ORIGINAL PAPER

A MODIFIED CLASS OF DUAL TO RATIO-TYPE ESTIMATORS FOR ESTIMATING THE POPULATION VARIANCE UNDER SIMPLE RANDOM SAMPLING SCHEME AND ITS APPLICATION TO REAL DATA

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Abstract. *This work is an extension to the work of [1] on ratio estimators of variance, by modification using dual to ratio method. The consistency conditions, bias, mean square error, optimum mean square error and efficiency have been derived and its performance is illustrated using natural populations. It is observed that the proposed class of estimators is most efficient at its optimum value, due to highest percent relative efficiency generated by it, when compared to the usual unbiased estimator for variance.*

Keywords: *auxiliary variable; dual to ratio-type estimators; mean square error.*

1. INTRODUCTION

The simplest estimator of mean (or variance) of a population under simple random sampling without replacement, is the simple random sample mean (or variance), when there is no auxiliary variable available. For improving the precision of the estimate of a parameter, auxiliary variable is widely used. A number of estimators such as ratio, product and linear regression estimators, and their combinations with other parameters are available in the literature. Ratio method of estimation is effective when the correlation between the study variable and the auxiliary variable is highly positive, whereas product method is used when correlation is negative. This manuscript deals with estimation of population variance with help of available auxiliary variable, using dual to ratio method, to improve its efficiency. Considerable amount of attention has been paid in survey sampling to address the problem of estimating population variance. In the papers [2-11] some authors have contributed extensively to the literature. Further work has been done by Singh and Nigam in [12] Recently, work performed on Enhanced estimators of population variance with the use of supplementary information in survey sampling and logarithmic type predictive estimators under simple random sampling [18-19] and Difference-cum-exponential efficient estimator for estimation population variance done by [20].

The outline of the paper is as follows: Section 2 describes the materials and methods which include notations used in the paper, literature survey and the existing estimators, proposed estimators using the work presented in [1]. The properties have also been derived

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here and some tables mention the consistent and inconsistent estimators. Efficiency comparison of the proposed estimator with other existing estimators have been made and numerically illustrated with real data sets. All comparisons are highlighted in tables. Results are interpreted in Section 3 and lastly, Section 4 summarizes the main conclusions.

2. MATERIALS AND METHODS

2.1. NOTATIONS

Consider a finite population $U = \{U_1, U_2, \dots, U_N\}$ of N distinct and identifiable units. Let Y be the study variable with value Y_i measured on $U_i, i = 1,2,3, \dots, N$. The problem is to estimate the population variance on the basis of a random sample selected from the population U.

N : Population size
n : Sample size
Y : Study variable
$\bar{Y}$: Mean of study variable
$S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{Y})^2$: Variance of the Study variable
$\bar{y}$: Sample mean of study variable
s_y^2 : Sample variance of study variable
$C_y = S_y/\bar{Y}$: Coefficient of variation of study variable
X : Auxiliary variable
$\bar{X}$: Mean of auxiliary variable
$\bar{x}$: Sample mean of auxiliary variable
$S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{X})^2$: Variance of the Auxiliary variable
s_x^2 : Sample variance
$C_x = S_x/\bar{X}$: Coefficient of variation of auxiliary variable
ρ : Correlation between Y and X
$\mu_{pr} = \frac{1}{N-1}\sum_{i}^{N}(y_i - \bar{Y})^p(x_i - \bar{X})^q$, $\lambda_{pq} = \frac{\mu_{pq}}{\mu_{20}^{\frac{p}{2}}\mu_{02}^{\frac{q}{2}}}$
$\lambda_{22} = \frac{\mu_{11}}{\sqrt{\mu_{02}\mu_{20}}}$: covariance between S_y^2 and S_x^2
$\lambda_{40} = \beta_2(y) = \frac{\mu_{40}}{\mu_{20}^2}$: kurtosis for population of study variable Y
$\lambda_{04} = \beta_2(x) = \frac{\mu_{04}}{\mu_{02}^2}$: kurtosis for population of auxiliary variable X
$\gamma = \frac{1}{n}$: sampling fraction
Q_1 : First quartile of the auxiliary variable
Q_3 : Third quartile of the auxiliary variable

$Q_r = Q_3 - Q_1$: Inter-quartile range of the auxiliary variable
$Q_d = (Q_3 - Q_1)/2$: Semi-quartile range of the auxiliary variable
$Q_a = (Q_3 + Q_1)/2$: Semi-quartile average of the auxiliary variable

2.2. EXISTING ESTIMATORS IN LITERATURE

The concept of ratio estimation was presented by [13]. He obtained the information from single-phase sampling and used it to develop the ratio estimator in order to estimate population mean. Murthy (in [14]) suggested a similar estimator for negatively correlated study and auxiliary variables, known as the product estimator.

