RATIONAL TYPE CONTRACTION MAPPING THEOREMS ON MULTIPLICATIVE *G*-METRIC SPACES

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Abstract. *In this article, we have extended the results of Y. U. Gaba by taking multiplicative G-metric space instead of G-metric space. In this way, we proved contraction mapping theorems on multiplicative, G-metric spaces using rational type contraction conditions for a single map. Later on, we extended our results in the settings of triplet maps. We have also given examples to support our results.*

Keywords: multiplicative G-metric space; contraction mappings; fixed point.

1. INTRODUCTION AND PRELIMINARIES

The notion of fixed point theory is most important in analysis. The first systematic way of finding fixed points of self mappings was initiated by polish mathematician Stefan Bannach in his famous contraction principle. This contraction principle is widely used in analysis of operators. It gives a sequence of approximate solutions and many other vital informations like, the rate of convergence toward the fixed point. For the solution of abstract and applied mathematical problems, fixed point iterations play a vital part in several algorithms. Many researchers proved common fixed point theorems on many generalizations of fixed point theory. Jamshaid ahmad et al [1] proved L-fuzzy mappings and common fixed point theorems. A. Azam and I. Beg [2] proved common fixed points of fuzzy maps. A. Azam [3] also proved fuzzy fixed point of fuzzy mapping by taking rational inequality. Fixed point theory has gained considerable importance after the celebrated Banach Contraction theorem [4]. Since then, a rich theory on the subject has emerged. Similarly, many generalizations of a metric space model have been defined by many researchers. S. Gahler [11-12] introduced the notion of 2-metric space. The 2-metric space was the first structure on three arguments before that metric space was the structure on two arguments. After this, B. C. Dhage [5], an Indian mathematician introduced the new structure on three arguments named as D-metric space. After this there was a spat of papers were published [6-9]. Z. Mustafa and B. Sims [14] gave some remarks on the structure of *D*-metric space. They claimed that the structure of *D*-metric space is not the generalization of usual metric space. Z. Mustafa and B. Sims [15] gave the more robust structure of generalized metric space named as *G*-metric space. They published many papers on this new structure like [17-21]. A new structure of metric space called multiplicative metric space, was introduced by Ozavasar and Ceikel [16]. In this structure, they introduced the contraction condition for multiplicative metric space which is quite different from the definition of contraction condition in usual metric space as well as they also proved fixed point theorems on this new structure.

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Bashirov et. al. [22] and M. Abbas [23] also proved the fixed point theorems on this structure. The concept of multiplicative G-metric space was introduced by S. Kumar [12] and generalized Banach fixed point theorem in the setting of multiplicative G-metric space. He also extended the generalization for two maps in the settings of multiplicative G-metric space. Y. U. Gaba [13] has given rational type contraction mapping theorems in the setting of G-metric space by using single map. Later on, he extended the idea by taking triplet maps in the setting of G-metric space. In this paper, we have extended the results of Y. U. Gaba by taking multiplicative G-metric spaces instead of G-metric space. In this way, we have proved contraction mapping theorems in multiplicative G-metric space using rational type contraction for a single maps. Later on, we have extended our results in the setting of triple maps. We have given the counter example to prove our theorems. First of all we give some basic definitions related to multiplicative G-metric spaces.

Definition 1.1. [12] A function $G: \Psi \times \Psi \times \Psi \to \mathbb{R}^+$ is multiplicative G-metric space if it satisfies the following axioms:

(i). $G(\alpha, \beta, \gamma) = 1$ iff $\alpha = \beta = \gamma$:

(ii). $1 < G(\alpha, \alpha, \beta)$ forall $\alpha, \beta \in \Psi$ with $\alpha \neq \beta$;

(iii). $G(\alpha, \alpha, \beta) \leq G(\alpha, \beta, \gamma)$ forall $\alpha, \beta, \gamma \in \Psi$, $\gamma \neq \beta$;

(iv). $G(\alpha, \beta, \gamma) = G(\alpha, \gamma, \beta) = G(\beta, \gamma, \alpha) = \cdots$ (Symmetry)

(v).

$G(\alpha, \beta, \gamma) \leq G(\alpha, a, a). G(a, \beta, \gamma)$ for all $\alpha, \beta, \gamma, a \in \Psi$ (rectangleinequality)

Then (Ψ, G) is multiplicative G-metric space.

Example 1.1. [12] Suppose $\mathbb{R}$ a set of all real numbers a function $G: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ by

$$G(\alpha, \beta, \gamma) = e^{|\alpha - \beta| + |\beta - \gamma| + |\gamma - \alpha|} \quad \text{for all} \quad \alpha, \beta, \gamma \in \mathbb{R}$$

Then $(\mathbb{R}, G)$ is multiplicative G-metric space.

Proposition 1.1. [12] Suppose Ψ be a multiplicative $G-$metric space then for all $\alpha, \beta, \gamma \in \Psi$ and $a \in \Psi$ the following conditions hold:

(i). $G(\alpha, \beta, \gamma) = 1$ if $\alpha = \beta = \gamma$;

(ii). $G(\alpha, \beta, \gamma) \leq G(\alpha, a, a). G(\beta, a, a). G(\gamma, a, a)$;

(iii). $G(\alpha, \beta, \gamma) \leq G(\alpha, \alpha, \beta). G(\alpha, \alpha, \gamma)$;

(iv). $G(\alpha, \beta, \beta) \leq G(\beta, \alpha, \alpha)$.

