ORIGINAL PAPER

GEOMETRY OF THE FOCAL SURFACES OF THE INVERSE SURFACE OF A REGULAR SURFACE IN E^3

MEHMET FIDAN[1], AYHAN SARIOĞLUGIL[1]

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Abstract. *The purpose of this paper, first, is to give a definition of the focal surfaces of the inverse of a given regular surface in* E^3. *Second, some new characteristic properties of the focal surfaces are to express depending on the algebraic invariants of the inverse surface of a given regular surface. In the last part of the study, we gave examples supporting our claims and plotted their graphics with the help of Maple software.*

Keywords: *inversion; surface; support function; principal curvatures; fundamental forms; Christoffel symbols of second kind.*

1. INTRODUCTION

This paper deal with the geometry of the focal surfaces the inverse surface of a regular surface, which is given in E^3, with respect to the unit sphere S^2. Let us start this section by ntroducing the inverse and focal surfaces, respectively. Early works on inversive geometry were made by Appollonius of Perga. Appollonius was famous for his work on astronomy rather than his well-known work on cone sections today. Unfortunately, Appollonius' original work on astronomy and most of his mathematical works except for the related conics have been lost. We can deduce this conclusion from the comments of Pappus of Alexandria. According to Pappus of Alexandria, Appollonius investigated one particular family of circles, and straight lines. Appollonius defined the curve as locus of points P provide the equation

$$c_k(A,B) \ldots \overrightarrow{PB} = k\,\overrightarrow{PA} \tag{1.1}$$

where A and B are two points in Euclidean plane and $k \in \mathbb{R}^+$ is constant. For $k = 1$ this curve is a straight line, otherwise is a circle, which is called Appollonian circle. Subsequently Appollonius proved that a circle c (with center C and radius r) belongs to the family $\{c_k(A,B)\}$ only if

$$\overline{BC}\ \ \overline{AC} = r^2 \tag{1.2}$$

where A and B are on the same ray C [1].

The nineteenth-century has great importance in view of the history of geometry. In this century, it has occurred important developments in geometry. In the early years of this century, French engineer and mathematician Jean-Victor Poncelet defined the inverse points with respect to a circle and called the reciprocal points. Swiss mathematician Jacob Steiner considered the inverse points in connection with the power of a point for a circle. The invention of the transformation of inversion is sometimes credited to Ludwig Immanuel Magnus, who published his work on the subject in 1831 [2]. A systematic development of

[1] Ondokuz Mayıs University, Faculty of Sciences and Art, Department of Mathematics, 55200 Atakum, Samsun.
E-mail: fsmehmet55@gmail.com .

inversion in a circle was first given by German mathematician and physicist Julius Plücker in his 1834 paper entitled "Analytisch-geometrische Aphorismen". Julius Plücker approximated the problem analytically and showed that the inversion preserved the magnitude of the angles between lines and circles. From here, it can be said that the inversion is a conform transformation [3, 4]. On the other hand, conformal transformations of the plane appeared in Swiss mathematican Leonhard Euler's 1770 paper [5]. In this paper, Euler considered linear fractional transformations of the complex plane. In 1845, Sir William Thomson (Lord Kelvin) used inversion to give geometrical proofs of some difficult propositions in the mathematical theory of elasticity. In 1847, French mathematician Joseph Liouville called inversion the transformation by reciprocal radii [6].

Definition 1.1. Let c be a circle with centered O and radius r, and let P be any point other than O. If R is the point on the line OP that lies on the same side of O as P and satisfies the equation

$$\overline{OP}\,\overline{OR}=\overline{OQ}^2=r^2 \tag{1.3}$$

then we call R the inverse of P with respect to the circle c. Similarly, we can say that P is the inverse of R with respect to the circle c (Fig. 1). On the other hand, if R lies on the opposite side of O from P then we have

$$\overline{OP}\,\overline{OR}=-\overline{OQ}^2=-r^2. \tag{1.4}$$

The point O is called the center of inversion, and c is called the circle of inversion, [7, 8].

From here we can write the following properties:

1. If P is inside in c then its inverse R is outside and vice versa.
2. Every point on c is its own inverse.
3. As P moves closer to the center of O, the inverse point P' approaches infinity.

In this case, the Euclidean plane with the added point of infinity is called inversive plane. Then inversion can be defined as a bijective transformation.

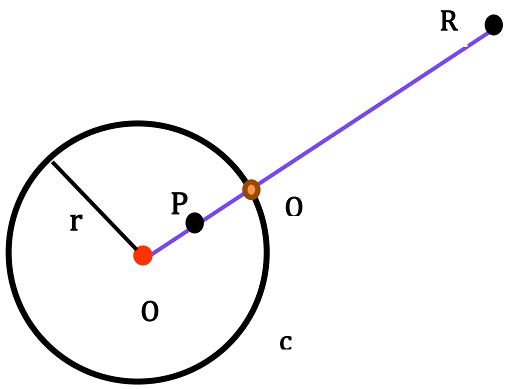

Figure 1. Inversion in circle.

Definition 1.2. For all $P=(x_1,y_1)\in\mathbb{R}^2-\{O\}$, the function

$$f:\mathbb{R}^2-\{O\}\rightarrow\mathbb{R}^2$$
$$P \quad \rightarrow f(P)=\left(\frac{x_1}{{x_1}^2+{y_1}^2},\frac{y_1}{{x_1}^2+{y_1}^2}\right) \tag{1.5}$$

is called an inversion transformation where [7, 8].

$$C=\{Q=(x,y):x^2+y^2=r^2\}\subset\mathbb{R}^2.$$

Circle inversion can be used to study several well-known problems and theorems in geometry such as the problem of Appollonius, Steiner porism, Feuerbach's theorem, The Pappus' chain theorem, Peaucellier's cell, Construction problems, Ptolemy's theorem, among others. In addition, inversion has important applications in areas of physics, engineering, astronomy, medicine, geometric modeling, etc.

Circle inversion is generalizable to sphere inversion in E^3. Inversion in a sphere was considered by Italian mathematician Giusto Bellavitis in his 1836 paper entitled "Teoria della figure inverse, e loro uso nella geometria elementare". In this paper [9], Bellavitis showed that stereographic projection is the inversion of a sphere into a plane. The Riemann sphere was introduced by Riemann in 1857 in paper [4]. In 1852, German mathematician August Ferdinand Mobius introduced the cross ratio of four points in the plane and studied the Mobius transformations of plane by Mobius in 1855 [10]. Liouville considered the Mobius transformations of 3-space in 1847. Subsequently, Mobius transforms were used by Henri Poincare, Eugenio Beltrami etc. to create a geometric model for non-Euclidean geometries, [2, 11]. Because of the inversion transformation has a conformal structure in plane (or space), many scientists studied in this subject. For example; considering the central conics in plane Allen [12], studied circles. Childress investigate the inversion with respect to the central conics [13]. Ottens studied geometry of Dupin cyclides under the inversion transformation [14]. Sarıoğlugil and Kuruoğlu, investigated inverse surfaces inE^3 [15, 16]. Tul et al. investigated inverse surface of a regular surface with respect to the unit sphere and found some new characteristic properties of its [17]. Furthermore, the inverse surface studied by Röthe [18].

The inversion of a point P in E^3 with respect to a sphere centered at a point O with radius r is r such that

$$\overline{OP}\,\overline{OR}=\overline{OQ}^2=r^2$$

where the points P and R lie on the same side of OP ray, Then, we can give the definition of spherical inversion.

Definition 1.3. For all $P=(x_1,y_1,z_1)\in\mathbb{R}^3-\{O\}$, the function

$$f:\mathbb{R}^3-\{O\}\rightarrow\mathbb{R}^3$$
$$P \quad \rightarrow f(P)=\left(\frac{r^2x_1}{{x_1}^2+{y_1}^2+{z_1}^2},\frac{r^2y_1}{{x_1}^2+{y_1}^2+{z_1}^2},\frac{r^2z_1}{{x_1}^2+{y_1}^2+{z_1}^2}\right) \tag{1.6}$$

is called an inversion transformation where (Fig. 2) [7, 8],

$$S_r^2=\{Q=(x,y,z):x^2+y^2+z^2=r^2\}\subset\mathbb{R}^3.$$

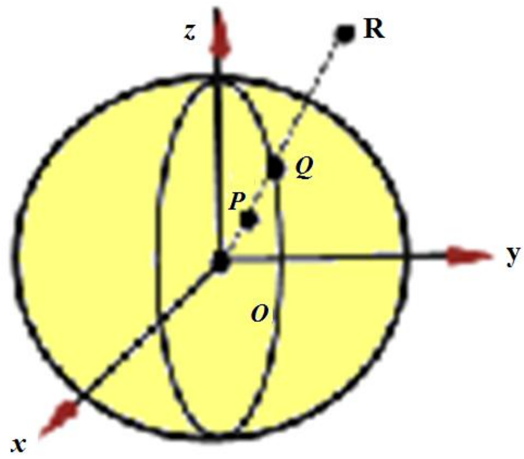

Figure 2. Inversion in sphere.

