ORIGINAL PAPER

NEW NARAYANA TRIANGLE

BAHAR KULOĞLU[1], ENGİN ÖZKAN[2,*]

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Abstract. *In this paper, inspiring Hosoya's triangle, we define a new Narayana triangle. Then, we represent this Narayana triangle geometrically on the plane. In addition, we give some identities and properties of the new Narayana triangle.*

Keywords: *Narayana sequence; Fibonacci sequence; Hosoya's triangle; Narayana triangle.*

1. INTRODUCTION

The Narayana sequence N_n is defined by Narayana, Indian mathematician, as the third order recurrence in the 14th century [1] by

$$N_{n+1} = N_n + N_{n-2},\ n \geq 2,\ N_0 = 0, N_1 = 1, N_2 = 1. \tag{1}$$

The recurrence relation involves the characteristic equation

$$x^3 - x^2 - 1 = 0.$$

If α, β and γ are the zeros of the characteristic equation of the Narayana sequence, then the following equalities hold $\alpha + \beta + \gamma = 1$, $\alpha\beta + \beta\gamma + \alpha\gamma = 0$ and $\alpha\beta\gamma = 1$.

The Binet formula [2] for the Narayana sequence is

$$N_n = \alpha^n a + \beta^n b + \gamma^n c, \qquad n \geq 0 \tag{2}$$

where

$$a = \frac{\alpha}{(\alpha - \beta)(\alpha - \gamma)}, \qquad b = \frac{\beta}{(\beta - \alpha)(\beta - \gamma)}, \qquad c = \frac{\gamma}{(\gamma - \alpha)(\gamma - \beta)} \tag{3}$$

The Narayana numbers and their properties have been studied by Özkan, Ramirez, Petersen, et al. [3-10]. In particular, Petersen [5] places them in the Euler-Macmahon-Carlitz/Riordan combinatorial spectrum. Shannon and Leon Bernstein have shown how third order lacunary recurrence relations can be related to the Jacob-Perron algorithm as a third dimension continued fraction algorithm [6].

[1] Erzincan Binali Yıldırım University, Graduate School of Natural and Applied Sciences, Department of Mathematics, Yalnızbağ Campus, 24100 Erzincan, Turkey. E-mail: bahar_kuloglu@hotmail.com.
[2] Erzincan Binali Yıldırım University, Faculty of Arts and Sciences, Department of Mathematics, Yalnızbağ Campus, 24100 Erzincan, Turkey.
[*] *Corresponding author: eozkan@erzincan.edu.tr.*

The Narayana numbers $N_{n,k}$ can be expressed by the Narayana Triangle which summarizes their symmetry $N_{n,k} = N_{n,n-k+1}$ as in the Pascal triangle.

$$N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1} = \frac{1}{k+1}\binom{n}{k}\binom{n-1}{k}$$

which can also be expressed as

$$N_{n,k} = \binom{n-1}{k}\binom{n+1}{k+1} - \binom{n}{k}\binom{n}{k+1}$$

which has echoes of the Catalan result

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}.$$

Many of the pertinent number theory relations and connections have been elaborated by Carlitz and Riordan [11] and Deveci, Sloane, and Dişkaya [12-14]. There are many links to the Online Encyclopedia of Integer Sequences [13] with its rich links to the literature, for example, (Triangular numbers are referenced in the On-Line Encyclopedia of Integer Sequences. They are numbered in OEIS as $N_{10,k} \equiv A001263$; $N_{n,1} = A000217$ [13]). With any of the above formulas for $N_{n,k}$, we obtain the data in Table 1.

Table 1. The Narayana Triangle.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
1	1									
2	1	1								
3	1	3	1							
4	1	6	6	1						
5	1	10	20	10	1					
6	1	15	50	50	15	1				
7	1	21	105	175	105	21	1			
8	1	28	196	490	490	196	28	1		
9	1	36	336	1176	1764	1176	336	36	1	
10	1	45	540	2520	5292	5292	2520	540	45	1

The first ten rows of the Narayana triangle read in Table1. The row sums in Table 1 are the Catalan Numbers, and the leading diagonal sums are the Generalized Catalan Numbers.