Definition 1.2. [12] Let Ψ be a multiplicative $G-$metric space then Ψ is said to be multiplicative-symmetric, if

$$G(\alpha, \alpha, \beta) = G(\alpha, \beta, \beta) \text{ for all } \quad \alpha, \beta \in \Psi \tag{1.1}$$

Definition 1.3. [12] Let Ψ be a multiplicative G −metric space and $\{\alpha_\xi\}$ be any sequence in Ψ we say the sequence is multiplicative G −convergent to α, if

$$\lim_{\xi,\mu\to\infty} G(\alpha_\xi, \alpha_\mu, \alpha) = 1$$

that is for every $\epsilon > 1$ there exist a number $n_0 \in N$, such that

$$G\big(\alpha_\xi, \alpha_\mu, \alpha\big) < \epsilon \ \forall \ \ \mu, \xi \geq n_0.$$

Definition 1.4. [12] Let Ψ be multiplicative G −metric space, a sequence $\{\alpha_\xi\}$ is said to be multiplicative G −Cauchy if for any $\epsilon > 1$, there exist $\xi_0 \in N$ such that

$$G(\alpha_\mu, \alpha_\xi, \alpha_\nu) \leq \epsilon$$

$\forall \ \mu, \xi, \nu \geq n_0$ if given

$$G(\alpha_\mu, \alpha_\xi, \alpha_\nu) \to 1$$

when $\xi, \mu, \nu \to \infty$

Definition 1.5. [12] Let Ψ and $\ \Psi^{'}$ be two multiplicative G −metric spaces and let $A \colon \Psi \to \Psi'$ be a function then A is said to be a multiplicative G −continuous at a point $a \in \alpha$ if given any $\epsilon > 1$, there exist $\delta > 1$, such that $\alpha, \beta \in \Psi$:

$$G(a, \alpha, \beta) < \delta \qquad \text{implies} \qquad G(Aa, A\alpha, A\beta) < \epsilon$$

A function A is said to be multiplicative G-continuous on Ψ if and only if it is multiplicative G −continuous at all $a \in \Psi$.

Definition 1.6. [12] A multiplicative G −metric space is said to be multiplicative G-complete if every multiplicative G-Cauchy sequence in Ψ is multiplicative G-Convergent in Ψ.

Definition 1.7 [12] let Ψ be a multiplicative G −metric space, if for any $\alpha \ \epsilon \ \Psi$ and $\epsilon > 1$ we define a set

$$B_\epsilon(\alpha_0) = \{\beta \in \alpha \colon G(\alpha_0, \beta, \beta) < \epsilon\}$$

which is called multiplicative G −open ball of radius ϵ and with center α_0

Similarly, the set

$$\overline{B}_\epsilon(\alpha_0) = \{\beta \in \alpha \colon G(\alpha_0, \beta, \beta) \leq \epsilon\}$$

is called multiplicative G −closed ball.

2. MAIN RESULTS

Definition 2.1. [9] Let Ψ be a multiplicative G −metric space. A mapping $A \colon \Psi \to \Psi$ is said to be multiplicative contraction if there exist $\lambda \in (0,1)$ such that

$$G(A\alpha, A\beta, A\gamma) \leq (G(\alpha, \beta, \gamma))^\lambda \ \text{for all} \quad \alpha, \beta, \gamma \ \in \Psi \tag{2.1}$$

Theorem 2.1. [9] Let Ψ be a complete multiplicative G−metric space and $A \colon \Psi \to \Psi$ a contraction mapping. Then A has a unique fixed point, if

$$G(A\alpha, A\beta, A\gamma) \leq (G(\alpha, \beta, \gamma))^\lambda \quad \text{for all} \quad \alpha, \beta, \gamma \in \Psi$$

Theorem 2.2. Let Ψ be a multiplicative $G-$complete metric space and $A: \Psi \to \Psi$ be a contraction mapping. Then A has a fixed point, if

$$G(A\alpha, A\beta, A\gamma) \leq \left\{ \frac{G(A\alpha, \beta, \gamma). G(\alpha, A\beta, \gamma). G(\alpha, \beta, A\gamma)}{G(A\alpha, A\alpha, \alpha). G(A\beta, A\beta, \beta). G(A\gamma, A\gamma, \gamma)} \right\}^\lambda \text{ forall } \alpha, \beta, \gamma \tag{2.2}$$
$$\in \Psi$$

Proof: Let α_0 be any arbitrary point in Ψ. We define a sequence $\{\alpha_\xi\}$ by the relation $\alpha_{\xi+1} = A(\alpha_\xi)$ and $\alpha_\xi \neq \alpha_\mu$ for all $\xi \neq \mu$.

Consider

$$G(\alpha_\xi, \alpha_{\xi+1}, \alpha_{\xi+1}) = G(A\alpha_{\xi-1}, A\alpha_\xi, A\alpha_\xi) \tag{2.3}$$

Applying Eq.(2.3) in Eq.(2.2), we have

$$G(A\alpha_{\xi-1}, A\alpha_\xi, A\alpha_\xi)$$

$$\leq \left\{ \frac{G(A\alpha_{\xi-1}, \alpha_\xi, \alpha_\xi). G(\alpha_{\xi-1}, A\alpha_\xi, \alpha_\xi). G(\alpha_{\xi-1}, \alpha_\xi, A\alpha_\xi)}{G(A\alpha_{\xi-1}, A\alpha_{\xi-1}, \alpha_{\xi-1}). G(A\alpha_\xi, A\alpha_\xi, \alpha_\xi). G(A\alpha_\xi, A\alpha_\xi, \alpha_\xi)} \right\}^\lambda$$

$$= \left\{ \frac{G(\alpha_\xi, \alpha_\xi, \alpha_\xi). G(\alpha_{\xi-1}, \alpha_{\xi+1}, \alpha_\xi). G(\alpha_{\xi-1}, \alpha_\xi, \alpha_{\xi+1})}{G(\alpha_\xi, \alpha_\xi, \alpha_{\xi-1}). G(\alpha_{\xi+1}, \alpha_{\xi+1}, \alpha_\xi). G(\alpha_{\xi+1}, \alpha_{\xi+1}, \alpha_\xi)} \right\} \tag{2.4}$$

$$= \left\{ \frac{1. G(\alpha_{\xi-1}, \alpha_{\xi+1}, \alpha_\xi). G(\alpha_{\xi-1}, \alpha_\xi, \alpha_{\xi+1})}{G(\alpha_\xi, \alpha_\xi, \alpha_{\xi-1}). G(\alpha_{\xi+1}, \alpha_{\xi+1}, \alpha_\xi). G(\alpha_{\xi+1}, \alpha_{\xi+1}, \alpha_\xi)} \right\}$$

$$= \left\{ \frac{G(\alpha_{\xi-1}, \alpha_{\xi+1}, \alpha_\xi). G(\alpha_{\xi-1}, \alpha_{\xi+1}, \alpha_\xi)}{G(\alpha_\xi, \alpha_\xi. \alpha_{\xi-1}). G(\alpha_{\xi+1}, \alpha_{\xi+1}, \alpha_\xi). G(\alpha_{\xi+1}, \alpha_{\xi+1}, \alpha_\xi)} \right\}^\lambda$$