The usual unbiased variance estimator, is given as

$$t_0 = s_y^2 \tag{2.1}$$

The above estimator is unbiased and up to the first order of approximation, its variance is:

$$V(t_0) = \gamma S_y^4(\lambda_{04} - 1) \tag{2.2}$$

In [3] the ratio type variance estimator for S_y^2 is proposed

$$t_1 = \frac{s_y^2}{s_x^2} S_x^2 \tag{2.3}$$

with Bias given as

$$B(t_1) = \gamma S_y^2(\lambda_{04} - 1)(1 - k) \tag{2.4}$$

where $k = \frac{\lambda_{22}-1}{\lambda_{04}-1}$ and the MSE given as

$$MSE(t_1) = \gamma S_y^4[(\lambda_{40} - 1) + (\lambda_{04} - 1)(1 - 2k)]$$

or (2.5)

$$MSE(t_1) = \gamma S_y^4[(\lambda_{40} - 1) + (\lambda_{04} - 1) - 2(\lambda_{22} - 1)].$$

A dual to ratio type estimator for S_y^2 was given by (Yadav & Kadilar, 2013) as

$$t_2 = s_y^2\left(\frac{s_x^{*2}}{S_x^2}\right) \tag{2.6}$$

where $s_x^{*2} = (NS_x^2 - ns_x^2)/(N - n) = (1 + g)S_x^2 - gs_x^2$, $g = \frac{n}{N-n}$ and MSE given as under

$$MSE(t_2) = \gamma {S_y}^4[(\lambda_{40} - 1) + g^2(\lambda_{04} - 1) - 2g(\lambda_{22} - 1)] \tag{2.7}$$

Yadav and Kadilar, in [15], also proposed the following ratio-cum-dual to ratio type estimator for S_y^2 as

$$t_3 = {s_y}^2\left[\alpha\left(\frac{{S_x}^2}{{s_x}^2}\right) + (1-\alpha)\left(\frac{{s_x}^{*2}}{{S_x}^2}\right)\right] \tag{2.8}$$

where α is a suitably chosen constant to be determined such that MSE of the proposed estimator is minimum. For $\alpha = 1$, the estimator (2.8) reduces to the ratio estimator in (2.3) and for $\alpha = 0$, it turns into the dual to ratio estimator in (2.6).

The MSE of t_3 is

$$MSE(t_3) = \gamma {S_y}^4[(\lambda_{40}-1) + \alpha^2(\lambda_{04}-1) - 2\alpha(\lambda_{22}-1)] \tag{2.9}$$

and for $\alpha = \frac{k-g}{1-g}$ the $MSE_{min}(t_3)$ is given as

$$MSE_{min}(t_3) = \gamma S_y^4\left[(\lambda_{40}-1) - \frac{(\lambda_{22}-1)^2}{(\lambda_{04}-1)}\right] \tag{2.10}$$

Singh et al. (in [1]) proposed the following class of estimator for S_y^2 as

$$t_3 = s_y^2\left(\frac{aS_x^2 + bs_x^2}{cs_x^2 + dS_x^2}\right) \tag{2.11}$$

where a, b, c, d are suitably chosen scalars such that $t_3 > 0$

The MSE is given as

$$\begin{aligned} MSE(t_3) = S_y^4[1 + A^2\{1 + \gamma[(\lambda_{40}-1) + \theta(\lambda_{04}-1)(\theta - 2\theta_2 + 4k)]\} \\ -\ 2A\{1 + \gamma\theta(\lambda_{04}-1)(k-\theta_2)\}\ ] \end{aligned} \tag{2.12}$$

where $A = (a+b)/(c+d)$,$\theta_1 = b/(a+b)$, $\theta_2 = c/(c+d)$

This estimator is consistent for $A = 1$, and $MSE(t_3)$is minimum at $\theta = -k = \theta_{opt}$

