\boldsymbol{T}^*, \boldsymbol{T}^*) = 1, \boldsymbol{S}(\boldsymbol{t_{T^*}}) = -\boldsymbol{t_{T^*}}$ and $g(\boldsymbol{S}(\boldsymbol{t_{T^*}}), \boldsymbol{t_{T^*}}) = 1$. From the Eq. (3) and (19), it follows that

$$\overline{\boldsymbol{\nabla}}_{\boldsymbol{t_{T^*}}}\boldsymbol{t_{T^*}} = (-\theta' cosh\theta \boldsymbol{T} + \theta' sinh\theta \boldsymbol{B})\frac{1}{\|\boldsymbol{W}\|} \tag{20}$$

Substituting (20) in the Eq. (18), we obtain

$$\boldsymbol{g_{T^*}} = \frac{\theta'}{\|\boldsymbol{W}\|} \tag{21}$$

By using $\boldsymbol{g_T} = coth\theta$, we obtain following relationship between $\boldsymbol{g_T}$ and $\boldsymbol{g_{T^*}}$:

$$\boldsymbol{g_{T^*}} = \frac{\boldsymbol{g_T}'\left(1-\boldsymbol{g_T}^2\right)}{\|\boldsymbol{W}\|} \tag{22}$$

If $\boldsymbol{W}$ is a timelike vector, then we have the same result. Thus we can give the following corollary:

Corollary 3.7. If the evolute curve α is a helix then we have for the geodesic curvature of the tangent indicatrix $(\boldsymbol{T}^*)$ of the involute curve β

$$\boldsymbol{g_{T^*}} = 0.$$

For the geodesic curvature $\boldsymbol{g_{N^*}}$ of the principal normal indicatrix $(\boldsymbol{N}^*)$ of the curve β with respect to $\boldsymbol{S}_1^2$, we can write

$$\boldsymbol{g_{N^*}} = \left\|\overline{\boldsymbol{\nabla}}_{\boldsymbol{t_{N^*}}}\boldsymbol{t_{N^*}}\right\| \tag{23}$$

If $\boldsymbol{W}$ is a spacelike vector, by using the Gauss equation and the Eq. (9), we can write

$$\overline{\boldsymbol{\nabla}}_{\boldsymbol{t}_{\boldsymbol{N}^*}}\boldsymbol{t}_{\boldsymbol{N}^*}=\left\{\left(\sigma' cosh\theta+\sigma\theta' sinh\theta-\frac{\boldsymbol{\kappa}}{\boldsymbol{k}_{\boldsymbol{N}}}-sinh\theta\|\boldsymbol{W}\|\boldsymbol{k}_{\boldsymbol{N}}\right)\boldsymbol{T}+\left(-\frac{\boldsymbol{k}_{\boldsymbol{N}}{}'}{\boldsymbol{k}_{\boldsymbol{N}}{}^{2}}\right)\boldsymbol{N}\right.$$
$$\left.+\left(-\sigma' sinh\theta-\sigma\theta' cosh\theta-\frac{\boldsymbol{\tau}}{\boldsymbol{k}_{\boldsymbol{N}}}+cosh\theta\|\boldsymbol{W}\|\boldsymbol{k}_{\boldsymbol{N}}\right)\boldsymbol{B}\right\}\frac{1}{\|\boldsymbol{W}\|\boldsymbol{k}_{\boldsymbol{N}}}. \tag{24}$$

By taking the norm of the last equation we obtain

$$\boldsymbol{g}_{\boldsymbol{N}^*}=\frac{1}{\|\boldsymbol{W}\|\boldsymbol{k}_{\boldsymbol{N}}}\sqrt{\left|\sigma'^{2}-\left[(\sigma\theta'+\|\boldsymbol{W}\|\boldsymbol{k}_{\boldsymbol{N}})+\frac{\|\boldsymbol{W}\|}{\boldsymbol{k}_{\boldsymbol{N}}}\right]^{2}+\frac{\boldsymbol{k}_{\boldsymbol{N}}{}'^{2}}{\boldsymbol{k}_{\boldsymbol{N}}{}^{4}}\right|}. \tag{25}$$

By using $\boldsymbol{g}_{\boldsymbol{N}}=\frac{\theta'}{\|\boldsymbol{W}\|}$, we get

$$\boldsymbol{g}_{\boldsymbol{N}^*}=\frac{1}{\|\boldsymbol{W}\|\boldsymbol{k}_{\boldsymbol{N}}}\sqrt{\left|\sigma'^{2}-\left[\frac{\|\boldsymbol{W}\|}{\boldsymbol{k}_{\boldsymbol{N}}}\left(\boldsymbol{g}_{\boldsymbol{N}}{}^{2}+\boldsymbol{k}_{\boldsymbol{N}}{}^{2}+1\right)\right]^{2}+\frac{\boldsymbol{k}_{\boldsymbol{N}}{}'^{2}}{\boldsymbol{k}_{\boldsymbol{N}}{}^{4}}\right|}. \tag{26}$$