We can see that triple number sequences such as Narayana have also found application in some branches of mathematics [15,16]. Some papers also used the techniques used in this paper. [17,18]

2. NEW NARAYANA TRIANGLE

H. Hosoya [19] defined a triangular array $\{f_{m,n}\}_{m\geq n\geq 0}$ of positive integers that is called Fibonacci triangle. The Fibonacci or Hosoya's triangle $\{f_{m,n}\}_{m\geq n\geq 0}$ is defined by the two recurrences

(i) $f_{m,n} = f_{m-1,n} + f_{m-2,n},\ m \geq 2,$

(ii) $f_{m,n} = f_{m-1,n-1} + f_{m-2,n-2},\ m \geq 2,$

with the initial conditions: $f_{0,0} = 1, f_{1,0} = 1, f_{1,1} = 1, f_{2,1} = 1.$

Hosoya shows that the $\{f_{m,n}\}$ is two dimensional of the Fibonacci sequence (for the details see [19]). Inspiring of Fibonacci triangle, we consider a new array by Narayana numbers. A new Narayana triangle $\{\lambda_n^m\}$ is defined by the recurrences

$$(i)\ \lambda_n^m = \lambda_{n-1}^m + \lambda_{n-3}^m,\ m \geq 3, \tag{4}$$

$$\text{(ii)}\ \lambda_n^m = \lambda_{n-1}^{m-1} + \lambda_{n-3}^{m-3},\ m, n \geq 3, \tag{5}$$

with the initial conditions

$$\begin{aligned} &\lambda_0^0 = 0, \lambda_1^0 = 1, \lambda_2^0 = 1, \\ &\lambda_1^1 = 0, \lambda_2^1 = 1, \lambda_3^1 = 1, \\ &\lambda_2^2 = 0, \lambda_3^2 = 1, \lambda_4^2 = 1. \end{aligned} \tag{6}$$

Let's λ_n^m denote the element in row n and column m. The numbers λ_n^m can be arranged triangulary as in Fig. 1 and Fig. 2.

$$\begin{array}{c}
\lambda_0^0 \\
\lambda_1^0 \quad \lambda_1^1 \\
\lambda_2^0 \quad \lambda_2^1 \quad \lambda_2^2 \\
\lambda_3^0 \quad \lambda_3^1 \quad \lambda_3^2 \quad \lambda_3^3 \\
\lambda_4^0 \quad \lambda_4^1 \quad \lambda_4^2 \quad \lambda_4^3 \quad \lambda_4^4 \\
\lambda_5^0 \quad \lambda_5^1 \quad \lambda_5^2 \quad \lambda_5^3 \quad \lambda_5^4 \quad \lambda_5^5 \\
\lambda_6^0 \quad \lambda_6^1 \quad \lambda_6^2 \quad \lambda_6^3 \quad \lambda_6^4 \quad \lambda_6^5 \quad \lambda_6^6 \\
\lambda_7^0 \quad \lambda_7^1 \quad \lambda_7^2 \quad \lambda_7^3 \quad \lambda_7^4 \quad \lambda_7^5 \quad \lambda_7^6 \quad \lambda_7^7 \\
\lambda_8^0 \quad \lambda_8^1 \quad \lambda_8^2 \quad \lambda_8^3 \quad \lambda_8^4 \quad \lambda_8^5 \quad \lambda_8^6 \quad \lambda_8^7 \quad \lambda_8^8 \\
⋰ \quad ⋰ \quad ⋰ \quad ⋰ \quad ⋰ \quad \ddots \quad \ddots \quad \ddots \quad \ddots \quad \ddots
\end{array}$$

Figure 1. New Binomial Narayana Triangle.

$$\begin{array}{c}
0 \\
1 \quad 0 \\
1 \quad 1 \quad 0 \\
1 \quad 1 \quad 1 \quad 0 \\
2 \quad 1 \quad 1 \quad 2 \quad 0 \\
3 \quad 2 \quad 1 \quad 2 \quad 3 \quad 0 \\
4 \quad 3 \quad 2 \quad 2 \quad 3 \quad 4 \quad 0 \\
6 \quad 4 \quad 3 \quad 4 \quad 3 \quad 4 \quad 6 \quad 0 \\
9 \quad 6 \quad 4 \quad 6 \quad 6 \quad 4 \quad 6 \quad 9 \quad 0 \\
13 \quad 9 \quad 6 \quad 8 \quad 9 \quad 8 \quad 6 \quad 9 \quad 13 \quad 0 \\
⋰ \quad ⋰ \quad ⋰ \quad ⋰ \quad ⋰ \quad \ddots \quad \ddots \quad \ddots \quad \ddots \quad \ddots
\end{array}$$

Figure 2. New Narayana Triangle numbers.