$$MSE_{min}(t_3) = \gamma S_y^4(\lambda_{40}-1)(1-\rho^{*2}) \tag{2.13}$$

Yadav et al., in [11], suggested a dual to ratio and product estimator for S_y^2

$$t_4 = {s_y}^2\left(\frac{\bar{x}^* + \alpha\bar{X}}{\bar{X} + \alpha\bar{x}^*}\right) \tag{2.14}$$

where α is a suitably chosen constant, obtained by minimizing the MSE of this estimator and

$$\underline{x}^* = (N\overline{\mathrm{X}} - n\overline{\mathrm{x}})/(N-n) = (1+g)\overline{\mathrm{X}} - g\overline{\mathrm{x}}$$

and

$$MSE(t_4) = \gamma {S_y}^4\left[(\lambda_{40}-1) + F^2g^2{C_x}^2 - 2Fg\lambda_{21}C_x\right] \tag{2.15}$$

where $F = (1-\alpha)/(1+\alpha)$ is minimum at $F = (\lambda_{21}/gC_x)$

The minimum mean squared error of t_4 for this optimum value of F is

$$MSE_{min}(t_4) = \gamma S_y^4\left[(\lambda_{40}-1)-{\lambda_{21}}^2\right] \tag{2.16}$$

where $\lambda_{21} = \frac{\mu_{21}}{\mu_{20}\sqrt{\mu_{02}}}$.

2.3. PROPOSED ESTIMATOR

Following [1] and [11] we propose the modified class of dual to ratio and product estimator for population variance as

$$t^* = s_y^2\left[\frac{as_x^{*2}+bS_x^2}{cS_x^2+ds_x^{*2}}\right] \tag{2.17}$$

where $a, b, c\ and\ d$ are suitable scalars such that $t^* > 0$. Scalars (a,b,c,d) are real values or parametric values such as $C_y, C_x, \rho, \beta_1(x), \beta_2(x)$ etc.

To obtain the bias and MSE, we write

$${s_y}^2 = {S_y}^2(1+e_0),$$

$${s_x}^2 = {S_x}^2(1+e_1),$$

such that

$$E(e_0) = E(e_1) = 0,$$

and

$$E({e_0}^2) = \gamma(\beta_2(y)-1) = \gamma(\lambda_{40}-1), E({e_1}^2) = \gamma(\beta_2(x)-1) = \gamma(\lambda_{04}-1)$$

$$E(e_0e_1) = \gamma(\lambda_{22}-1)$$

Equation (2.17) in terms of $e's$ is as in (2.18)

$$\begin{aligned} t^* &= S_y^2(1+e_0)\left[\frac{aS_x^2(1-ge_1)+bS_x^2}{cS_x^2+dS_x^2(1-ge_1)}\right] \\ &= S_y^2(1+e_0)\left[\frac{a(1-ge_1)+b}{c+d(1-ge_1)}\right] \\ &= S_y^2(1+e_0)\left(\frac{a+b}{c+d}\right)\left[1-\frac{age_1}{a+b}\right]\left[1-\frac{dge_1}{c+d}\right]^{-1} \\ &= \left(\frac{a+b}{c+d}\right)S_y^2(1+e_0)(1-g\theta_1e_1)(1-g\theta_2e_1)^{-1} \\ &= AS_y^2(1+e_0)(1-g\theta_1e_1)(1-g\theta_2e_1)^{-1} \end{aligned} \tag{2.18}$$

Expanding equation (2.18), neglecting the higher terms of $e's$ gives

$$t^* = AS_y^2[1+e_0-g\theta_1e_1-g\theta_1e_0e_1+g\theta_2e_1+g\theta_2e_0e_1-g^2\theta_1\theta_2e_0e_1+g^2\theta_2^2e_1^2\ldots]$$

$$\cong AS_y^2[1+e_0-(\theta_1-\theta_2)ge_1-(\theta_1-\theta_2)ge_0e_1-(\theta_1-\theta_2)g^2\theta_2e_1^2]$$

Let $(\theta_1-\theta_2) = \theta$.

$$t^* = AS_y^2[1 + e_0 - \theta(ge_1 + ge_0e_1 + g^2\theta_2e_1^2)] \tag{2.19}$$

$$\left(t^* - S_y^2\right) = S_y^2[A\{1 + e_0 - \theta(ge_1 + ge_0e_1 + g^2\theta_2e_1^2)\} - 1] \tag{2.20}$$

Taking expectation of equation (2.20), the bias up to the first degree of approximation is given as

$$B(t^*) = S_y^2[A\{1 - \gamma g\theta(\lambda_{04} - 1)(k + g\theta_2)\} - 1] \tag{2.21}$$

where $k = \frac{\lambda_{22}-1}{\lambda_{04}-1}$.