Now, consider

$$G(\alpha_{\xi-1}, \alpha_{\xi+1}, \alpha_\xi) \leq G(\alpha_{\xi-1}, \alpha_\xi, \alpha_\xi). G(\alpha_\xi, \alpha_\xi, \alpha_{\xi+1})$$
$$= G(\alpha_{\xi-1}, \alpha_\xi, \alpha_\xi). G(\alpha_\xi, \alpha_{\xi+1}, \alpha_{\xi+1}) \tag{2.5}$$

So, using Eq. (2.5) in Eq.(2.4), we have

$$= \left\{ \frac{G(\alpha_{\xi-1}, \alpha_\xi, \alpha_\xi). G(\alpha_\xi, \alpha_{\xi+1}, \alpha_{\xi+1}). G(\alpha_{\xi-1}, \alpha_\xi, \alpha_\xi). G(\alpha_\xi, \alpha_{\xi+1}, \alpha_{\xi+1})}{G(\alpha_\xi, \alpha_\xi, \alpha_{\xi-1}). G(\alpha_{\xi+1}, \alpha_{\xi+1}, \alpha_\xi). G(\alpha_{\xi+1}, \alpha_{\xi+1}, \alpha_\xi)} \right\}^\lambda$$

$$= G(\alpha_{\xi-1}, \alpha_\xi, \alpha_\xi)^\lambda$$

Consider again,

$$G(\alpha_\xi, \alpha_{\xi-1}, \alpha_{\xi-1}) = G(A\alpha_{\xi-1}, A\alpha_{\xi-2}, A\alpha_{\xi-2}) \tag{2.6}$$

Using Eq. (2.6) in Eq.(2.2), we have,

$$G(A\alpha_{\xi-1}, A\alpha_{\xi-2}, A\alpha_{\xi-2})$$

$$\leq \left\{ \frac{G(A\alpha_{\xi-1}, \alpha_{\xi-2}, \alpha_{\xi-2}).G(\alpha_{\xi-1}, A\alpha_{\xi-2}, \alpha_{\xi-2}).G(\alpha_{\xi-1}, \alpha_{\xi-2}, A\alpha_{\xi-2})}{G(A\alpha_{\xi-1}, A\alpha_{\xi-1}, \alpha_{\xi-1}).G(A\alpha_{\xi-2}, A\alpha_{\xi-2}, \alpha_{\xi-2}).G(A\alpha_{\xi-2}.A\alpha_{\xi-2}, \alpha_{\xi-2})} \right\}^{\lambda} \tag{2.7}$$

$$= \left\{ \frac{G(\alpha_{\xi}, \alpha_{\xi-2}, \alpha_{\xi-2}).G(\alpha_{\xi-1}, \alpha_{\xi-1}, \alpha_{\xi-2}).G(\alpha_{\xi-1}, \alpha_{\xi-2}, \alpha_{\xi-1})}{G(\alpha_{\xi}, \alpha_{\xi}, \alpha_{\xi-1}).G(\alpha_{\xi-1}, \alpha_{\xi-1}, \alpha_{\xi-2}).G(\alpha_{\xi-1}, \alpha_{\xi-1}, \alpha_{\xi-2})} \right\}^{\lambda}$$

Now, consider

$$G(\alpha_{\xi}, \alpha_{\xi-2}, \alpha_{\xi-2}) \leq G(\alpha_{\xi}, \alpha_{\xi-1}, \alpha_{\xi-1}).G(\alpha_{\xi-1}, \alpha_{\xi-1}, \alpha_{\xi-2})$$
$$= G(\alpha_{\xi}, \alpha_{\xi}, \alpha_{\xi-1}).G(\alpha_{\xi-1}, \alpha_{\xi-1}, \alpha_{\xi-2}) \tag{2.8}$$

Using Eq.(2.8) in Eq.(2.7), we get

$$= \left\{ \frac{G(\alpha_{\xi}, \alpha_{\xi}, \alpha_{\xi-1}).G(\alpha_{\xi-1}, \alpha_{\xi-1}, \alpha_{\xi-2}).G(\alpha_{\xi-1}, \alpha_{\xi-1}, \alpha_{\xi-2}).G(\alpha_{\xi-1}, \alpha_{\xi-1}, \alpha_{\xi-2})}{G(\alpha_{\xi}, \alpha_{\xi}, \alpha_{\xi-1}).G(\alpha_{\xi-1}, \alpha_{\xi-1}, \alpha_{\xi-2}).G(\alpha_{\xi-1}, \alpha_{\xi-1}, \alpha_{\xi-2})} \right\}^{\lambda}$$

$$\leq G(\alpha_{\xi-1}, \alpha_{\xi-1}, \alpha_{\xi-2})^{\lambda}$$

Continuing in this way, we get

$$G(\alpha_{\xi}, \alpha_{\mu}, \alpha_{\mu}) \leq G(\alpha_0, \alpha_0, \alpha_1)^{\lambda^{\xi+\mu}}$$