Narayana Triangle is a triangular arrangement of numbers (like Pascal's triangle) based on the Narayana numbers. Each number is the sum of itself and two following terms in either the left diagonal or the right diagonal.

Each element written as the product of two Fibonacci numbers is similar to the combinatorial representation of each element in Pascal's triangle. This situation manifests itself in the Narayana triangle, similar to the Hosoya and Pascal's triangle. Also, if you look at the subscripts in Hosoya's triangle it has the same pattern of increasing and decreasing subscripts as the binomial expansion in Pascal's triangle. Similarly, it can be said that the Narayana triangle, the Hosoya triangle, and therefore the Fibonacci numbers have application areas such as in nature, science and art. If we exclude the first diagonal on the right each element is the product of the two Narayana numbers namely:

$$\begin{array}{ccccccccccc}
 & & & & & N_1N_1 & & & & & \\
 & & & & N_2N_1 & & N_1N_2 & & & & \\
 & & & N_3N_1 & & N_2N_2 & & N_1N_3 & & & \\
 & & N_4N_1 & & N_3N_2 & & N_2N_3 & & N_1N_4 & & \\
 & N_5N_1 & & N_4N_2 & & N_3N_3 & & N_2N_4 & & N_1N_5 & \\
N_6N_1 & & N_5N_2 & & N_4N_3 & & N_3N_4 & & N_2N_5 & & N_1N_6 \\
\iddots & & \iddots & & \iddots & \vdots & & \ddots & & \ddots &
\end{array}$$

Figure 3. New Narayana Triangle as Products of Narayana Numbers.

These explanations will be expressed more clearly in the future as visual figures. From the relation (4), we write

$$\lambda_n^0 = \lambda_{n-1}^0 + \lambda_{n-3}^0$$

with $\lambda_0^0 = 0 = N_0$, $\lambda_1^0 = 1 = N_1$, $\lambda_2^0 = 1 = N_2$. So, we conclude $\lambda_n^0 = N_n$.

Likewise, since $\lambda_n^n = \lambda_{n-1}^n + \lambda_{n-3}^n$, it follows that

$$\lambda_{n+1}^n = N_{n+1}.$$

Similarly, we can show that

$$\lambda_n^0 = \lambda_n^{n-1} = N_n$$

or

$$\lambda_n^1 = \lambda_n^{n-2} = N_{n-1}.$$

Successive applications of the recurrence (4) give interesting patterns

$$\begin{aligned}
\lambda_n^m &= \lambda_{n-1}^m + \lambda_{n-3}^m \\
&= \lambda_{n-3}^m + \lambda_{n-2}^m + \lambda_{n-4}^m \\
&= \lambda_{n-3}^m + \lambda_{n-5}^m + \lambda_{n-3}^m + \lambda_{n-4}^m \\
&= 2\lambda_{n-3}^m + \lambda_{n-5}^m + \lambda_{n-4}^m \\
&= 2\lambda_{n-3}^m + \lambda_{n-5}^m + \lambda_{n-7}^m + \lambda_{n-5}^m \\
&= 2\lambda_{n-3}^m + 2\lambda_{n-5}^m + \lambda_{n-7}^m \\
&= \cdots \\
&= 6\lambda_{n-6}^m + 3\lambda_{n-7}^m + 4\lambda_{n-8}^m \\
&= \cdots \\
&= 19\lambda_{n-9}^m + 9\lambda_{n-10}^m + 13\lambda_{n-11}^m.
\end{aligned}$$

So, we find a close relation between λ_n^m and Narayana numbers as follows.

Theorem 2.1. $\lambda_n^m = N_{k+1}\lambda_{n-k}^m + N_{k-1}\lambda_{n-k-1}^m + N_k\lambda_{n-k-2}^m,\ 2 \le k \le n-m-2.$

Proof: In particular, let $k = n-m-2$. Then, we have

$$\begin{aligned}\lambda_n^m &= N_{n-m-1}\lambda_{m+2}^m + N_{n-m-3}\lambda_{m+1}^m + N_{n-m-2}\lambda_m^m \\ &= N_{n-m-1}N_{m+1} + N_{n-m-3}N_{m+1} + N_{n-m-2}0 \\ &= N_{n-m}N_{m+1}.\end{aligned} \tag{7}$$

Thus, every element in the array is a product of two Narayana numbers. For example, $\lambda_6^4 = N_2N_5 = 1.3 = 3$ and $\lambda_9^3 = N_6N_4 = 2.4 = 8$.