Now, let us investigate the geometry of focal surfaces of a regular surface in E^3. For a better understanding of focal surfaces, which are known as special rectilinear congruences, the concept of the rectilinear congruence surfaces should be well-known in all aspects. So first, let's take a brief look at the history of rectilinear congruence surfaces. Rectiinear congruences were first studied by Monge [19]. Monge has divided his works into two parts such that rectilinear congruence and rectilinear normal congruences. Monge's work on congruences had a great impact. For this reason, many scientists have made important contributions by writing works on these subjects.For example, while the relationship between linear congruences and optics was investigated by Malus [20], Dupin [21], Hamilton [22] and simulteneously Kummer [23], Darboux [24] and Weingarten [25] were developing the general theory, as a result of these developements, studies on the theory of the linear congruence surfaces have continued and are still continuing (i.e., Alexiou [26], Amur [27], Behari [28], Maekawa [29], Ogura [30], Papadoulous [31, 32], Taliaferro [33], Hagen et al. [34-36] and Hahman [37]). With the aim of getting further information about inversion geometry, the interested readers are referred to [19-37].

Then, we can give the definition of the rectilinear congruence surface in 3-dimensional Euclidean space E^3.

Definition 1.4. Let M be a regular surface which is defined by

$$\begin{aligned} X: D \subset E^2 &\longrightarrow X(D) \subset E^3 \\ (u,v) &\longrightarrow X(u,v) = \big(x(u,v), y(u,v), c(u,v)\big) \end{aligned} \tag{1.7}$$

where $x, y, z \in C^3(D, \mathbb{R})$ and $N(u,v)$ is the unit normal vector of M.

A differentiable 2-parameter family of lines is called line congruence. At each point of a parametric surface $X(u,v)$, a line passing through this point is assigned. Its direction is given by the unit vector $E(u,v)$. The line congruence $\overline{\overline{M}}$ therefore has the parameteric representation [35, 37],

$$R(u,v,\lambda) = X(u,v) + \lambda(u,v)E(u,v) \tag{1.8}$$

For each value of $(u,v) \in D$, Eq. (1.8) is a generating line of $\overline{M}$ and λ is the parameter of its points which measures the distance to the corresponding point of X. The surface $\overline{M}$ determined by the eq. (1.8) is called the rectilinear congruence surface. Here, M is the generator surface [19]. On the other hand, if the unit normal vektör N of M is taken instead of the unit vector E in (1.8), we get

$$R(u,v,\lambda) = X(u,v) + \lambda(u,v)N(u,v). \tag{1.9}$$

In this case, the rectilinear congruence surface $\bar{M}$ is called the normal rectilinear congruence surface or offset surface and λ is called offset function [34-37]. It is well-known that the most important concept is curvature. Curvatures give us important information about the shapes of geometric objects (curves and surfaces) in space.

Substituting by $\lambda(u,v) = \frac{1}{k_i(u,v)}, 1 \leq i \leq 2$, into (1.9), we have

$$\bar{M}_1 \ldots R_i(u,v,) = X(u,v) + \frac{1}{k_i(u,v)}, N(u,v). \tag{1.10}$$

where k_1 and k_2 are the principal curvatures of M Then, the normal rectilinear congruence surfaces $\bar{M}_1$ and $\bar{M}_2$ are called focal surfaces of M [34-37], (Fig. 3).

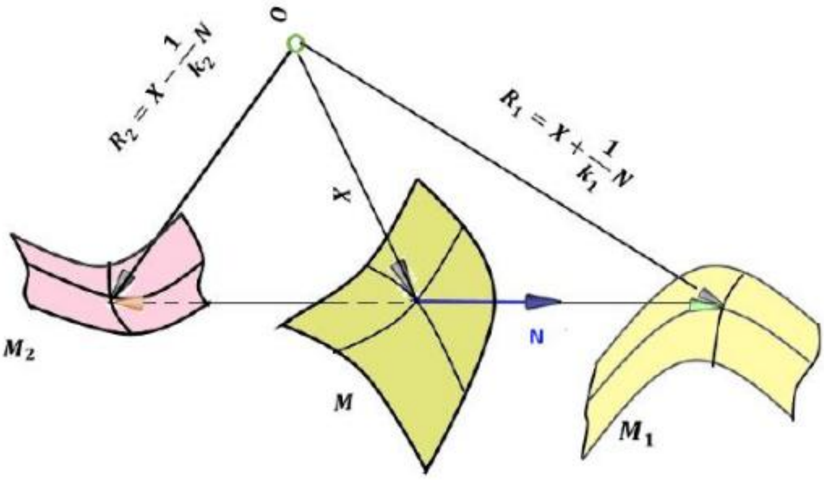

Figure 3. The focal surfaces of *M*.

Furthermore, setting $\lambda(u,v) = af(k_1,k_2), a\tau \in \mathbb{R}$, in (1.9) we get

$$\bar{M} \cdots R(u,v) = X(u,v) + af(k_1,k_2), N(u,v). \tag{1.11}$$

Hence, the surface $\overline{M}$ which is determined by (11) is called the generalized focal surface and f is called offset function [19].The offset function are given in the following form depending on the principal curvatures k_1 and k_2 of M:

1. $f = k_1k_2$ (Gauss curvature),
2. $f = \frac{k_1+k_2}{2}$ (Mean curvature),
3. $f = (k_1)^2{+}{+}(k_2)^2$ (Energy functiona),
4. $f = k_1k_2$ (Gauss curvature),
5. $f = |k_1| + |k_2|$ (Absolute curvature),
6. $f = k_i 1 \leq i \leq 2$, (Principal curvature),
7. $f = constant$ (Parallel surfaces).

This method was introduces by Hagen and Hahmann [34-37] and is based on concept of focal surfaces which are known from line geometry. Now, here are a few examples of selecting offset functions:

1. If we choose the flat points of the surface, then a flat point is a special umbilic point, with $k_1 = k_2 = 0$.The detection of flat points can be done by choosing one of the offset functions [35-37],

$$f = |k_1| + |k_2|$$

or

$$f = (k_1)^2 + (k_2)^2.$$

2. A surface is locally convex at $X(u, v)$, if the Gaussian curvature is positive at this point. Often a surface called non convex, if there is a change in sign of the Gaussian curvature. If one takes the offset functions

$$f = k_1 k_2 = K$$

The surfaces $X(u, v)$ and $R(u, v)$ intersects at the parabolic points [35-37]. This paper is organized as follows: Section 1 is reserved for the introduction, where the inverse and focal surfaces are introduced, the problem is presented, and a brief literature summary is included. Section 2 occurs of the basic concepts of the curves and surfaces theory to be used during throughout our study. In section 3, some characteristic properties of the inverse surface of a regular surface in E^3 are given related to the support function and the length of the sition vector of M. Finally, invers surfaces of some surfaces are found and their graphics are drawn with the help of the maple software program.

2. PRELIMINARIES

In this section, we will give the basic concepts of the differential geometry for later use. Let us consider the three dimensional Euclidean space E^3 with the inner product $g = \langle , \rangle$. Here, g is a metric and is called the Euclidean metric. This metric is given by

$$g = dx^2 + dy^2 + dz^2 \tag{2.1}$$

where $\{x, y, z\}$ is the local coordinate system in E^3. On the other hand, the vectoral product of the vectors $E_i = \sum_{i=1}^{3} x_i \, e_i$ and $E_j = \sum_{j=1}^{3} y_j \, e_y$ is defined by

$$E_i \wedge E_j = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \tag{2.2}$$

where $\left\{ e_i = \frac{\partial}{\partial x_i}\Big|_P , 1 \leq \dot{\text{I}} \leq 3 \right\}$ is a standard basis of E^3 [38].

Let M be a regular surface in E^3 given by

$$\begin{aligned} X &: D \subset E^2 \longrightarrow X(D) \subset E^3 \\ (u, v) &\longrightarrow X(u, v) = \big(x(u, v), y(u, v), z(u, v)\big) \end{aligned} \tag{2.3}$$

be the parametric equation. Here

$$X_u \wedge X_v \neq 0 \tag{2.4}$$

where $X_u = \frac{\partial}{\partial u}(X)$ and $X_v = \frac{\partial}{\partial v}(X)$ are the partial derivates of X. From here, the unit normal vector of M is

$$N = \frac{1}{\|X_u \wedge X_v\|}(X_u \wedge X_v) \tag{2.5}$$

Then we can say that

$$T_M(P) = Span\{X_u, X_v\}. \tag{2.6}$$

In this case, the set $\{X_u, X_v, N\}$ is called the frame of Gauss of M [38].On the other hand, the regular curve on M is defined by [39],

$$\begin{aligned} \beta = X \circ \alpha : I \subset \mathbb{R} &\longrightarrow M \subset E^3 \\ s &\longrightarrow \beta(s) = (X \circ \alpha)(s) = X(\alpha(s)) \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} \alpha : I \subset \mathbb{R} &\longrightarrow M \subset E^3 \\ s &\longrightarrow \alpha(s) = (u(s), v(s)) \end{aligned} \tag{2.8}$$

we keep the first parameter u constant, $v \to X(u,v)$ is a curve on M. Similarly, if v is constant, $u \to X(u,v)$ is a curve on M. These curves called the parameter (or grid) curves of M [42] (Fig. 4).