Morover by using triangle inequality, we can get

$$G(\alpha_{\xi}, \alpha_{\mu}, \alpha_{\mu}) \leq G(\alpha_{\xi}, \alpha_{\xi+1}, \alpha_{\xi+1}).G(\alpha_{\xi+1}, \alpha_{\xi+2}, \alpha_{\xi+2}) \dots .G(\alpha_{\mu-1}, \alpha_{\mu}, \alpha_{\mu})$$

$$\leq \{G(\alpha_0, \alpha_1, \alpha_1)^{\lambda^{\xi}}\}.\{G(\alpha_0, \alpha_1, \alpha_1)^{\lambda^{\xi+1}}\} \dots \{G(\alpha_0, \alpha_1, \alpha_1)^{\lambda^{\mu-1}}\}$$

$$= G(\alpha_0, \alpha_1, \alpha_1)^{\lambda^{\xi} + \lambda^{\xi+1} + \dots + \lambda^{\mu-1}}$$

$$= G(\alpha_0, \alpha_1, \alpha_1)^{\lambda^{\xi}(1 + \lambda + \lambda^2 + \dots + \lambda^{\mu-\xi-1})}$$

$$= G(\alpha_0, \alpha_1, \alpha_1)^{\lambda^{\xi}(\frac{1-\lambda^{\xi-\mu}}{1-\lambda})}$$

As $\lambda < 1$, so $1 - \lambda^{\xi-\mu} < 1$, and $\frac{1-\lambda^{\xi-\mu}}{1-\lambda} < 1$,   so, $\lambda^{\xi}(\frac{1-\lambda^{\xi-\mu}}{1-\lambda}) \to 1$

$$G(\alpha_{\xi}, \alpha_{\mu}, \alpha_{\mu}) \to 1 \quad \text{when} \quad \mu, \xi \to \infty$$

So, the sequence $\{\alpha_{\xi}\}$ is a cauchy sequence.
Now, we claim that ω be a fixed point of A.
For the triplet $(\alpha_{\xi+1}, A\omega, A\omega)$ using in Eq.(2.2) , we have

$$G(\alpha_{\xi+1}, A\omega, A\omega) \leq \left\{ \frac{G(\alpha_{\xi+1}, \omega, \omega). G(\alpha_\xi, A\omega, \omega). G(\alpha_\xi, \omega, A\omega)}{G(\alpha_{\xi+1}, \alpha_{\xi+1}, \alpha_\xi). G(A\omega, A\omega, \omega). G(A\omega, A\omega, \omega)} \right\}^\lambda \tag{2.9}$$

Applying limit on both sides as $\alpha_\xi, \alpha_{\xi+1} \to \omega$ using in Eq.(2.9), we get

$$G(\omega, A\omega, A\omega) \leq \left\{ \frac{G(\omega, \omega, \omega). G(\omega, A\omega, \omega). G(\omega, \omega, A\omega)}{G(\omega, \omega, \omega). G(A\omega, A\omega, \omega). G(A\omega, A\omega, \omega)} \right\}^\lambda$$

i.e. $G(\omega, A\omega, A\omega) = 1$ so $A\omega = \omega$

Hence, ω is a fixed point of A.

Corollary 2.1. Let Ψ be a multiplicative G −complete metric space and $A: \Psi \to \Psi$ be a contraction mapping. Then A has at least one fixed point, if

$$G(A\alpha, A\beta, A\gamma) \leq \left\{ \frac{G(A\alpha, \beta, \gamma). G(\alpha, A\beta, \gamma). G(\alpha, \beta, A\gamma)}{G(A\alpha, A\alpha, \alpha). G(A\beta, A\beta, \beta). G(A\gamma, A\gamma, \gamma)} \right\}^{\frac{1}{2}} \text{ forall } \alpha, \beta, \gamma \tag{2.10}$$
$$\in \Psi$$

Corollary 2.2. Let Ψ be a multiplicative G −complete metric space and $A: \Psi \to \Psi$ be a contraction mapping.Then A has a fixed point if

$$G(A\alpha, A\beta, A\gamma) \leq \left\{ \frac{G(A\alpha, \beta, \gamma). G(\alpha, A\beta, \gamma). G(\alpha, \beta, A\gamma)}{G(A\alpha, A\alpha, \alpha). G(A\beta, A\beta, \beta). G(A\gamma, A\gamma, \gamma)} \right\}^\lambda \tag{2.11}$$

for all $\alpha, \beta, \gamma \in \bar{B}(\alpha_0, r)$

Morover, if

$$G(\alpha_0, A(\alpha_0), A(\alpha_0)) \leq r^{1-\lambda}$$

Example 2.1. *Let* $\Psi = \{0, \frac{1}{2}, 1\}$ *and* $G: \Psi^3 \to (0, \infty)$ *be defined by*

$$G(0,1,1) = 6 = G(1,0,0)$$

$$G(0, \frac{1}{2}, \frac{1}{2}) = 4 = G(\frac{1}{2}, 0, 0)$$

$$G(\frac{1}{2}, 1, 1) = 5 = G(1, \frac{1}{2}, \frac{1}{2})$$

$$G(0, \frac{1}{2}, 1) = \frac{15}{2}$$

$$G(\alpha, \alpha, \alpha) = 1 \text{ for all } \alpha \in \Psi$$

Let $A: \Psi \to \Psi$ be defined by

$$A(0) = 0, A(\frac{1}{2}) = \frac{1}{2}, A(1) = 0$$

$$G(A(0), A(\frac{1}{2}), A(\frac{1}{2})) = G(0, \frac{1}{2}, \frac{1}{2}) = 4$$

$$G(A(0), A(1), A(1)) = G(0,0,0) = 1$$

$$G(A(\tfrac{1}{2}), A(1), A(1)) = G(\tfrac{1}{2}, 0, 0) = 4$$

$$G(A(0), A(\tfrac{1}{2}), A(1)) = G(0, \tfrac{1}{2}, 0) = 4$$

Now,

$$4 = G(A(0), A(\tfrac{1}{2}), A(\tfrac{1}{2})) = G(0, \tfrac{1}{2}, \tfrac{1}{2}) \tag{2.12}$$