Since $\lambda_n^{n-m-1} = \lambda_n^m$, from equation (7), it follows that

$$\lambda_n^{n-m-1} = \lambda_n^m = N_{n-m}N_{m+1}.$$

Let $n = 2r$ and $m = r$. The equality (7) yields

$$\lambda_{2r}^r = N_rN_{r+1}.$$

Thus, λ_{2r}^r is the consecutive product of Narayana numbers.

Theorem 2.2. *The Binet formula for the Narayana triangle is*

$$\lambda_n^m = a_m\alpha^{n+1} + b_m\beta^{n+1} + c_m\gamma^{n+1}$$

where

$$\begin{aligned}a_m &= a^2 + ab\beta^{2m+2}\gamma^{m+1} + ac\beta^{m+1}\gamma^{2m+2}, \\ b_m &= b^2 + ab\alpha^{2m+2}\gamma^{m+1} + bc\alpha^{m+1}\gamma^{2m+2}, = N_{n-m}N_{m+1}, \\ c_m &= c^2 + ac\alpha^{2m+2}\beta^{m+1} + bc\alpha^{m+1}\beta^{2m+2}\end{aligned} \tag{8}$$

and a, b, c are defined as in (3).

Proof: By using the relation in (8), we can write that

$$\begin{aligned}\lambda_n^m &= N_{n-m}N_{m+1} \\ \lambda_n^m &= (\alpha^{n-m}a + \beta^{n-m}b + \gamma^{n-m}c)(\alpha^{m+1}a + \beta^{m+1}b + \gamma^{m+1}c) \\ &= a^2\alpha^{n+1} + ab\alpha^{n-m}\beta^{m+1} + ac\alpha^{n-m}\gamma^{m+1} + ab\alpha^{m+1}\beta^{n-m} + b^2\beta^{n+1} + bc\beta^{n-m}\gamma^{m+1} \\ &\quad + ac\alpha^{m+1}\gamma^{n-m} + bc\beta^{m+1}\gamma^{n-m} + c^2\gamma^{n+1} \\ &= \alpha^{n+1}(a^2 + ab\beta^{2m+2}\gamma^{m+1} + ac\beta^{m+1}\gamma^{2m+2}) \\ &\quad + \beta^{n+1}(b^2 + ab\alpha^{2m+2}\gamma^{m+1} + bc\alpha^{m+1}\gamma^{2m+2}) + \gamma^{n+1}(c^2 + ac\alpha^{2m+2}\beta^{m+1} \\ &\quad + bc\alpha^{m+1}\beta^{2m+2}) \\ &= a_m\alpha^{n+1} + b_m\beta^{n+1} + c_m\gamma^{n+1}.\end{aligned}$$

Theorem 2.3. *The generating function for the partial sum of the Narayana triangle numbers is*

$$G_\lambda(x) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\lambda_n^m\right)x^n = \frac{x+2x^2}{(1-x-x^3)^2}.$$

Proof: Let

$$G_\lambda(x) = \lambda_0^0 + (\lambda_1^0 + \lambda_1^1)x + (\lambda_2^0 + \lambda_2^1 + \lambda_2^2)x^2 + \cdots + (\lambda_n^0 + \lambda_n^1 + \cdots + \lambda_n^n)x^n + \cdots$$

If we multiply every term of this function by $x^2, x^6, -2x, 2x^4, -2x^3$, respectively, we get

$$x^2G_\lambda(x) = \lambda_0^0x^2 + (\lambda_1^0+\lambda_1^1)x^3 + \cdots + (\lambda_n^0 + \lambda_n^1 + \cdots + \lambda_n^n)x^{n+2} + \cdots,$$
$$x^6G_\lambda(x) = \lambda_0^0x^6 + (\lambda_1^0+\lambda_1^1)x^7 + \cdots + (\lambda_n^0 + \lambda_n^1 + \cdots + \lambda_n^n)x^{n+6} + \cdots,$$
$$-2xG_\lambda(x) = -2\lambda_0^0x - 2(\lambda_1^0+\lambda_1^1)x^2 - \cdots - 2(\lambda_n^0 + \lambda_n^1 + \cdots + \lambda_n^n)x^{n+1} + \cdots,$$
$$2x^4G_\lambda(x) = 2\lambda_0^0x^4 + 2(\lambda_1^0+\lambda_1^1)x^5 + \cdots + 2(\lambda_n^0 + \lambda_n^1 + \cdots + \lambda_n^n)x^{n+4} + \cdots,$$
$$-2x^3G_\lambda(x) = -2\lambda_0^0x^3 - 2(\lambda_1^0+\lambda_1^1)x^4 - \cdots - 2(\lambda_n^0 + \lambda_n^1 + \cdots + \lambda_n^n)x^{n+3} + \cdots.$$