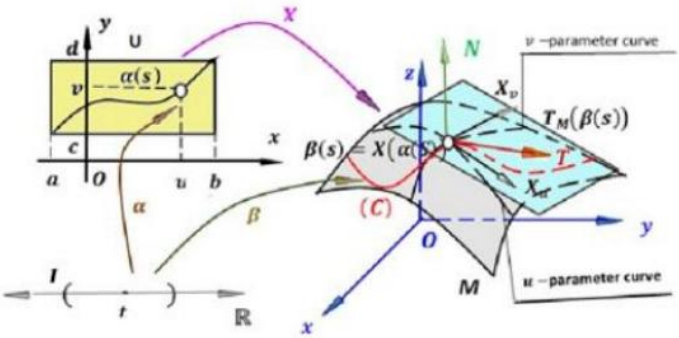

Figure 4. The curves on surfaces.

Definition 2.1. Let M be a regular surface in E^3. The support function of M is defined by

$$\begin{aligned} h : M &\to \mathbb{R} \\ P &\to h(P) = \langle X, N \rangle \end{aligned} \tag{2.9}$$

where X is the position vector of M, [15-39]. From here, it can be said that h is the length of the projection of X in direction of N, (Fig. 5),

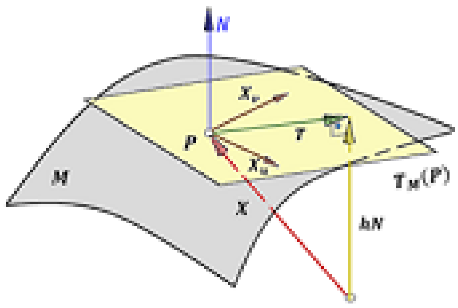

Figure 5. The support function of *M*.

Definition 2.2. Let M be a regular surface in E^3 and N be a unit normal vector of M. Then, the shape operator of M is defined by

$$\begin{aligned} S : T_M(P) &\to T_M(P) \\ X &\to S(X) = -D_X N \end{aligned} \tag{2.10}$$

where D is the affine connection on M. Furthermore, the shape operator is linear and self-adjoint. Then the matrix of the shape oprator S is

$$S = \begin{bmatrix} -\dfrac{\det(X_{uu}, X_u, X_v)}{\|X_u\|^3 \|X_v\|} & -\dfrac{\det(X_{uv}, X_u, X_v)}{\|X_u\|^2 \|X_v\|^2} \\ -\dfrac{\det(X_{uv}, X_u, X_v)}{\|X_u\|^2 \|X_v\|^2} & -\dfrac{\det(X_{vv}, X_u, X_v)}{\|X_u\| \|X_v\|^3} \end{bmatrix} \tag{2.11}$$

where $T_M(P) = Sp\{X_u, X_v\}$,]40].

Definition 2.3. Let M be a regular surface in E^3 and S be shape operator of M. Then, the q th fundamental form of M is defined by

$$\begin{aligned} I^q : T_M(P) \times T_M(P) &\to C^\infty(M, \mathbb{R}), 1 \le q \le 3, \\ (X, Y) &\to I^q(X, Y) = \langle S^{q-1}(X), Y \rangle \end{aligned} \tag{2.12}$$

where N is the unit normal vector of M [40].

For q = 1, the map

$$\begin{aligned} I : T_M(P) \times T_M(P) &\to C^\infty(M, \mathbb{R}) \\ (X, Y) &\to I(X, Y) = \langle X, Y \rangle \end{aligned} \tag{2.13}$$

is called the first fundamental form of M.

For q = 2, the map

$$\begin{aligned} II : T_M(P) \times T_M(P) &\to C^\infty(M, \mathbb{R}) \\ (X, Y) &\to II(X, Y) = \langle S(X), Y \rangle \end{aligned} \tag{2.14}$$

is called the second fundamental form of M.

For q = 3, the map

$$\begin{aligned} III: T_M(P)\times T_M(P) &\to C^\infty(M,\mathbb{R}) \\ (X,Y) \quad &\to \quad III(X,Y)=\langle S^2(X),Y\rangle \end{aligned} \tag{2.15}$$

is called the third fundamental form of M. Then, for the coefficients of the first, second and third fundamental forms we have [42],

$$\begin{cases} g_{11} = \langle X_u, X_u\rangle \\ g_{12} = \langle X_u, X_v\rangle\,, \\ g_{22} = \langle X_v, X_v\rangle \end{cases} \tag{2.16}$$

$$\begin{cases} b_{11} = -\langle X_u, N_u\rangle = \langle X_{uu}, N\rangle \\ ,b_{12} = -\langle X_u, N_v\rangle = \langle X_{uv}, N\rangle, \\ b_{22} = -\langle X_v, N_v\rangle = \langle X_{vv}, N\rangle \end{cases} \tag{2.17}$$

and

$$\begin{cases} n_{11} = \langle N_u, N_u\rangle \\ n_{12} = \langle N_u, N_v\rangle. \\ n_{22} = \langle N_v, N_v\rangle \end{cases} \tag{2.18}$$

From here, the first, second and third fundamental forms are [42],

$$I = g_{11}du^2 + 2g_{12}du\,dv + g_{22}dv^2 \tag{2.19}$$

$$II = b_{11}du^2 + 2b_{12}du\,dv + b_{22}dv^2 \tag{2.20}$$

$$III = n_{11}du^2 + 2n_{12}du\,dv + n_{22}dv^2 \tag{2.21}$$

Theorem 2.1. Let M be a regular surface in E^3. Then the Christoffel symbols of the second kind are

$$\begin{cases} \Gamma_{11}^1 = \frac{g_{22}(g_{11})_u - 2g_{12}(g_{12})_u + g_{12}(g_{11})_v}{2W^2} \\ \Gamma_{12}^1 = \Gamma_{21}^1 = \frac{g_{22}(g_{11})_v - g_{12}(g_{22})_u}{2W^2} \\ \Gamma_{22}^1 = -\frac{g_{22}(g_{22})_u - 2g_{22}(g_{12})_v + g_{12}(g_{22})_v}{2W^2} \\ \Gamma_{11}^2 = -\frac{g_{11}(g_{11})_v - 2g_{11}(g_{12})_u + g_{12}(g_{11})_u}{2W^2} \\ \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{g_{11}(g_{22})_v - g_{12}(g_{11})_v}{2W^2} \\ \Gamma_{22}^2 = \frac{g_{11}(g_{11})_v - 2g_{12}(g_{12})_v + g_{12}(g_{22})_u}{2W^2} \end{cases} \tag{2.22}$$

where [38],

$$W^2 = g_{11}g_{22} - (g_{12})^2 \neq 0\,. \tag{2.23}$$

If the parameter curves is orthogonal, then we have $g_{12}=0$. From here, we can write [38]:

$$\begin{cases} \Gamma_{11}^{1} = \frac{(g_{11})_u}{2g_{11}} & , \quad \Gamma_{12}^{1} = \Gamma_{21}^{1} = \frac{(g_{11})_v}{2g_{11}} \\ \Gamma_{22}^{1} = -\frac{(g_{22})_u}{2g_{11}} & , \quad \Gamma_{11}^{2} = -\frac{(g_{11})_v}{2g_{22}}. \\ \Gamma_{12}^{2} = \Gamma_{21}^{2} = \frac{(g_{22})_v}{2g_{22}} & , \quad \Gamma_{22}^{2} = \frac{(g_{11})_v}{2g_{22}} \end{cases} \tag{2.24}$$

Definition 2.4. Let M be a regular surface in E^3 and S be shape operator of *M*. The characteristic values of S are called the principal curvatures of *M*. Non-zero vectors corresponding to these characteristic values are called the principal directions of M. Then we have

$$S(X) = kX \tag{2.25}$$

where X is a principal direction corresponding to the principal curvature k [38].

Definition 2.5. Let C be a regular curve on M and T be a unit tangent vector of C. If $S(T) = kT$, then C is called the line of curvature on M, [41].

The equation of the lines of curvature is

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ g_{11} & g_{12} & g_{22} \\ b_{11} & b_{12} & b_{22} \end{vmatrix} = 0 \tag{2.26}$$

where g_{ij} and $b_{ij}, 1 \le i, j \le 2,$ are the coefficients of the first and the second fundamental forms of M, respectively [41].