Applying Eq. (2.12) in Eq. (2.10), we get

$$\leq \left\{ \frac{G(A(0), \tfrac{1}{2}, \tfrac{1}{2}).\, G(0, A(\tfrac{1}{2}), \tfrac{1}{2}).\, G(0, \tfrac{1}{2}, A(\tfrac{1}{2}))}{G(A(0), A(0), 0).\, G(A(\tfrac{1}{2}), A(\tfrac{1}{2}), \tfrac{1}{2}).\, G(A(\tfrac{1}{2}), A(\tfrac{1}{2}), \tfrac{1}{2})} \right\}^{1/2}$$

$$= \left\{ \frac{G(0, \tfrac{1}{2}, \tfrac{1}{2}).\, G(0, \tfrac{1}{2}, \tfrac{1}{2}).\, G(0, \tfrac{1}{2}, \tfrac{1}{2})}{G(0, 0, 0).\, G(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}).\, G(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})} \right\}^{1/2}$$

$$= \left\{ \frac{4.4.4}{1.1.1} \right\}^{\tfrac{1}{2}}$$

$$4 < 8$$

Now,

$$1 = G(A(0), A(1), A(1)) = G(0, 0, 0) \tag{2.13}$$

Applying Eq.(2.13) in Eq. (2.10), we get

$$\leq \left\{ \frac{G(A(0), 1, 1).\, G(0, A(1), 1).\, G(0, 1, A(1))}{G(A(0), A(0), 0).\, G(A(1), A(1), 1).\, G(A(1), A(1), 1)} \right\}^{1/2}$$

$$= \left\{ \frac{G(0, 1, 1).\, G(0, 0, 1).\, G(0, 1, 0)}{G(0, 0, 0).\, G(0, 0, 1).\, G(0, 0, 1)} \right\}^{1/2}$$

$$= \left\{ \frac{6.6.6}{1.6.6.} \right\}^{1/2}$$

$$1 < \{6\}^{1/2}$$

Again, consider

$$4 = G(A(\tfrac{1}{2}), A(1), A(1) = G(\tfrac{1}{2}, 0, 0) \tag{2.14}$$

Applying Eq. (2.14) in Eq.(2.10), we get,

$$\leq \left\{ \frac{G(A(\frac{1}{2}),1,1) \cdot G(\frac{1}{2},A(1),1) \cdot G(\frac{1}{2},1,A(1))}{G(A(\frac{1}{2}),A(\frac{1}{2}),\frac{1}{2}) \cdot G(A(1),A(1),1) \cdot G(A(1),A(1),1)} \right\}^{1/2}$$

$$= \left\{ \frac{G(\frac{1}{2},1,1) \cdot G(\frac{1}{2},0,1) \cdot G(\frac{1}{2},1,0)}{G(\frac{1}{2},\frac{1}{2},\frac{1}{2}) \cdot G(0,0,1) \cdot G(0,0,1)} \right\}^{1/2}$$

$$= \left\{ \frac{5 \cdot \frac{15}{2} \cdot \frac{15}{2}}{1.6.6} \right\}^{1/2}$$

$$= \left\{ \frac{1125}{144} \right\}^{1/2}$$

$$4 < \left\{ \frac{1125}{144} \right\}^{1/2}$$

So, A has two fixed points $\{0, \frac{1}{2}\}$.

3. TRIPLET MAPPING

Theorem 3.1. Let Ψ be a multiplicative G−complete metric space and $A, B, C: \Psi \rightarrow \Psi$ be a contraction mapping. Then A, B, C has a fixed point if

$$G(A\alpha, B\beta, C\gamma) \leq \{G(\alpha, \beta, \gamma)\}^\lambda \quad \text{for all} \quad \alpha, \beta, \gamma \in \Psi \text{ and } \lambda \in (0,1) \tag{3.1}$$

Proof: Let α_0 be any arbitrary point in Ψ. Define a sequence $\{\alpha_\xi\}$ in Ψ by the relation

$$\alpha_{3\xi+1} = A(\alpha_{3\xi}), \ \alpha_{3\xi+2} = B(\alpha_{3\xi+1}), \ \alpha_{3\xi+3} = C(\alpha_{3\xi+2})$$

Apart from this, its supposed that $\alpha_\xi \neq \alpha_\mu$ for $\xi \neq \mu$
Consider,

$$G(\alpha_{3\xi+1}, \alpha_{3\xi+2}, \alpha_{3\xi+3}) = G(A\alpha_{3\xi}, B\alpha_{3\xi+1}, C\alpha_{3\xi+2}) \tag{3.2}$$

Using Eq. (3.2) in Eq. (3.1), we get

$$G(A\alpha_{3\xi}, B\alpha_{3\xi+1}, C\alpha_{3\xi+2}) \leq G(\alpha_{3\xi}, \alpha_{3\xi+1}, \alpha_{3\xi+2})^\lambda$$

$$= G(A\alpha_{3\xi-1}, B\alpha_{3\xi}, C\alpha_{3\xi+1})^\lambda$$

$$\leq \{G(\alpha_{3\xi-1}, \alpha_{3\xi}, \alpha_{3\xi+1})^\lambda\}^\lambda$$

$$= G(\alpha_{3\xi-1}, \alpha_{3\xi}, \alpha_{3\xi+1})^{\lambda^2}$$

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$$\leq G(\alpha_0, \alpha_1, \alpha_2)^{\lambda^{3\xi}}$$