Squaring equation (2.20) and neglecting terms of $e's$ having power greater than two we have

$$B\left(t^* - S_y^2\right)^2 \cong S_y^4\left[1 + A^2\begin{Bmatrix}1 + 2e_0 - 2g\theta e_1 + e_0^2 + \theta(\theta - 2\theta_2)g^2e_1^2 \\ -4\theta ge_0e_1\end{Bmatrix} - 2A\{1 + e_0 - \theta(ge_1 + ge_0e_1 + g^2\theta_2e_1^2)\}\right] \tag{2.22}$$

Taking expectation, we get the MSE up to the first degree of approximation as

$$MSE(t^*) = S_y^4[1 + A^2\{1 + \gamma[(\lambda_{40} - 1) + g\theta(\lambda_{04} - 1)(g\theta - 2g\theta_2 - 4k)]\} - 2A1-\gamma g\theta\lambda04-1k+g\theta2 \tag{2.23}$$

The proposed class of estimators reduces to some existing consistent estimators of S_y^2 (Table 2.1).

Table 2.1. Dual of some existing Consistent estimators

Constants				Estimator
a	b	c	d	
1	1	1	1	$t_0^* = s_y^2$ [The usual unbiased estimator]
1	0	1	0	$t_1^* = s_y^2\left(\frac{s_x^{*2}}{s_x^2}\right)$ [1]
1	$-\frac{C_x}{S_x^2}$	$\frac{S_x^2 - C_x}{S_x^2}$	0	$t_2^* = s_y^2\left(\frac{s_x^{*2}-C_x}{s_x^2-C_x}\right)$ [6]
1	$\frac{-\beta_2(x)}{S_x^2}$	$\frac{S_x^2 - \beta_2(x)}{S_x^2}$	0	$t_3^* = s_y^2\left(\frac{s_x^{*2}-\beta_2(x)}{S_x^2-\beta_2(x)}\right)$ [6]
$\beta_2(x)$	$-\frac{C_x}{S_x^2}$	$\frac{S_x^2\beta_2(x) - C_x}{S_x^2}$	0	$t_4^* = s_y^2\left(\frac{s_x^{*2}\beta_2(x)-C_x}{S_x^2\beta_2(x)-C_x}\right)$ [6]
C_x	$-\frac{\beta_2(x)}{S_x^2}$	$\frac{S_x^2C_x - \beta_2(x)}{S_x^2}$	0	$t_5^* = s_y^2\left(\frac{s_x^{*2}C_x-\beta_2(x)}{S_x^2C_x-\beta_2(x)}\right)$ [6]
1	$\frac{Q_1}{S_x^2}$	$\frac{S_x^2 + Q_1}{S_x^2}$	0	$t_6^* = s_y^2\left(\frac{s_x^{*2}+Q_1}{S_x^2+Q_1}\right)$ [8]
1	$\frac{Q_3}{S_x^2}$	$\frac{S_x^2 + Q_3}{S_x^2}$	0	$t_7^* = s_y^2\left(\frac{s_x^{*2}+Q_3}{S_x^2+Q_3}\right)$ [8]
1	$\frac{Q_r}{S_x^2}$	$\frac{S_x^2 + Q_r}{S_x^2}$	0	$t_8^* = s_y^2\left(\frac{s_x^{*2}+Q_r}{S_x^2+Q_r}\right)$ [8]
1	$\frac{Q_d}{S_x^2}$	$\frac{S_x^2 + Q_d}{S_x^2}$	0	$t_9^* = s_y^2\left(\frac{s_x^{*2}+Q_d}{S_x^2+Q_d}\right)$ [8]
1	$\frac{Q_a}{S_x^2}$	$\frac{S_x^2 + Q_a}{S_x^2}$	0	$t_{10}^* = s_y^2\left(\frac{s_x^{*2}+Q_a}{S_x^2+Q_a}\right)$ [8]
ρ	$\frac{Q_3}{S_x^2}$	$\frac{S_x^2\rho + Q_3}{S_x^2}$	0	$t_{11}^* = s_y^2\left(\frac{s_x^{*2}\rho+Q_3}{S_x^2\rho+Q_3}\right)$ [9]