Definition 2.6. Let C be a regular curve on M and T be a unit tangent vector of C. The tangent vector T is called the asymptotic direction satisfied by

$$\langle S(T), T \rangle = II \langle T, T \rangle = 0 \tag{2.27}$$

The differential equation of the asymptotic lines is

$$b_{11}du^2 + 2b_{12}dudv + b_{22}dv^2 = 0 \tag{2.28}$$

where $b_{ij}, 1 \le i, j \le 2,$ is the coefficients of the second fundamental forms of M, respectively, [40].

Theorem 2.2 (Gauss Equations). Let M be a regular surface in E^3. Then we have

$$\begin{cases} X_{uu} = \Gamma_{11}^{1} X_u + \Gamma_{11}^{2} X_v + b_{11} N \\ X_{uv} = X_{vu} = \Gamma_{12}^{1} X_u + \Gamma_{12}^{2} X_v + b_{12} N \\ X_{vv} = \Gamma_{22}^{1} X_u + \Gamma_{22}^{2} X_v + b_{22} N \end{cases} \tag{2.29}$$

where b_{ij} and $\Gamma_{ij}^{k}, 1 \le i, j, k \le 2,$ are the coefficients of the second fundamental form and, the Christoffel symbols of the second kind of *M*, respectively [38].

Theorem 2.3 (Weingarten Equations). Let M be a regular surface in E^3. Then the shape operator S of M is given in terms of the basis $\{X_u, X_v\}$ by

$$\begin{cases} -S(X_u) = N_u = \dfrac{b_{12}g_{12} - b_{11}g_{22}}{W^2} X_u + \dfrac{b_{11}g_{12} - b_{12}g_{11}}{W^2} X_v \\ -S(X_v) = N_v = \dfrac{b_{22}g_{12} - b_{12}g_{22}}{W^2} X_u + \dfrac{b_{12}g_{12} - b_{22}g_{11}}{W^2} X_v \end{cases} \tag{2.30}$$

where g_{ij} and $b_{ij}, 1 \le i, j \le 2,$ are the coefficients of the first and the second fundamental forms of M, respectively, [38]. On the other hand, if the parameter curves of M are the curvature lines then we can write $g_{12} = b_{12} = 0$. Thus, by (31) we have

$$\begin{cases} N_u = -\dfrac{b_{11}}{g_{11}} X_u \\ N_v = -\dfrac{b_{22}}{g_{22}} X_v \end{cases} \tag{2.31}$$

and

$$\begin{cases} k_1 = \dfrac{b_{11}}{g_{11}} \\ k_2 = \dfrac{b_{22}}{g_{22}} \end{cases} \tag{2.32}$$

Here, k_1 and k_2 are called the principal curvatures of M. The curvatures k_1 and k_2 are the principal values of the shape operator S . Thus, the matrix of the shape operrator S

$$S = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}. \tag{2.33}$$

Then, the Gauss curvature and the mean curvature of M are defined by [42],

$$K = \frac{1}{2} det(S) = \frac{1}{2}(k_1 k_2) \tag{2.34}$$

and

$$H = \frac{1}{2} trace(S) = \frac{1}{2}(k_1 + k_2) \tag{2.35}$$

On the other hand, from the definition K and H the principal curvatures are the roots of the equation [42],

$$k^2 - 2Hk + K = 0 \tag{2.36}$$

from which

$$\begin{cases} k_1 = H + \sqrt{H^2 - K} \\ k_2 = H - \sqrt{H^2 - K} \end{cases}. \tag{2.37}$$

Theorem 2.4. Let M be a regular surface in E^3. Then the Gauss curvature and the mean curvature of M, respectively, are

$$K = \frac{b_{11}b_{22} - b_{12}{}^2}{g_{11}g_{22} - g_{12}{}^2} \tag{2.38}$$

and

$$H = \frac{b_{11}g_{22} - 2b_{12}g_{12} + b_{22}g_{11}}{2(g_{11}g_{22} - g_{12}{}^2)} \tag{2.39}$$

where g_{ij} and $b_{ij}, 1 \le i, j \le 2,$ are the coefficients of the first and second fundametal forms of M, respectively [38].

Theorem 2.5. Let M be a regular surface in E^3.Then we have

1. $N \Lambda X_u = \frac{1}{W}[g_{11}X_v - g_{12}X_u]$ (2.40)
2. $N \Lambda X_v = \frac{1}{W}[g_{12}X_v - g_{22}X_u]$ (2.41)
3. $N \Lambda [N \Lambda X_u] = -X_u$ (2.42)
4. $N \Lambda [N \Lambda X_v] = -X_v$ (2.43)
5. $X_v \Lambda [N \Lambda X_u] = g_{12}N$ (2.44)
6. $X_u \Lambda [N \Lambda X_v] = g_{12}N$ (2.45)
7. $X_v \Lambda [N \Lambda X_v] = g_{22}N$ (2.46)
8. $N_u \Lambda N_v = K(X_u \Lambda X_v)$ (2.47)
9. $N_u \Lambda X_u = \frac{b_{12}g_{11} - b_{11}g_{12}}{W^2}(X_u \Lambda X_v)$ (2.48)
10. $N_u \Lambda X_v = -\frac{b_{11}g_{22} - b_{12}g_{12}}{W^2}(X_u \Lambda X_v)$ (2.49)
11. $N_v \Lambda X_u = \frac{b_{22}g_{11} - b_{12}g_{12}}{W^2}(X_u \Lambda X_v)$ (2.50)
12. $N_v \Lambda X_v = -\frac{b_{12}g_{22} - b_{22}g_{12}}{W^2}(X_u \Lambda X_v)$ (2.51)
13. $N_u \Lambda N_v = \mathrm{K}(X_u \Lambda X_v)$ (2.52)
14. $X_u \Lambda N_v + N_u \Lambda X_v = -2\mathrm{H}(X_u \Lambda X_v)$ (2.53)

where N is the unit normal vector of M [41].

Theorem 2.6. Let M be a regular surface in E^3.Then we have

$$X = hN - \frac{\rho}{W^2}\left[(\rho_v g_{12} - \rho_u g_{22})X_u + (\rho_u g_{12} - \rho_v g_{11})X_v\right] \tag{2.54}$$

where $T_M(P) = Sp\{X_u, X_v\}$ [15-17].

Proposition 2.1. Let M be a regular surface in E^3. Then we have

$$\rho^2 = \frac{h^2}{1 - \overset{I}{\nabla}(\rho, \rho)} \tag{2.55}$$

where

$$\overset{I}{\nabla}(\rho,\rho)=\frac{(\rho_v)^2 g_{11}-2\rho_u\rho_v g_{12}+(\rho_u)^2 g_{22}}{g_{11}g_{22}-(g_{22})^2} \tag{2.56}$$

Here, $\overset{I}{\nabla}(\rho,\rho)$ is the Beltrami operator with respect to the coefficients of the first fundamental form of M [15, 39].

3. GEOMETRY OF THE FOCAL SURFACES OF THE INVERSE SURFACE OF A REGULAR SURFACE IN E^3

Definition 3.1. Let $M\ldots X=X(u,v)$ be a regular surface in $E^3$and Let $\bar{M}\ldots\bar{X}=\bar{X}(u,v)$ be an inverse surface of M with respect to the unit sphere S^2. For the focal surfaces in $\widetilde{M}_1$ ve $\widetilde{M}_2$ of M we have

$$\widetilde{M}^{"1} \cdots \tilde{R}_1(u,v)=\bar{X}(u,v)+(\tfrac{1}{\bar{k}^1})\bar{N}(u,v) \tag{3. 1}$$

and

$$\widetilde{M}^2 \cdots R_2(u,v)=\bar{X}(u,v)+\left(\tfrac{1}{\bar{k}^2}\right)\bar{N}(u,v). \tag{3. 2}$$

Throughout our study, parameter curves will be considered as curvature lines. In this case we can write

$$\bar{b}_{12}=\bar{g}_{12}=0 \tag{3. 3}$$

or

$$\begin{cases}\bar{N}_u=-\bar{k}_1\bar{X}_u\\ \bar{N}_v=-\bar{k}_2\bar{X}_v\end{cases}. \tag{3. 4}$$

Theorem 3.1. Let $M\ldots X=X(u,v)$ be a regular surface in $E^3$and Let $\bar{M}\ldots\bar{X}=\bar{X}(u,v)$ be an inverse surface of M with respect to the unit sphere S^2. For the unit vectors $\widetilde{N_1}$ and $\widetilde{N_2}$ of the focal surfaces $\widetilde{M}_1$ ve $\widetilde{M}_2$ of M we have

$$\widetilde{N_1}(u,v)=\frac{1}{\sqrt{\bar{g}_{11}}}\bar{X}_u \tag{3. 5}$$

and

$$\widetilde{N_2}(u,v)=\frac{1}{\sqrt{\bar{g}_{22}}}\bar{X}_v \tag{3. 6}$$

where $g_{11}$and g_{22} are the coefficients of the first fundamental form I.