Moreover, by using rectangle inequality, we have

$$G(\alpha_{3\xi}, \alpha_\mu, \alpha_\nu) \leq G(\alpha_{3\xi}, \alpha_{3\xi+1}, \alpha_{3\xi+2}). G(\alpha_{3\xi+1}, \alpha_{3\xi+2}, \alpha_{3\xi+3}) \ldots G(\alpha_{3\mu-1}, \alpha_{3\mu}, \alpha_{3\nu})$$

$$= G(\alpha_0, \alpha_1, \alpha_2)^{\lambda^{3\xi}}. G(\alpha_0, \alpha_1, \alpha_2)^{\lambda^{3\xi+1}} \ldots G(\alpha_0, \alpha_1, \alpha_2)$$

$$= G(\alpha_0, \alpha_1, \alpha_2)^{\lambda^{3\xi}+\lambda^{3\xi+1}+\ldots+\lambda^{3\mu-1}}$$

$$= G(\alpha_0, \alpha_1, \alpha_2)^{\lambda^{3\xi}(1+3\xi+\ldots+3\mu-3\xi-1)}$$

$$= G(\alpha_0, \alpha_1, \alpha_2)^{\lambda^{3\xi}\left(\frac{1-\lambda^{3\mu-3\xi}}{1-\lambda}\right)}$$

as $\lambda < 1$, so $1 - \lambda^{3\mu-3\xi} < 1$, and $\frac{1-\lambda^{3\mu-3\xi}}{1-\lambda} < 1$. Thus, $\frac{\lambda^{3\xi}(1-\lambda^{3\mu-3\xi}}{1-\lambda} \to 1$

$$G(\alpha_{3\xi}, \alpha_{3\mu}, \alpha_{3\nu}) \to 1 \quad \text{when} \quad \mu, \xi, \nu \to \infty$$

so, the sequence $\{\alpha_\xi\}$ is a Cauchy sequence.t

Since,

$$\lim_{\xi \to \infty} G(\omega, \alpha_{3\xi+2}, \alpha_{3\xi+3}) = \lim_{\xi \to \infty} G(\alpha_{3\xi+1}, \omega, \alpha_{3\xi+3}) = \lim_{\xi \to \infty} G(\alpha_{3\xi+1}, \alpha_{3\xi+2}, \omega) = 1.$$
$$\text{That is,} \quad G(A\omega, B\omega, C\omega) = 1$$

implies that

$$A\omega = B\omega = C\omega$$

Hence, ω is a common fixed point.

Theorem 3.2. Let Ψ be a multiplicative G−complete metric space and $A, B, C: \Psi \to \Psi$ be a contraction mapping. Then A, B, C has a common fixed point, if

$$G(A\alpha, B\beta, C\gamma)$$
$$\leq \left\{ \frac{G(A\alpha, \alpha, \alpha). G(B\beta, \beta, \beta). G(C\gamma, \gamma, \gamma). G(\alpha, \beta, \gamma)}{G(A\alpha, A\alpha, \alpha). G(B\beta, B\beta, \beta). G(C\gamma, C\gamma, \gamma)} \right\}^\lambda \text{ for all } \alpha, \beta, \gamma \in \Psi \tag{3.3}$$

Proof: Let α_0 be any arbitrary point in Ψ. Define a sequence $\{\alpha_\xi\}$ in Ψ by the relation

$$\alpha_{3\xi+1} = A(\alpha_{3\xi}), \alpha_{3\xi+2} = B(\alpha_{3\xi+1}), \alpha_{3\xi+3} = C(\alpha_{3\xi+2})$$

Apart from this, its supposed that $\alpha_\xi \neq \alpha_\mu$ for $\xi \neq \mu$
Consider,

$$G(\alpha_{3\xi+1}, \alpha_{3\xi+2}, \alpha_{3\xi+3}) = G(A\alpha_{3\xi}, B\alpha_{3\xi+1}, C\alpha_{3\xi+2}) \tag{3.4}$$

Using Eq. (3.4) in Eq.(3.3), we have

$$G\left(A\alpha_{3\xi}, B\alpha_{3\xi+1}, C\alpha_{3\xi+2}\right) \leq$$

$$\left\{\frac{\begin{array}{c}G(A\alpha_{3\xi}, \alpha_{3\xi}, \alpha_{3\xi}).G(B\alpha_{3\xi+1}, \alpha_{3\xi+1}, \alpha_{3\xi+1}).\\ G(C\alpha_{3\xi+2}, \alpha_{3\xi+2}, \alpha_{3\xi+2}).G(\alpha_{3\xi}, \alpha_{3\xi+1}, \alpha_{3\xi+2})\end{array}}{G(A\alpha_{3\xi}, A\alpha_{3\xi}, \alpha_{3\xi}).G(B\alpha_{3\xi+1}, B\alpha_{3\xi+1}, \alpha_{3\xi+1}).G(C\alpha_{3\xi+2}, C\alpha_{3\xi+2}, \alpha_{3\xi+2})}\right\}^{\lambda}$$

$$= \left\{\frac{\begin{array}{c}G(\alpha_{3\xi+1}, \alpha_{3\xi}, \alpha_{3\xi}).G(\alpha_{3\xi+2}, \alpha_{3\xi}, \alpha_{3\xi+1}).\\ G(\alpha_{3\xi+3}, \alpha_{3\xi}, \alpha_{3\xi+2}).G(\alpha_{3\xi}, \alpha_{3\xi+1}, \alpha_{3\xi+2})\end{array}}{G(\alpha_{3\xi+1}, \alpha_{3\xi+1}, \alpha_{3\xi}).G(\alpha_{3\xi+2}, \alpha_{3\xi+2}, \alpha_{3\xi+1}).G(\alpha_{3\xi+3}, \alpha_{3\xi+3}, \alpha_{3\xi+2})}\right\}^{\lambda} \qquad (3.5)$$

$$= \left\{\frac{\begin{array}{c}G(\alpha_{3\xi}, \alpha_{3\xi}, \alpha_{3\xi+1}).G(\alpha_{3\xi+1}, \alpha_{3\xi+1}, \alpha_{3\xi+2}).\\ G(\alpha_{3\xi+2}, \alpha_{3\xi+2}, \alpha_{3\xi+3}).G(\alpha_{3\xi}, \alpha_{3\xi+1}, \alpha_{3\xi+2})\end{array}}{G(\alpha_{3\xi}, \alpha_{3\xi+1}, \alpha_{3\xi+1}).G(\alpha_{3\xi+1}, \alpha_{3\xi+2}, \alpha_{3\xi+2}).G(\alpha_{3\xi+2}, \alpha_{3\xi+2}, \alpha_{3\xi+3})}\right\}^{\lambda}$$