From equation (2.21), we see that $B(t^*) = S_y^2(A-1)$, i.e., the proposed class of estimators is not consistent. To make it consistent we put $A = 1$, which gives the equation as follows:

$$t_c^* = S_y^2[1 + e_0 - \theta(ge_1 + ge_0e_1 + g^2\theta_2 e_1^2)] \tag{2.24}$$

The bias and MSE of the proposed class of consistent (t_c^*) estimators up to the first degree of approximation are thus respectively given as

$$B(t_c^*) = -\gamma g S_y^2 \theta(\lambda_{04} - 1)(k + g\theta_2) \tag{2.25}$$

$$MSE(t_c^*) = \gamma S_y^4[(\lambda_{40} - 1) + g\theta(\lambda_{04} - 1)(g\theta - 2k)] \tag{2.26}$$

To obtain the optimum value of θ, we differentiate the $MSE(t_c^*)$ with respect to θ and equate the derivative to zero, θ_{opt} thus obtained is given by,

$$\theta_{opt} = kg^{-1}$$

Using the value of θ_{opt} in equation (2.26) we get the $MSE_{min}(t_c^*)$given as

$$\begin{aligned} MSE_{min}(t_c^*) &= \gamma S_y^4[(\lambda_{40} - 1) - (\lambda_{04} - 1)k^2] \\ &= \gamma S_y^4(\lambda_{40} - 1)(1 - \rho^{*2}) \end{aligned} \tag{2.27}$$

where

$$\rho^* = \frac{(\lambda_{22} - 1)}{\sqrt{(\lambda_{40} - 1)(\lambda_{04} - 1)}} \cong \frac{Cov(s_y^2, s_x^2)}{\sqrt{V(s_y^2)V(s_x^2)}}$$

Table 2.2. Some Consistent estimators of the proposed class of estimators (t_c^*)

Constants				Estimator
a	b	c	d	
$\beta_2(x)$	ρ	$\beta_2(x)$	ρ	$t_{c_1}^* = s_y^2\left(\frac{\beta_2(x)s_x^{*2} + \rho S_x^2}{\beta_2(x)S_x^2 + \rho s_x^{*2}}\right)$
$\beta_2(x)$	C_x	$\beta_2(x)$	C_x	$t_{c_2}^* = s_y^2\left(\frac{\beta_2(x)s_x^{*2} + C_x S_x^2}{\beta_2(x)S_x^2 + C_x s_x^{*2}}\right)$
1	ρ^2	1	ρ^2	$t_{c_3}^* = s_y^2\left(\frac{s_x^{*2} + \rho^2 S_x^2}{S_x^2 + \rho^2 s_x^{*2}}\right)$
ρ	f	ρ	f	$t_{c_4}^* = s_y^2\left(\frac{\rho s_x^{*2} + f S_x^2}{\rho S_x^2 + f s_x^{*2}}\right)$
C_x	f	C_x	f	$t_{c_5}^* = s_y^2\left(\frac{C_x s_x^{*2} + f S_x^2}{C_x S_x^2 + f s_x^{*2}}\right)$
$\beta_2(x)$	1	$\beta_2(x)$	1	$t_{c_6}^* = s_y^2\left(\frac{\beta_2(x)s_x^{*2} + S_x^2}{\beta_2(x)S_x^2 + s_x^{*2}}\right)$

Considering the case when $A \neq 1$ or $(a + b) \neq (c + d)$ in equation (2.19) we derive the expression for the inconsistent case by minimizing its mean square error (2.23) with respect A. The optimum value of A is given as:

$$A_{opt} = \left[\frac{\{1 - g\theta\gamma(\lambda_{04} - 1)(g\theta_2 + k)\}}{\{1 + \gamma[(\lambda_{40} - 1) + g\theta(\lambda_{04} - 1)(g\theta - 2g\theta_2 - 4k)]\}}\right] \tag{2.28}$$

Therefore, optimum value for MSE equation (2.23) (inconsistent case) is given as:

$$MSE_{min}(t_{ic}^*) = S_y^4 \left[1 - \frac{\{1 - g\theta\gamma(\lambda_{04} - 1)(g\theta_2 + k)\}^2}{\{1 + \gamma[(\lambda_{40} - 1) + g\theta(\lambda_{04} - 1)(g\theta - 2g\theta_2 - 4k)]\}}\right] \tag{2.28}$$

Table 2.3. Some Inconsistent estimators of the proposed class of estimators (t_{ic}^*)