Proof: From (9) the unit vectors $\widetilde{N_1}$ and $\widetilde{N_2}$ of the focal surfaces $\widetilde{M}_1$ ve $\widetilde{M}_2$ of M we can write

$$\widetilde{N_1}=\frac{1}{\|\ (\tilde{R}_1)_u\wedge(\tilde{R}_1)_v\|}\left((\tilde{R}_1)_u\wedge(\tilde{R}_1)_v\right) \tag{3.7}$$

and

$$\widetilde{N_2}=\frac{1}{\|(\tilde{R}_2)_u\wedge(\tilde{R}_2)_v\|}\left((\tilde{R}_2)_u\wedge(\tilde{R}_2)_v\right). \tag{3.8}$$

If the partial derivatives of equations (3.1) and (3. 2) are calculated according to the "u" and "v" parameters we obtain

$$\begin{cases}\left(\tilde{R}_1\right)_u = \bar{X}_u - \frac{(\bar{k}_1)_u}{(\bar{k}_1)^2}\bar{N} + \frac{1}{\bar{k}_1}\bar{N}_u \\ \left(\tilde{R}_1\right)_v = \bar{X}_v - \frac{(\bar{k}_1)_v}{(\bar{k}_1)^2}\bar{N} + \frac{1}{\bar{k}_1}\bar{N}_v\end{cases}. \tag{3. 9}$$

and

$$\begin{cases}\left(\tilde{R}_2\right)_u = \bar{X}_u - \frac{(\bar{k}_2)_u}{(\bar{k}_2)^2}\bar{N} + \frac{1}{\bar{k}_2}\bar{N}_u \\ \left(\tilde{R}_2\right)_v = \bar{X}_v - \frac{(\bar{k}_2)_v}{(\bar{k}_2)^2}\bar{N} + \frac{1}{\bar{k}_2}\bar{N}_v\end{cases} \tag{3.10}$$

Substituting (3.3) into (3.7) and (3.8) and rearranging we get

$$\begin{cases}\left(\tilde{R}_1\right)_u = -\frac{(\bar{k}_1)_u}{(\bar{k}_1)^2}\bar{N} \\ \left(\tilde{R}_1\right)_v = \frac{1}{(\bar{k}_1)^2}\left[\left(\bar{k}_1\right)^2 - \bar{K}\right]\bar{X}_v + \left(\bar{k}_1\right)_v\bar{N}\end{cases} \tag{3.11}$$

and

$$\begin{cases}\left(\tilde{R}_2\right)_u = \frac{1}{(\bar{k}_2)^2}\left[\left(\bar{k}_2\right)^2 - \bar{K}\right]\bar{X}_v + \left(\bar{k}_2\right)_v\bar{N} \\ \left(\tilde{R}_2\right)_v = -\frac{(\bar{k}_2)_v}{(\bar{k}_2)^2}\bar{N}\end{cases}. \tag{3. 12}$$

Computing the vectors $\left(\tilde{R}_1\right)_u \wedge \left(\tilde{R}_1\right)_v$ and $\left(\tilde{R}_2\right)_u \wedge \left(\tilde{R}_2\right)_v$, respectively, we have

$$\left(\tilde{R}_1\right)_u \wedge \left(\tilde{R}_1\right)_v = -\frac{(\bar{k}_1)_u}{(\bar{k}_1)^2}\left[1 - \frac{\bar{k}_2}{\bar{k}_1}\right]\frac{1}{\bar{W}}\bar{g}_{11}\bar{X}_v \tag{3.13}$$

and

$$\left(\tilde{R}_2\right)_u \wedge \left(\tilde{R}_2\right)_v = -\frac{(\bar{k}_2)_v}{(\bar{k}_2)^2}\left[1 - \frac{\bar{k}_1}{\bar{k}_2}\right]\frac{1}{\bar{W}}\bar{g}_{22}\bar{X}_u. \tag{3.14}$$

If we take the norms of the vectors $\left(\tilde{R}_1\right)_u \wedge \left(\tilde{R}_1\right)_v$ and $\left(\tilde{R}_2\right)_u \wedge \left(\tilde{R}_2\right)_v$, respectively, then we obtain

$$\left\| \left(\tilde{R}_1\right)_u \wedge \left(\tilde{R}_1\right)_v \right\| = \sqrt{\bar{\boldsymbol{g}}_{\mathbf{11}}} \tag{3.15}$$

and

$$\left\| \left(\tilde{R}_2\right)_u \wedge \left(\tilde{R}_2\right)_v \right\| = \sqrt{\bar{\boldsymbol{g}}_{\mathbf{22}}}. \tag{3. 16}$$

Substituting (3.13), (3.14), (3.15) and (3.16) into (3.7) and (3.8) we get the desired results.

Theorem 3.2. Let $M \ldots X = X(u,v)$ be a regular surface in $E^3$and Let $\bar{M} \ldots \bar{X} = \bar{X}(u,v)$ be an inverse surface of M with respect to the unit sphere S^2. For the partial derivatives of the unit vectors $\widetilde{N_1}$ and $\widetilde{N_2}$ of the focal surfaces $\widetilde{M}_1$ ve $\widetilde{M}_2$ of M we have

$$\begin{cases}\left(\widetilde{N_1}\right)_u = \frac{1}{\sqrt{\bar{g}_{11}}}\left[(1 - \bar{g}_{11})\bar{\Gamma}_{11}^1\bar{X}_u + \bar{\Gamma}_{11}^2\bar{X}_v + \bar{b}_{11}\bar{N}\right] \\ \left(\widetilde{N_1}\right)_v = \frac{1}{\sqrt{\bar{g}_{11}}}[(\bar{\Gamma}_{12}^1 + \bar{g}_{11}\bar{\Gamma}_{11}^2)\bar{X}_u + \bar{\Gamma}_{11}^2\bar{X}_v]\end{cases} \tag{3.17}$$

and

$$\begin{cases}\left(\widetilde{N_2}\right)_u = \frac{1}{\sqrt{\bar{g}_{22}}}[\bar{\Gamma}_{12}^1\bar{X}_u + (\bar{\Gamma}_{12}^2 + \bar{g}_{22}\bar{\Gamma}_{22}^1)\bar{X}_v] \\ \left(\widetilde{N_2}\right)_v = \frac{1}{\sqrt{\bar{g}_{22}}}\left[\bar{\Gamma}_{12}^1\bar{X}_u + (1 - \bar{g}_{22})\bar{\Gamma}_{11}^2\bar{X}_v + \bar{b}_{22}\bar{N}\right]\end{cases} \tag{3.18}$$

where $g_{i1}$and b_{ij} are the coefficients of the first fundamental form I and the second fundamental form II.

Proof: If the derivatives of (3.6) and (3.7) are taken according to the u and v parameters and the obtained equations are rearranged, we get

$$\begin{cases} \left(\widetilde{N_1}\right)_u = \frac{1}{\sqrt{\bar{g}_{11}}}\left[(1-\bar{g}_{11})\bar{\Gamma}_{11}^1\bar{X}_u + \bar{\Gamma}_{11}^2\bar{X}_v + \bar{b}_{11}\bar{N}\right] \\ \left(\widetilde{N_1}\right)_v = \frac{1}{\sqrt{\bar{g}_{11}}}\left[(\bar{\Gamma}_{12}^1 + \bar{g}_{11}\bar{\Gamma}_{11}^2)\bar{X}_u + \bar{\Gamma}_{11}^2\bar{X}_v\right] \end{cases}$$

and

$$\begin{cases} \left(\widetilde{N_2}\right)_u = \frac{1}{\sqrt{\bar{g}_{22}}}\left[\bar{\Gamma}_{12}^1\bar{X}_u + (\bar{\Gamma}_{12}^2 + \bar{g}_{22}\bar{\Gamma}_{22}^1)\bar{X}_v\right] \\ \left(\widetilde{N_2}\right)_v = \frac{1}{\sqrt{\bar{g}_{22}}}\left[\bar{\Gamma}_{12}^1\bar{X}_u + (1-\bar{g}_{22})\bar{\Gamma}_{11}^2\bar{X}_v + \bar{b}_{22}\bar{N}\right] \end{cases}.$$

This is completed the proof.