Using Eq.(1.1) in the numerator of Eq. (3.5), we have

$$G(\alpha_{3\xi}, \alpha_{3\xi}, \alpha_{3\xi+1}) = G(\alpha_{3\xi}, \alpha_{3\xi+1}, \alpha_{3\xi+1}) \qquad (3.6)$$

$$G(\alpha_{3\xi+1}, \alpha_{3\xi+1}, \alpha_{3\xi+2}) = G(\alpha_{3\xi+1}, \alpha_{3\xi+2}, \alpha_{3\xi+2}) \qquad (3.7)$$

and

$$G(\alpha_{3\xi+2}, \alpha_{3\xi+2}, \alpha_{3\xi+3}) = G(\alpha_{3\xi+2}, \alpha_{3\xi+3}, \alpha_{3\xi+3}) \qquad (3.8)$$

Applying the results of (3.6), (3.7) & (3.8) in (3.5), we have

$$G\left(A\alpha_{3\xi}, B\alpha_{3\xi+1}, C\alpha_{3\xi+2}\right)$$

$$= \left\{\frac{\begin{array}{c}G(\alpha_{3\xi}, \alpha_{3\xi+1}, \alpha_{3\xi+1}).G(\alpha_{3\xi+1}, \alpha_{3\xi+2}, \alpha_{3\xi+2}).\\ G(\alpha_{3\xi+2}, \alpha_{3\xi+3}, \alpha_{3\xi+3}).G(\alpha_{3\xi}, \alpha_{3\xi+1}, \alpha_{3\xi+2})\end{array}}{G(\alpha_{3\xi}, \alpha_{3\xi+1}, \alpha_{3\xi+1}).G(\alpha_{3\xi+1}, \alpha_{3\xi+2}, \alpha_{3\xi+2}).G(\alpha_{3\xi+2}, \alpha_{3\xi+2}, \alpha_{3\xi+3})}\right\}^{\lambda}$$

$$= \left\{(\alpha_{3\xi}, \alpha_{3\xi+1}, \alpha_{3\xi+2})\right\}^{\lambda}$$

$$.$$
$$.$$
$$.$$

$$= G(\alpha_0, \alpha_1, \alpha_2)^{\lambda^{3\xi}}$$

Moreover by using rectangle inequality, we have

$$G\left(\alpha_{3\xi}, \alpha_\mu, \alpha_{3\nu}\right) \leq G\left(\alpha_{3\xi}, \alpha_{3\xi+1}, \alpha_{3\xi+2}\right).G\left(\alpha_{3\xi+1}, \alpha_{3\xi+2}, \alpha_{3\xi+3}\right)\ldots G\left(\alpha_{3\mu-1}, \alpha_{3\mu}, \alpha_{3\nu}\right)$$

$$= G(\alpha_0, \alpha_1, \alpha_2)^{\lambda^{3\xi}}.G(\alpha_0, \alpha_1, \alpha_2)^{\lambda^{3\xi+1}}\ldots G(\alpha_0, \alpha_1, \alpha_2)^{\lambda^{3\mu-1}}$$

$$= G(\alpha_0, \alpha_1, \alpha_2)^{\lambda^{3\xi}+\lambda^{3\xi+1}+\cdots+\lambda^{3\mu-1}}$$

$$= G(\alpha_0, \alpha_1, \alpha_2)^{\lambda^{3\xi}(1+3\xi+\ldots+3\mu-3\xi-1)}$$

$$= G(\alpha_0, \alpha_1, \alpha_2)^{\lambda^{3\xi}(\frac{1-\lambda^{3\mu-3\xi}}{1-\lambda})}$$

as $\lambda < 1$, so $1 - \lambda^{3\xi-3\mu} < 1$,   and $\dfrac{1-\lambda^{3\xi-3\mu}}{1-\lambda} < 1$. Thus $\dfrac{\lambda^{3\xi}(1-\lambda^{3\mu-3\xi}}{1-\lambda}) \to 1$

$$G(\alpha_{3\mu}, \alpha_{3\xi}, \alpha_{3\nu}) \to 1 \quad \text{when} \quad \xi, \mu, \nu \to \infty$$

so, the sequence $\{\alpha_\xi\}$ is a Cauchy sequence.

Now, we claim that ω is a fixed point of A, B, C.

Since,

$$G(\omega, \alpha_{3\xi+2}, \alpha_{3\xi+3}) = G(A\omega, B\alpha_{3\xi+1}, C\alpha_{3\xi+2})$$

$$\leq \left\{ \frac{\begin{array}{c} G(A\omega, \omega, \omega). G(B\alpha_{3\xi+1}, \alpha_{3\xi+1}, \alpha_{3\xi+1}). \\ G(C\alpha_{3\xi+2}, \alpha_{3\xi+2}, \alpha_{3\xi+2}). G(\alpha_{3\xi}, \alpha_{3\xi+1}, \alpha_{3\xi+2}) \end{array}}{G(A\omega, A\omega, \omega). G(B\alpha_{3\xi+1}, B\alpha_{3\xi+1}, \alpha_{3\xi+1}). G(C\alpha_{3\xi+2}, C\alpha_{3\xi+2}, \alpha_{3\xi+2})} \right\}^{\lambda}$$

$$= \left\{ \frac{\begin{array}{c} G(\omega, \omega, \omega). G(\alpha_{3\xi+2}, \alpha_{3\xi+1}, \alpha_{3\xi+1}). \\ G(\alpha_{3\xi+3}, \alpha_{3\xi+2}, \alpha_{3\xi+2}). G(\alpha_{3\xi}, \alpha_{3\xi+1}, \alpha_{3\xi+2}) \end{array}}{G(\omega, \omega, \omega). G(\alpha_{3\xi+2}, \alpha_{3\xi+2}, \alpha_{3\xi+1}). G(\alpha_{3\xi+3}, \alpha_{3\xi+3}, \alpha_{3\xi+2})} \right\}^{\lambda}$$

Taking limit as $\alpha_\xi \to \omega$ when $\xi \to \infty$, we have.

$$G(\omega, \omega, \omega) \leq \left\{ \frac{G(\omega, \omega, \omega). G(\omega, \omega, \omega). G(\omega, \omega, \omega). G(\omega, \omega, \omega)}{G(\omega, \omega, \omega). G(\omega, \omega, \omega). G(\omega, \omega, \omega)} \right\}^{\lambda} = 1$$

Similarly,

$$\lim_{\xi \to \infty} G(\alpha_{3\xi+1}, \omega, \alpha_{3\xi+3}) = 1$$

and,

$$\lim_{\xi \to \infty} G(\alpha_{3\xi+1}, \alpha_{3\xi+2}, \omega) = 1$$

So, $A\omega = B\omega = C\omega = \omega$

$\Rightarrow A, B, C$ have a common fixed point ω.