Constants				Estimator
a	b	c	d	
Q_d	1	Q_1	1	$t_{ic_1}^* = s_y^2\left(\frac{Q_d(x)s_x^{*2} + S_x^2}{Q_1 S_x^2 + s_x^{*2}}\right)$
Q_r	1	Q_a	1	$t_{ic_2}^* = s_y^2\left(\frac{Q_r(x)s_x^{*2} + S_x^2}{Q_a S_x^2 + s_x^{*2}}\right)$
Q_a	Q_d	Q_3	1	$t_{ic_3}^* = s_y^2\left(\frac{Q_a(x)s_x^{*2} + Q_d S_x^2}{Q_3 S_x^2 + s_x^{*2}}\right)$
$\beta_2(x)$	C_x	$\beta_2(x)$	ρ	$t_{ic_4}^* = s_y^2\left(\frac{\beta_2(x)s_x^{*2} + C_x S_x^2}{\beta_2(x)S_x^2 + \rho s_x^{*2}}\right)$
$\beta_2(x)$	C_x	$\beta_2(x)$	ρ	$t_{ic_4}^* = s_y^2\left(\frac{\beta_2(x)s_x^{*2} + C_x S_x^2}{\beta_2(x)S_x^2 + \rho s_x^{*2}}\right)$
ρ	f	C_x	f	$t_{ic_5}^* = s_y^2\left(\frac{\rho s_x^{*2} + f S_x^2}{C_x S_x^2 + f s_x^{*2}}\right)$
$\beta_2(x)$	ρ	$\beta_2(x)$	1	$t_{ic_6}^* = s_y^2\left(\frac{\beta_2(x)s_x^{*2} + \rho S_x^2}{\beta_2(x)S_x^2 + s_x^{*2}}\right)$

2.4. EFFICIENCY COMPARISON

Case 1: $when\ A = 1, and\ \theta_{opt} = kg^{-1}$

From equation (2.2), (2.5), (2.7) and (2.27), comparing the mean square errors of $V(t_0)$, $MSE(t_1)$, $MSE(t_2)$and $MSE_{min}(t_c^*)$, we have

$$V(t_0) - MSE_{min}(t_c^*) = \gamma S_y^4(\lambda_{40} - 1)\rho^{*2} \geq 0 \tag{2.29}$$

$$\begin{aligned} MSE(t_1) - MSE_{min}(t_c^*) &= \gamma S_y^4[(\lambda_{04} - 1)(1 - 2k) + (\lambda_{40} - 1)\rho^{*2}] \\ &= \gamma S_y^4\left[\sqrt{(\lambda_{04} - 1)} - \rho^*\sqrt{(\lambda_{40} - 1)}\right]^2 \geq 0 MSE(t_1) - MSE_{min}(t_c^*) \\ &= \gamma S_y^4[(\lambda_{04} - 1)(1 - 2k) + (\lambda_{40} - 1)\rho^{*2}] \end{aligned} \tag{2.30}$$

$$\begin{aligned} MSE(t_2) - MSE_{min}(t_c^*) &= \gamma S_y^4[g(\lambda_{04} - 1)(g - 2k) + (\lambda_{40} - 1)\rho^{*2}] \\ &= \gamma S_y^4\left[g\sqrt{(\lambda_{04} - 1)} - \rho^*\sqrt{(\lambda_{40} - 1)}\right]^2 \geq 0 \end{aligned} \tag{2.31}$$

It follows from (2.29), (2.30) and (2.31) that the proposed class of consistent estimator is more efficient than the usual unbiased estimator(t_0), the classical ratio estimator for variance (t_1) by [3] and dual to ratio estimator(t_2).

Case 2: $when\ A = 1, and\ \theta_{opt} \neq kg^{-1}$

When optimum value of θ, does not coincide with its exact optimum value kg^{-1}, then from (2.2) and (2.26), we have

$$MSE(t_c^*) < V(t_0)$$

for which, we get either $\frac{2k}{g} < \theta < 0$ or $0 < \theta < \frac{2k}{g}$ or equivalently $\left\{0, \frac{2k}{g}\right\} < \theta < \left\{0, \frac{2k}{g}\right\}$

2.5. NUMERICAL ILLUSTRATIONS

For numerical illustrations, we have taken the data [16] p.108 and [17] p.228. The data is reproduced as under:

Table 2.4. Data Sets

Data Set 1: [16] Y: Population in 1981 X: Cultivated area (in acres) in 1981				Data set 2: [17] Y: Output of factories X: No. of workers			
N	70	C_y	0.6254	N	80	C_y	0.3542
n	25	S_x	140.8572	n	20	S_x	8.4542
$\bar{Y}$	96.7000	C_x	0.8037	$\bar{Y}$	51.8264	C_x	0.7507
$\bar{X}$	175.2671	λ_{04}	7.0952	$\bar{X}$	11.2624	λ_{04}	2.8664
ρ	0.7293	λ_{40}	4.7596	ρ	0.9413	λ_{40}	2.2667
S_y	60.7140	λ_{22}	4.6038	S_y	18.3569	λ_{22}	2.2209
Q_1	80.1500	Q_a	152.5875	Q_1	9.318	Q_a	11.0625
Q_2	160.3000	Q_d	72.4375	Q_2	7.5750	Q_d	5.9125
Q_3	225.0250	Q_r	144.8750	Q_3	16.975	Q_r	11.82

The MSEs of the proposed consistent, the inconsistent estimators and the existing consistent estimators with the usual unbiased ratio estimator have been calculated. Also, their Percent Relative Efficiency (PREs) have been calculated (Table 2.5, 2.6, 2.7 & 2.8).

Table 2..: MSEs for Data Set 1

$MSE(t_0^*)$	2037662	$MSE(t_{c1}^*)$	945443.6
$MSE(t_1^*)$	885502.6	$MSE(t_{c2}^*)$	954486.2
$MSE(t_2^*)$	885497.3	$MSE(t_{c3}^*)$	1468844
$MSE(t_3^*)$	885456	$MSE(t_{c4}^*)$	1412978
$MSE(t_4^*)$	885501.8	$MSE(t_{c5}^*)$	1352718
$MSE(t_5^*)$	885444.6	$MSE(t_{c6}^*)$	980263.8
$MSE(t_6^*)$	886045.5	$MSE(t_{ic1}^*)$	889583.4
$MSE(t_7^*)$	887098.2	$MSE(t_{ic2}^*)$	887433.6
$MSE(t_8^*)$	886504.6	$MSE(t_{ic3}^*)$	1036948
$MSE(t_9^*)$	885992	$MSE(t_{ic4}^*)$	949890.4
$MSE(t_{10}^*)$	886567.5	$MSE(t_{ic5}^*)$	1382395
$MSE(t_{11}^*)$	887745.4	$MSE(t_{ic6}^*)$	961913.5

Table 2.6. PREs of Data Set 1

$PRE(t_0^*, s_y^2)$	100.00	$PRE(t_{c1}^*, s_y^2)$	214.46
$PRE(t_1^*, s_y^2)$	229.72	$PRE(t_{c2}^*, s_y^2)$	212.44
$PRE(t_2^*, s_y^2)$	229.72	$PRE(t_{c3}^*, s_y^2)$	176.87
$PRE(t_3^*, s_y^2)$	**230.12**	$PRE(t_{c4}^*, s_y^2)$	143.75
$PRE(t_4^*, s_y^2)$	229.75	$PRE(t_{c5}^*, s_y^2)$	150.09
$PRE(t_5^*, s_y^2)$	229.75	$PRE(t_{c6}^*, s_y^2)$	206.86
$PRE(t_6^*, s_y^2)$	229.14	$PRE(t_{ic1}^*, s_y^2)$	228.17
$PRE(t_7^*, s_y^2)$	228.85	$PRE(t_{ic2}^*, s_y^2)$	**228.76**
$PRE(t_8^*, s_y^2)$	229.41	$PRE(t_{ic3}^*, s_y^2)$	195.09
$PRE(t_9^*, s_y^2)$	228.11	$PRE(t_{ic4}^*, s_y^2)$	213.18
$PRE(t_{10}^*, s_y^2)$	229.41	$PRE(t_{ic5}^*, s_y^2)$	146.63
$PRE(t_{11}^*, s_y^2)$	228.65	$PRE(t_{ic6}^*, s_y^2)$	210.46