Theorem 3.3. Let $M \ldots X = X(u,v)$ be a regular surface in $E^3$and Let $\bar{M} \ldots \bar{X} = \bar{X}(u,v)$ be an inverse surface of M with respect to the unit sphere S^2. For the support functions $\tilde{h}_1$ and $\tilde{h}_2$ of the focal surfaces $\widetilde{M}_1$ ve $\widetilde{M}_2$ of M we have

$$\tilde{h}_1 = \frac{\bar{\rho}\bar{\rho}_u}{\sqrt{\bar{g}_{11}}} \tag{3.19}$$

and

$$\tilde{h}_2 = \frac{\bar{\rho}\bar{\rho}_v}{\sqrt{\bar{g}_{22}}}. \tag{3.20}$$

Proof: From (13) the support functions $\tilde{h}_1$ and $\tilde{h}_2$ of the focal surfaces $\widetilde{M}_1$ ve $\widetilde{M}_2$ of $M$are

$$\tilde{h}_1 = \langle(\widetilde{R_1}), (\widetilde{N_1})\rangle \tag{3.21}$$

and

$$\tilde{h}_2 = \langle(\widetilde{R_2}), (\widetilde{N_2})\rangle. \tag{3.22}$$

Substituting (3.1), (3.2), (3.5) and (3.6) into (3.21) and (3.22), and if necessary arrangements are made, we get

$$\tilde{h}_1 = \frac{1}{\sqrt{\bar{g}_{11}}}\langle\bar{X}, \bar{X}_u\rangle \tag{3.23}$$

and

$$\tilde{h}_2 = \frac{1}{\sqrt{\bar{g}_{22}}}\langle\bar{X}, \bar{X}_v\rangle. \tag{3.24}$$

On the other hand, writing $\langle\bar{X}, \bar{X}_u\rangle = \bar{\rho}\bar{\rho}_u$ and $\langle\bar{X}, \bar{X}_v\rangle = \bar{\rho}\bar{\rho}_v$into the above equations we get the desired results.

Theorem 3.4. Let $M \ldots X = X(u,v)$ be a regular surface in $E^3$and Let $\bar{M} \ldots \bar{X} = \bar{X}(u,v)$ be an inverse surface of M with respect to the unit sphere S^2. For coefficients $\bar{g}_{ij}^1$ and $\bar{g}_{ij}^2$ of the first fundamental forms $\tilde{I}_1$ and $\tilde{I}_2$of the focal surfaces $\widetilde{M}_1$ ve $\widetilde{M}_2$ of M we have

$$\begin{cases} \tilde{g}^1_{11} = \frac{(\bar{k}_1)^2_u}{\bar{k}^4_1} \\ \tilde{g}^1_{12} = \frac{(\bar{k}_1)_u(\bar{k}_1)_v}{\bar{k}^4_1} \\ \tilde{g}^1_{22} = \frac{(\bar{k}_1-\bar{k}_2)^2}{\bar{k}^2_1}\,\bar{g}_{22} + \frac{(\bar{k}_1)^2_v}{\bar{k}^4_1} \end{cases} \tag{3.25}$$

and

$$\begin{cases} \tilde{g}^2_{11} = \frac{(\bar{k}_2-\bar{k}_1)^2}{\bar{k}^2_2}\,\bar{g}_{11} + \frac{(\bar{k}_2)^2_u}{\bar{k}^4_2} \\ \tilde{g}^2_{12} = \frac{(\bar{k}_2)_u(\bar{k}_2)_v}{\bar{k}^4_2} \;. \\ \tilde{g}^2_{22} = \frac{(\bar{k}_2)^2_u}{\bar{k}^4_2} \end{cases} \tag{3.26}$$

Proof: From (20) we may write

$$\begin{cases} \tilde{g}^1_{11} = \langle \left(\widetilde{R_1}\right)_u, \left(\widetilde{R_1}\right)_u \rangle \\ \tilde{g}^1_{12} = \langle \left(\widetilde{R_1}\right)_u, \left(\widetilde{R_1}\right)_v \rangle \\ \tilde{g}^1_{22} = \langle \left(\widetilde{R_1}\right)_v, \left(\widetilde{R_1}\right)_v \rangle \end{cases} \tag{3.27}$$

and

$$\begin{cases} \tilde{g}^2_{11} = \langle \left(\widetilde{R_2}\right)_u, \left(\widetilde{R_2}\right)_u \rangle \\ \tilde{g}^2_{12} = \langle \left(\widetilde{R_2}\right)_u, \left(\widetilde{R_2}\right)_v \rangle \;. \\ \tilde{g}^2_{22} = \langle \left(\widetilde{R_2}\right)_v, \left(\widetilde{R_2}\right)_v \rangle \end{cases} \tag{3.28}$$

Substituting (3.11) and (3.12) into (3.27) and (3.28) and the obtained equations are rearranged, we get

$$\begin{cases} \tilde{g}^1_{11} = \frac{(\bar{k}_1)^2_u}{\bar{k}^4_1} \\ \tilde{g}^1_{12} = \frac{(\bar{k}_1)_u(\bar{k}_1)_v}{\bar{k}^4_1} \\ \tilde{g}^1_{22} = \frac{(\bar{k}_1-\bar{k}_2)^2}{\bar{k}^2_1}\,\bar{g}_{22} + \frac{(\bar{k}_1)^2_v}{\bar{k}^4_1} \end{cases}$$

and

$$\begin{cases} \tilde{g}^2_{11} = \frac{(\bar{k}_2-\bar{k}_1)^2}{\bar{k}^2_2}\,\bar{g}_{11} + \frac{(\bar{k}_2)^2_u}{\bar{k}^4_2} \\ \tilde{g}^2_{12} = \frac{(\bar{k}_2)_u(\bar{k}_2)_v}{\bar{k}^4_2} \;. \\ \tilde{g}^2_{22} = \frac{(\bar{k}_2)^2_u}{\bar{k}^4_2} \end{cases}$$

This is completed proof.

Then we can give the following corrolaries.

Corollary 3.1. Let $M \ldots X = X(u,v)$ be a regular surface in $E^3$and Let $\bar{M} \ldots \bar{X} = \bar{X}(u,v)$ be an inverse surface of M with respect to the unit sphere S^2. For the first fundamental forms $\tilde{I}_{\,1}$ and $\tilde{I}_{\,2}$of the focal surfaces $\widetilde{M}_1$ ve $\widetilde{M}_2$ of M we have

$$\tilde{I}_1 = \left(\frac{(\bar{k}_1)_u^2}{\bar{k}_1^4}\right)du^2 + \left(\frac{(\bar{k}_1)_u(\bar{k}_1)_v}{\bar{k}_1^4}\right)dudv + \left(\frac{(\bar{k}_1 - \bar{k}_2)^2}{\bar{k}_1^2}\,\bar{g}_{22} + \frac{(\bar{k}_1)_v^2}{\bar{k}_1^4}\right)dv^2$$

and

$$\tilde{I}_2 = \left(\frac{(\bar{k}_2 - \bar{k}_1)^2}{\bar{k}_2^2}\,\bar{g}_{11} + \frac{(\bar{k}_2)_u^2}{\bar{k}_2^4}\right)du^2 + \left(\frac{(\bar{k}_2)_u(\bar{k}_2)_v}{\bar{k}_2^4}\right)dudv + \left(\frac{(\bar{k}_2)_u^2}{\bar{k}_2^4}\right)dv^2$$

Corollary 3.2. Let $M \ldots X = X(u,v)$ be a regular surface in $E^3$and Let $\bar{M} \ldots \bar{X} = \bar{X}(u,v)$ be an inverse surface of M with respect to the unit sphere S^2. For the determinants of the matrices corresponding to the first fundamental forms $\tilde{I}_1$ and $\tilde{I}_2$of the focal surfaces $\widetilde{M}_1$ ve $\widetilde{M}_2$ of M we have

$$\widetilde{W}_1^{\ 2} = \tilde{g}_{11}^1\tilde{g}_{22}^1 - (\tilde{g}_{12}^1)^2 = \frac{(\bar{k}_1-\bar{k}_2)^2(\bar{k}_1)_u^2}{\bar{k}_1^6}\,\bar{g}_{22} \tag{3.29}$$

and

$$\widetilde{W}_2^{\ 2} = \tilde{g}_{11}^2\tilde{g}_{22}^2 - (\tilde{g}_{12}^2)^2 = \frac{(\bar{k}_2-\bar{k}_1)^2(\bar{k}_1)_u^2}{\bar{k}_2^6}\,\bar{g}_{11}\ . \tag{3.30}$$