In the case of timelike $\boldsymbol{W}$, similar results can be easily obtained as follow,

$$\boldsymbol{g}_{\boldsymbol{N}^*}=\frac{1}{\|\boldsymbol{W}\|\boldsymbol{k}_{\boldsymbol{N}}}\sqrt{\left|-\sigma'^{2}+\left[(\sigma\theta'+\varepsilon_0\|\boldsymbol{W}\|\boldsymbol{k}_{\boldsymbol{N}})-\frac{\|\boldsymbol{W}\|}{\boldsymbol{k}_{\boldsymbol{N}}}\right]^{2}+\frac{\boldsymbol{k}_{\boldsymbol{N}}{}'^{2}}{\boldsymbol{k}_{\boldsymbol{N}}{}^{4}}\right|}, \tag{27}$$

$$\boldsymbol{g}_{\boldsymbol{N}^*}=\frac{1}{\|\boldsymbol{W}\|\boldsymbol{k}_{\boldsymbol{N}}}\sqrt{\left|-\sigma'^{2}+\left[\frac{\|\boldsymbol{W}\|}{\boldsymbol{k}_{\boldsymbol{N}}}\left(\boldsymbol{g}_{\boldsymbol{N}}{}^{2}+\varepsilon_0\boldsymbol{k}_{\boldsymbol{N}}{}^{2}-1\right)\right]^{2}+\frac{\boldsymbol{k}_{\boldsymbol{N}}{}'^{2}}{\boldsymbol{k}_{\boldsymbol{N}}{}^{4}}\right|}. \tag{28}$$

where $\varepsilon_0=g(\boldsymbol{S}(\boldsymbol{t}_{\boldsymbol{N}^*}),\boldsymbol{t}_{\boldsymbol{N}^*})=\pm 1$.

Corollary 3.8. If the evolute curve α is a helix then we have for the geodesic curvature of the principal normal indicatrix $(\boldsymbol{N}^*)$ of the involute curve β

$$\boldsymbol{g}_{\boldsymbol{N}^*}=0 \text{ or } \boldsymbol{g}_{\boldsymbol{N}^*}=2.$$

For the geodesic curvature $\boldsymbol{g}_{\boldsymbol{B}^*}$ of the principal normal indicatrix $(\boldsymbol{B}^*)$ of the curve β with respect to $\boldsymbol{H}_0^2$, we can write

$$\boldsymbol{g}_{\boldsymbol{B}^*}=\left\|\overline{\overline{\boldsymbol{\nabla}}}_{\boldsymbol{t}_{\boldsymbol{B}^*}}\boldsymbol{t}_{\boldsymbol{B}^*}\right\| \tag{29}$$

If $\boldsymbol{W}$ is a spacelike vector, by using the Gauss equation and the Eq. (13), we can write

$$\overline{\overline{\boldsymbol{\nabla}}}_{\boldsymbol{t}_{\boldsymbol{B}^*}}\boldsymbol{t}_{\boldsymbol{B}^*}=-cosh\theta\boldsymbol{T}-\frac{\|\boldsymbol{W}\|}{\theta'}\boldsymbol{N}+sinh\theta\boldsymbol{B}-\boldsymbol{B}^* \tag{30}$$

By using $\boldsymbol{B}^*=-cosh\theta\boldsymbol{T}+sinh\theta\boldsymbol{\theta}\boldsymbol{B}$ given in the Lemma 1. (i), we obtain

$$\overline{\overline{\boldsymbol{\nabla}}}_{\boldsymbol{t}_{\boldsymbol{B}^*}}\boldsymbol{t}_{\boldsymbol{B}^*}=-\frac{\|\boldsymbol{W}\|}{\theta'}\boldsymbol{N}. \tag{31}$$

By taking the norm of the last equation we obtain

$$\boldsymbol{g}_{\boldsymbol{B}^*} = \frac{\|\boldsymbol{W}\|}{\theta'}. \tag{32}$$

By using $\boldsymbol{g}_{\boldsymbol{B}} = tanh\theta$, we obtain following relationship between $\boldsymbol{g}_{\boldsymbol{B}}$ and $\boldsymbol{g}_{\boldsymbol{B}^*}$:

$$\boldsymbol{g}_{\boldsymbol{B}^*} = \frac{\|\boldsymbol{W}\|\left(1-\boldsymbol{g}_{\boldsymbol{B}}{}^2\right)}{\boldsymbol{g}_{\boldsymbol{B}}{}'}. \tag{33}$$

If $\boldsymbol{W}$ is a timelike vector, then we have the same result. Thus we can give the following corollary:

Corollary 3.9. If the evolute curve α is a helix then the geodesic curvature $\boldsymbol{g}_{\boldsymbol{B}^*}$ of the binormal indicatrix $(\boldsymbol{B}^*)$ of the involute curve β is undefined.

4. CONCLUSION

In this study, we have give some important relationships between arc lengths and geodesic curvatures of the spherical indicatrices of involute-evolute curve couple in Minkowski 3-space. Also, we give some important results about curve couple. We expect that the ideas and techniques used in this paper may open new horizons for researchers studying on spherical indicatrices of special curves.

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