Table 2.7. MSEs for Data Set 2

$MSE(t_0^*)$	860368.9	$MSE(t_{c1}^*)$	616867.7
$MSE(t_1^*)$	448383.8	$MSE(t_{c2}^*)$	585194
$MSE(t_2^*)$	448373.5	$MSE(t_{c3}^*)$	827480.7
$MSE(t_3^*)$	448344.6	$MSE(t_{c4}^*)$	640160.6
$MSE(t_4^*)$	448380.2	$MSE(t_{c5}^*)$	681750.3
$MSE(t_5^*)$	448331.6	$MSE(t_{c6}^*)$	626322.5
$MSE(t_6^*)$	448511.1	$MSE(t_{ic1}^*)$	522105
$MSE(t_7^*)$	448615.7	$MSE(t_{ic2}^*)$	495656.6
$MSE(t_8^*)$	448545.3	$MSE(t_{ic3}^*)$	580886.7
$MSE(t_9^*)$	448464.6	$MSE(t_{ic4}^*)$	600809.2
$MSE(t_{10}^*)$	448534.9	$MSE(t_{ic5}^*)$	660640
$MSE(t_{11}^*)$	448630.1	$MSE(t_{ic6}^*)$	621576.7

Table 2.8. PREs of Data Set 2

$PRE(t_0^*, s_y^2)$	100.00	$PRE(t_{c1}^*, s_y^2)$	139.47
$PRE(t_1^*, s_y^2)$	191.88	$PRE(t_{c2}^*, s_y^2)$	147.02
$PRE(t_2^*, s_y^2)$	191.88	$PRE(t_{c3}^*, s_y^2)$	103.97
$PRE(t_3^*, s_y^2)$	**191.89**	$PRE(t_{c4}^*, s_y^2)$	134.39
$PRE(t_4^*, s_y^2)$	191.88	$PRE(t_{c5}^*, s_y^2)$	126.2
$PRE(t_5^*, s_y^2)$	191.90	$PRE(t_{c6}^*, s_y^2)$	137.36
$PRE(t_6^*, s_y^2)$	191.82	$PRE(t_{ic1}^*, s_y^2)$	164.78
$PRE(t_7^*, s_y^2)$	191.78	$PRE(t_{ic2}^*, s_y^2)$	**173.58**
$PRE(t_8^*, s_y^2)$	191.81	$PRE(t_{ic3}^*, s_y^2)$	148.11
$PRE(t_9^*, s_y^2)$	191.84	$PRE(t_{ic4}^*, s_y^2)$	143.2
$PRE(t_{10}^*, s_y^2)$	191.81	$PRE(t_{ic5}^*, s_y^2)$	130.23
$PRE(t_{11}^*, s_y^2)$	191.77	$PRE(t_{ic6}^*, s_y^2)$	138.41

The MSEs and PREs using the optimum values have also been derived (Table 2.9).

$$i.e. if\ we\ use\ \theta = \theta_{opt} = kg^{-1}$$

Table 2.9. MSE and PRE using optimum value of θ

Dataset 1		**Dataset 2**	
$MSE_{min}(t_c^*)$	881314	$MSE_{min}(t_c^*)$	317910.6
$PRE\left(t_0^*, t_{c(min)}^*\right)$	**231.20**	$PRE\left(t_0^*, t_{c(min)}^*\right)$	**270.63**

3. RESULTS AND DISCUSSION

The mean square errors of the proposed class of estimators have been derived for two datasets mentioned in Table 2.4. The MSEs for dataset 1 are displayed in Table 2.5 and Table 2.7 for dataset 2. It can be observed in Table 2.6 (PREs-Dataset 1) and Table 2.8 (PREs-Dataset 2) that the proposed consistent estimators and inconsistent estimators are better than the usual unbiased estimator of variance, the ratio estimator, and other mentioned estimators. Further from Table 2.9 it can be deduced that the proposed class of estimators is most efficient at its optimum value, as it gives the highest PRE when compared to the usual unbiased estimator for variance.

4. CONCLUSIONS

An improved class of dual to ratio and product estimators for estimating population variance using auxiliary information has been proposed. Calculations have been made using combination of different parameters i.e., coefficient of variation of the auxiliary variable, the correlation coefficient, sampling fraction, the kurtosis of the auxiliary variable for the consistent estimators; and combination of the correlation coefficient, kurtosis of the auxiliary variable, coefficient of variation of auxiliary variable, sampling fraction, quartiles, interquartile range, semi-quartile range and semi-quartile average for the inconsistent estimators. From the results of the empirical study and theoretical discussions, it is inferred that the proposed estimator for estimating the population variance of the study variable under the optimum condition performs better than the sample variance estimator, traditional ratio estimator for variance, and the dual to ratio-type estimator, therefore it can preferably be used for the estimation of population variance.

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