Theorem 3.5. Let $M \ldots X = X(u,v)$ be a regular surface in $E^3$and Let $\bar{M} \ldots \bar{X} = \bar{X}(u,v)$ be an inverse surface of M with respect to the unit sphere S^2. For coefficients $\bar{b}_{ij}^1$ and $\bar{b}_{ij}^2$ of the second fundamental forms $\widetilde{II}_1$ and $\widetilde{II}_2$of the focal surfaces $\widetilde{M}_1$ ve $\widetilde{M}_2$ of M we have

$$\begin{cases} \tilde{b}_{11}^1 = \frac{-(\bar{k}_1)_u}{\bar{K}_1^2\sqrt{\bar{g}_{11}}}\ \bar{b}_{11} \\ \tilde{b}_{12}^1 = 0 \\ \tilde{b}_{22}^1 = \frac{1}{\sqrt{\bar{g}_{11}}}\left(\frac{\bar{k}_1-\bar{k}_2}{\bar{k}_1}\Gamma_{12}^2\bar{g}_{22}\right) \end{cases} \tag{3.31}$$

and

$$\begin{cases} \tilde{b}_{11}^2 = \frac{1}{\sqrt{\bar{g}_{22}}}\left(\frac{\bar{k}_2-\bar{k}_1}{\bar{k}_2}\Gamma_{12}^1\bar{g}_{11}\right) \\ \tilde{b}_{12}^2 = 0 \quad . \\ \tilde{b}_{22}^2 = \frac{-(\bar{k}_2)_v}{\bar{k}_2^2\sqrt{\bar{g}_{22}}}\ \bar{b}_{22} \end{cases} \tag{3.32}$$

Proof: From (21) we may write

$$\begin{cases} \tilde{b}_{11}^1 = -\langle(\widetilde{R_1})_u, (\widetilde{N_1})_u\rangle \\ \tilde{b}_{12}^1 = -\langle(\widetilde{R_1})_u, (\widetilde{N_1})_v\rangle \\ \tilde{b}_{22}^1 = -\langle(\widetilde{R_1})_v, (\widetilde{N_1})_v\rangle \end{cases} \tag{3.33}$$

and

$$\begin{cases} \tilde{b}_{11}^2 = \langle(\widetilde{R_2})_u, (\widetilde{N_2})_u\rangle \\ \tilde{b}_{12}^2 = -\langle(\widetilde{R_2})_u, (\widetilde{N_2})_v\rangle\ . \\ \tilde{b}_{22}^2 = -\langle(\widetilde{R_2})_v, (\widetilde{N_2})_v\rangle \end{cases} \tag{3.34}$$

Substituting (3.11), (3.12), (3. 17) and (3.18) into (3.33) and (3.34), and the obtained equations are rearranged, we get

$$\begin{cases} \tilde{b}_{11}^1 = \dfrac{-(\bar{k}_1)_u}{\bar{k}_1^2\sqrt{\bar{g}_{11}}}\,\bar{b}_{11} \\ \tilde{b}_{12}^1 = 0 \\ \tilde{b}_{22}^1 = \dfrac{1}{\sqrt{\bar{g}_{11}}}\left(\dfrac{\bar{k}_1 - \bar{k}_2}{\bar{k}_1}\bar{\Gamma}_{12}^2\bar{g}_{22}\right) \end{cases}$$

and

$$\begin{cases} \tilde{b}_{11}^2 = \dfrac{1}{\sqrt{\bar{g}_{22}}}\left(\dfrac{\bar{k}_2 - \bar{k}_1}{\bar{k}_2}\bar{\Gamma}_{12}^1\bar{g}_{11}\right) \\ \tilde{b}_{12}^2 = 0 \quad . \\ \tilde{b}_{22}^2 = \dfrac{-(\bar{k}_2)_v}{\bar{k}_2^2\sqrt{\bar{g}_{22}}}\,\bar{b}_{22} \end{cases}$$

This is completed the proof.
Thus, we can g.ve the following corollary.

Corollary 3.3. Let $M \ldots X = X(u,v)$ be a regular surface in $E^3$and Let $\bar{M} \ldots \bar{X} = \bar{X}(u,v)$ be an inverse surface of M with respect to the unit sphere S^2. For the secondfundamental forms $\widetilde{II}_1$ and $\widetilde{II}_2$of the focal surfaces $\widetilde{M}_1$ ve $\widetilde{M}_2$ of M we have

$$\widetilde{II}_1 = \left(\frac{-(\bar{k}_1)_u}{\bar{K}_1^2\sqrt{\bar{g}_{11}}}\,\bar{b}_{11}\right)du^2 + \left(\frac{1}{\sqrt{\bar{g}_{11}}}\left(\frac{\bar{k}_1-\bar{k}_2}{\bar{k}_1}\bar{\Gamma}_{12}^2\bar{g}_{22}\right)\right)dv^2$$

and

$$I\tilde{I}_2 = \left(\frac{1}{\sqrt{\bar{g}_{22}}}\left(\frac{\bar{k}_2-\bar{k}_1}{\bar{k}_2}\Gamma_{12}^1\bar{g}_{11}\right)\right)du^2 + \left(\frac{-(\bar{k}_2)_v}{\bar{k}_2^2\sqrt{\bar{g}_{22}}}\,\bar{b}_{22}\right)dv^2$$

Theorem 3.7. Let $M \ldots X = X(u,v)$ be a regular surface in $E^3$and Let $\bar{M} \ldots \bar{X} = \bar{X}(u,v)$ be an inverse surface of M with respect to the unit sphere S^2. For coefficients $\bar{n}_{ij}^1$ and $\bar{n}_{ij}^2$ of the third fundamental forms $\widetilde{III}_1$ and $\widetilde{III}_2$of the focal surfaces $\widetilde{M}_1$ ve $\widetilde{M}_2$ of M we have

$$\begin{cases} \tilde{n}_{11}^1 = \dfrac{1}{\overline{g_{11}}}\left[(1-\bar{g}_{11})^2(\bar{\Gamma}_{11}^1)^2\bar{g}_{11} + (\bar{\Gamma}_{11}^2)^2 + \left(\bar{b}_{11}\right)^2\right] \\ \tilde{n}_{12}^1 = \dfrac{1}{\overline{g_{11}}}[(1-\bar{g}_{11})(\bar{g}_{22}\bar{\Gamma}_{11}^2 + \bar{\Gamma}_{12}^1)(\bar{\Gamma}_{11}^1)\bar{g}_{11} + (\bar{\Gamma}_{12}^2\bar{\Gamma}_{11}^2)\bar{g}_{22}] \\ n_{22}^1 = \dfrac{1}{\overline{g_{11}}}[(\bar{g}_{22}\bar{\Gamma}_{11}^2 + \bar{\Gamma}_{12}^1)^2\bar{g}_{11} + (\bar{\Gamma}_{12}^2)^2\bar{g}_{22}] \end{cases} \quad (3.35)$$

and

$$\begin{cases} \tilde{n}_{11}^2 = \dfrac{1}{\bar{g}_{22}}[(\bar{g}_{22}\bar{\Gamma}_{22}^1 + \bar{\Gamma}_{12}^2)^2\bar{g}_{11} + (\bar{\Gamma}_{12}^1)^2\bar{g}_{22}] \\ \tilde{n}_{12}^2 = \dfrac{1}{\bar{g}_{22}}[\Gamma_{12}^1\Gamma_{22}^1\bar{g}_{11} + (\bar{g}_{22}\bar{\Gamma}_{22}^1 + \bar{\Gamma}_{12}^2)(1-\bar{g}_{22})\bar{g}_{22}] \,. \\ \tilde{n}_{22}^2 = \dfrac{1}{\bar{g}_{22}}\left[(\bar{\Gamma}_{22}^1)^2\bar{g}_{11} + (1-\bar{g}_{22})^2(\Gamma_{22}^2)^2 + \left(\bar{b}_{22}\right)^2\right] \end{cases} \quad (3.36)$$

Proof: From (22) we may write

$$\begin{cases} \tilde{n}_{11}^1 = \langle (\widetilde{N_1})_u, (\widetilde{N_1})_u \rangle \\ \tilde{n}_{12}^1 = \langle (\widetilde{N_1})_u, (\widetilde{N_1})_v \rangle \\ n_{22}^1 = \langle (\widetilde{N_1})_v, (\widetilde{N_1})_v \rangle \end{cases} \tag{3. 37}$$

and

$$\begin{cases} \tilde{n}_{11}^2 = \langle (\widetilde{N_2})_u, (\widetilde{N_2})_u \rangle \\ \tilde{n}_{12}^2 = \langle (\widetilde{N_2})_u, (\widetilde{N_2})_v \rangle \\ \tilde{n}_{22}^2 = \langle (\widetilde{N_2})_v, (\widetilde{N_2})_v \rangle \end{cases} . \tag{3. 38}$$