Corollary 3.1. Let Ψ be a multiplicative G $-$complete metric space and $A, B, C: \Psi \to \Psi$ be a contraction mapping. Then A, B, C has a common fixed point, if

$$G(A\alpha, B\beta, C\gamma)$$

$$\leq \left\{ \frac{G(A\alpha, \alpha, \alpha). G(B\beta, \beta, \beta). G(C\gamma, \gamma, \gamma). G(\alpha, \beta, \gamma)}{G(A\alpha, A\alpha, \alpha). G(B\beta, B\beta, \beta). G(C\gamma, C\gamma, \gamma)} \right\}^{\frac{1}{3}} \quad \text{for all } \alpha, \beta, \gamma \in \Psi \tag{3.9}$$

Example 3.1. Let $\Psi = \{0, \frac{1}{2}, 1\}$ and $A, B, C: \Psi^3 \to (0, \infty)$ be defined by

$$G(0,1,1) = 6 = G(1,0,0)$$

$$G(0, \frac{1}{2}, \frac{1}{2}) = 4 = G(\frac{1}{2}, 0, 0)$$

$$G(\frac{1}{2}, 1, 1) = 5 = G(1, \frac{1}{2}, \frac{1}{2})$$

$$G(0, \frac{1}{2}, 1) = \frac{15}{2}$$

$$G(\alpha, \alpha, \alpha) = 1 \text{ for all } \alpha \in \Psi$$

let $A, B, C : \Psi \to \Psi$ be defined by

$$A(0) = 0, A(\frac{1}{2}) = \frac{1}{2}, A(1) = 0$$

$$B(0) = 0, B(\frac{1}{2}) = \frac{1}{2}, B(1) = 0$$

$$C(0) = 0, C(\frac{1}{2}) = \frac{1}{2}, C(1) = 0$$

$$G(A(0), B(\frac{1}{2}), C(\frac{1}{2})) = G(0, \frac{1}{2}, \frac{1}{2}) = 4$$

$$G(A(0), B(1), C(1)) = G(0, 0, 0) = 1$$

$$G(A(\frac{1}{2}), B(1), C(1)) = G(\frac{1}{2}, 0, 0) = 4$$

$$G(A(0), B(\frac{1}{2}), C(1)) = G(0, \frac{1}{2}, 0) = 4$$

Now,

$$4 = G(A(0), B(\frac{1}{2}), C\frac{1}{2})) = G(0, \frac{1}{2}, \frac{1}{2}) \tag{3.10}$$

Applying Eq. (3.10) in Eq. (3.9), we get

$$\leq \left\{ \frac{G(A(0), 0, 0) . G(B(\frac{1}{2}), \frac{1}{2}, \frac{1}{2}) . G(C(\frac{1}{2}), \frac{1}{2}, \frac{1}{2})}{G(A(0), A(0), 0) . G(B(\frac{1}{2}), B(\frac{1}{2}), \frac{1}{2}) . G(C(\frac{1}{2}), C(\frac{1}{2}), \frac{1}{2})} \right\}^{1/3}$$

$$= \left\{ \frac{G(0, 0, 0) . G(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) . G(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})}{G(0, 0, 0) . G(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) . G(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})} \right\}^{1/3}$$

$$= \left\{ \frac{1.1.1}{1.1.1} \right\}^{1/3} = 1$$

Now,

$$1 = G(A(0), B(1), C(1)) = G(0,0,0) \tag{3.11}$$

Applying Eq. (3.11) in Eq. (3.10), we get

$$\leq \left\{ \frac{G(A(0),0,0).\,G(B(1),1,1).\,G(C(1),1,1)}{G(A(0),A(0),0).\,G(B(1),B(1),1).\,G(C(1),1,1)} \right\}^{1/3}$$

$$= \left\{ \frac{G(0,0,0).\,G(0,1,1).\,G(0,1,1)}{G(0,0,0).\,G(0,0,1).\,G(0,0,1)} \right\}^{1/3} = 1$$

Thus,

$$4 = G(A(\tfrac{1}{2}), B(1), C(1)) = G(\tfrac{1}{2}, 0, 0) \tag{3.12}$$

Applying Eq. (3.12) in Eq. (3.9), we get

$$\leq \left\{ \frac{G(A(\tfrac{1}{2}),\tfrac{1}{2},\tfrac{1}{2}).\,G(B(1),1,1).\,G(C(1),1,1)}{G(A(\tfrac{1}{2}),\tfrac{1}{2},\tfrac{1}{2}).\,G(B(1),B(1),1).\,G(C(1),C(1),1)} \right\}^{1/3}$$

$$= \left\{ \frac{G(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}).\,G(0,1,1).\,G(0,1,1)}{G(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}).\,G(0,0,1).\,G(0,0,1)} \right\}^{1/3}$$

$$= \left\{ \frac{1.6.6}{1.6.6} \right\}^{1/3} \not< 1$$

Hence, A, B, C has only one fixed point $\{\tfrac{1}{2}\}$.

4. CONCLUSION

We introduced rational type contraction conditions for multiplicative G-metric spaces which are more general then Gaba results [10]. Still, multiplicative G-metric space has to be explored further with its applications in Science and Technology. It looks that the further research in this field will open new applications of fixed point theory. Our results are more robust and they opened a new gate for the researchers in this new field.

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