Substituting (3. 17) and (3.18) into (3.37) and (3.38), and the obtained equations are rearranged, we get

$$\begin{cases} \tilde{n}_{11}^1 = \frac{1}{\bar{g}_{11}} \left[(1-\bar{g}_{11})^2 (\bar{\Gamma}_{11}^1)^2 \bar{g}_{11} + (\bar{\Gamma}_{11}^2)^2 + (\bar{b}_{11})^2 \right] \\ \tilde{n}_{12}^1 = \frac{1}{\bar{g}_{11}} [(1-\bar{g}_{11})(\bar{g}_{22}\bar{\Gamma}_{11}^2 + \bar{\Gamma}_{12}^1)(\bar{\Gamma}_{11}^1)\bar{g}_{11} + (\bar{\Gamma}_{12}^2 \bar{\Gamma}_{11}^2)\bar{g}_{22}] \\ n_{22}^1 = \frac{1}{\bar{g}_{11}} [(\bar{g}_{22}\bar{\Gamma}_{11}^2 + \bar{\Gamma}_{12}^1)^2 \bar{g}_{11} + (\bar{\Gamma}_{12}^2)^2 \bar{g}_{22}] \end{cases}$$

and

$$\begin{cases} \tilde{n}_{11}^2 = \frac{1}{\bar{g}_{22}} [(\bar{g}_{22}\bar{\Gamma}_{22}^1 + \bar{\Gamma}_{12}^2)^2 \bar{g}_{11} + (\bar{\Gamma}_{12}^1)^2 \bar{g}_{22}] \\ \tilde{n}_{12}^2 = \frac{1}{\bar{g}_{22}} [\Gamma_{12}^1 \Gamma_{22}^1 \bar{g}_{11} + (\bar{g}_{22}\bar{\Gamma}_{22}^1 + \bar{\Gamma}_{12}^2)(1-\bar{g}_{22})\bar{g}_{22}] . \\ \tilde{n}_{22}^2 = \frac{1}{\bar{g}_{22}} \left[(\bar{\Gamma}_{22}^1)^2 \bar{g}_{11} + (1-\bar{g}_{22})^2 (\Gamma_{22}^2)^2 + (\bar{b}_{22})^2 \right] \end{cases}$$

This is completed the proof.

Theorem 3.8. Let $M \ldots X = X(u,v)$ be a regular surface in $E^3$and Let $\bar{M} \ldots \bar{X} = \bar{X}(u,v)$ be an inverse surface of M with respect to the unit sphere S^2. For Gaussian curvatures $\widetilde{K}_1$ and $\widetilde{K}_2$of the focal surfaces $\widetilde{M}_1$ ve $\widetilde{M}_2$ of M we have

$$\widetilde{K}_1 = -\frac{\bar{k}_1^4 (\bar{\Gamma}_{12}^2)}{(\bar{k}_1)_u} \tag{3.39}$$

and

$$\widetilde{K}_2 = -\frac{\bar{k}_2^4 (\bar{\Gamma}_{12}^1)}{(\bar{k}_2)_u}, \tag{3.40}$$

Proof: From (42) we may write

$$\widetilde{K}_1 = \frac{\tilde{b}_{11}^1 \tilde{b}_{22}^1}{\widetilde{W}_1{}^2} \tag{3.41}$$

and

$$\widetilde{K}_2 = \frac{\tilde{b}_{11}^2 \tilde{b}_{22}^2}{\widetilde{W}_2{}^2}. \tag{3.42}$$

Substituting (3. 29) ,(3.30)), (3.31)and (3.32) into (3.41) and (3.42), and the obtained equations are rearranged, we get desired results.

Theorem 3.9. Let $M \ldots X = X(u,v)$ be a regular surface in $E^3$and Let $\bar{M} \ldots \bar{X} = \bar{X}(u,v)$ be an inverse surface of M with respect to the unit sphere S^2. For Gaussian curvatures $\widetilde{H}_1$ and $\widetilde{H}_2$ of the focal surfaces $\widetilde{M}_1$ ve $\widetilde{M}_2$ of M we have

$$\widetilde{H}_1 = \frac{\bar{k}_1^2}{(\bar{k}_1-\bar{k}_2)^2\bar{g}_{22}\sqrt{\bar{g}_{11}}}\left(\left(\frac{\bar{k}_1-\bar{k}_2}{\bar{k}_1}\Gamma^2_{12}\bar{g}_{22} + \left(\bar{k}_1-\bar{k}_2\right)^2\bar{g}_{22}\,\bar{b}_{11} + \frac{(\bar{k}_1)^2_v}{k_1^2}\bar{b}_{11}\right)\right) \tag{3.43}$$

and

$$\widetilde{H}_2 = \frac{\bar{k}_2^2}{(\bar{k}_2-\bar{k}_1)^2\bar{g}_{11}\sqrt{\bar{g}_{22}}}\left(\left(\frac{\bar{k}_2-\bar{k}_1}{\bar{k}_2}\Gamma^1_{12}\bar{g}_{11} + \left(\bar{k}_2-\bar{k}_1\right)^2\bar{g}_{11}\,\bar{b}_{22} + \frac{(\bar{k}_2)^2_v}{k_2^2}\bar{b}_{22}\right)\right) \tag{3.44}$$

Proof: From (42) we may write

$$\widetilde{H}_1 = \frac{\tilde{b}^1_{11}\tilde{g}^1_{22}+\tilde{b}^1_{22}\tilde{g}^1_{11}}{\widetilde{W}_1^{\,2}} \tag{3.45}$$

and

$$H_2 = \frac{\tilde{b}^2_{11}\tilde{g}^2_{22}+\tilde{b}^2_{22}\tilde{g}^2_{11}}{2\widetilde{W}_2^{\,2}}. \tag{3.46}$$

Substituting (3. 29), (3.30), (3.31) and (3.32) into (3.45) and (3.46), and the obtained equations are rearranged, we get desired results.

Example 3.1. Given the surface $M \ldots X = X(u,v) = (2\cos v, 2\sin v, u)$. Then for the parametric equation of the inverse surface of M with respect to the unit sphere S^2 we have

$$\bar{M} \cdots \bar{X}(u,v) = \frac{1}{u^2+4}(2\cos v, 2\sin v, u)\ . \tag{3.50}$$

Using (3.1) and (3.2) the equations of the focal surfaces $\widetilde{M}_1$ and $\widetilde{M}_2$ are

$$\widetilde{M}_1 \ldots \widetilde{R}_1(u,v) = \left(\frac{1}{4}cosv, \frac{1}{4}cosv, 0\right) \tag{3.51}$$

and

$$\widetilde{M}_2 \ldots.\widetilde{R}_2(u,v) = \frac{1}{u^2+4}\left(\cos v, \sin v, \frac{u^3}{u^2-4}\right). \tag{3.52}$$

So the figure is as below:

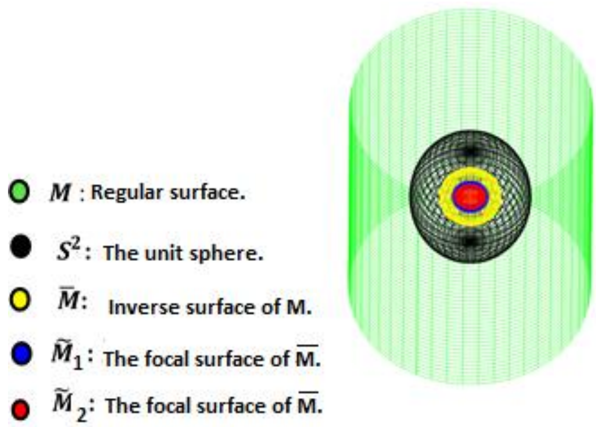

Figure 6. Invers surface and it’s focal surfaces.

Example 3.2. Given the surface $M \ldots X(u,v) = (\cosh u \sin v, \sinh v, \cosh u \cos v)$. Then for the parametric equation of the inverse surface of M with respect to the unit sphere S^2 we have

$$\bar{M} \cdots \bar{X}(u,v) = \frac{1}{2\cosh^2 u - 1}(\cosh u \sin v, \sinh v, \cosh u \cos v). \tag{3. 53}$$

Using (3.1) and (3.2) the equations of the focal surfaces $\widetilde{M}_1$ and $\widetilde{M}_2$ are

$$\widetilde{M}_1 \,..\, \tilde{R}_1(u,v) = \frac{1}{(2\cosh^2 u+1)}\,(2coshusinv, 0, 2coshucosv, 0) \tag{3. 54}$$

and

$$\widetilde{M}_2 \,....\, \tilde{R}_2(u,v) = \frac{2}{(3-2\cosh^2 u)}(0, -2\sinh u\,, 0). \tag{3.55}$$

So the figure is as below:

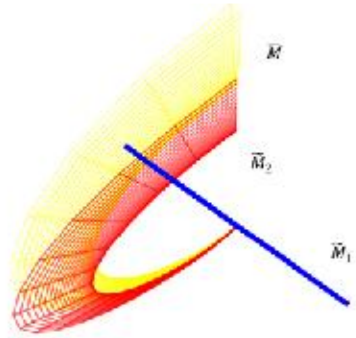

Figure 7. Inverse surface and its focals.

4. CONCLUSION

As it is known, focal (offset) and inversion (inversive) geometry are two of the methods used to approach a surface. In this study, focals of inverse of a surface were investigated and good results were obtained. Subsequently, examples supporting our claims were given and their graphs were drawn.

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