If we add the above equations, we have

$$(1-x-x^3)^2G_\lambda(x) = \lambda_0^0 + (\lambda_0^0 + \lambda_1^1)x + (\lambda_2^0+\lambda_2^1+\lambda_2^2 + \lambda_0^0)x^2 + 0 + \cdots + 0 + \cdots.$$

Then we obtain that

$$G_\lambda(x) = \frac{x + 2x^2}{(1-x-x^3)^2}$$

Thus, the proof is completed.

3. SOME IDENTITIES OF NEW NARAYANA TRIANGLE

The properties of various configurations within the triangle array for the Fibonacci triangle were investigated in [4]. In this section, we examine similar properties for the Narayana triangle.

Proposition 3.1. *The following relations are satisfied.*

1. $\lambda_n^m = \lambda_{n-4}^{m-1}+\lambda_{n-4}^{m-3} + (\lambda_{n-6}^{m-3} + \lambda_{n-2}^{m-1})$,
2. $\lambda_{n+2}^{m+1}\lambda_{n-2}^{m-1} = \lambda_n^{m+1}\lambda_n^{m-1}$,
3. $\lambda_{n-6}^{m-3} = \lambda_{n-2}^{m-1}-\lambda_{n-1}^{m-1} - \lambda_{n-1}^m + \lambda_n^m$,
4. $\lambda_{2n}^n = \lambda_n^0\lambda_{n+1}^n$.

Proof:

1. Using (7) and Fig. 2, we get Fig. 4.

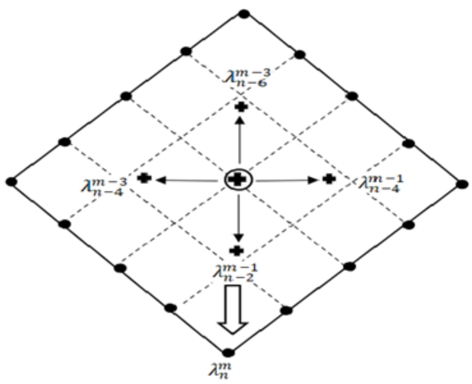

Figure 4. Planar view of $\lambda_n^m = \lambda_{n-4}^{m-1}+\lambda_{n-4}^{m-3} + (\lambda_{n-6}^{m-3} + \lambda_{n-2}^{m-1})$

We obtain

$$\begin{aligned}\lambda_n^m &= \lambda_{n-4}^{m-1}+\lambda_{n-4}^{m-3} + (\lambda_{n-6}^{m-3} + \lambda_{n-2}^{m-1})\\ &= N_{n-m-3}N_m + N_{n-m-1}N_{m-2} + (N_{n-m-3}N_{m-2} + N_{n-m-1}N_m)\\ &= N_m(N_{n-m-3} + N_{n-m-1}) + N_{m-2}(N_{n-m-1} + N_{n-m-3})\\ &= N_mN_{n-m}+N_{m-2}N_{n-m}\\ &= N_{n-m}N_{m+1}.\end{aligned}$$

2. Using (7) and Fig. 2, we obtain the Fig. 5.
We obtain

$$\lambda_{n+2}^{m+1}\lambda_{n-2}^{m-1} = \lambda_n^{m+1}\lambda_n^{m-1},$$
$$N_{n-m+1}N_{m+2}N_{n-m-1}N_m = N_{n-m-1}N_{m+2}N_{n-m+1}N_m.$$

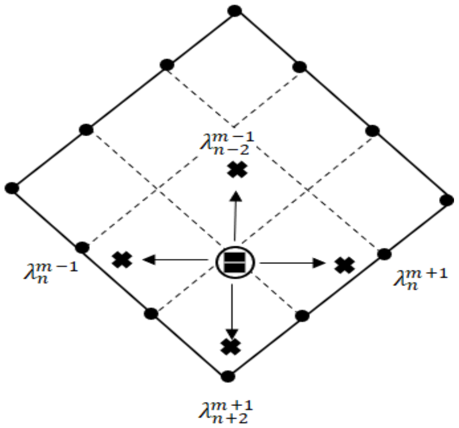

Figure 5. Planar view of $\lambda_{n+2}^{m+1}\lambda_{n-2}^{m-1} = \lambda_n^{m+1}\lambda_n^{m-1}$.

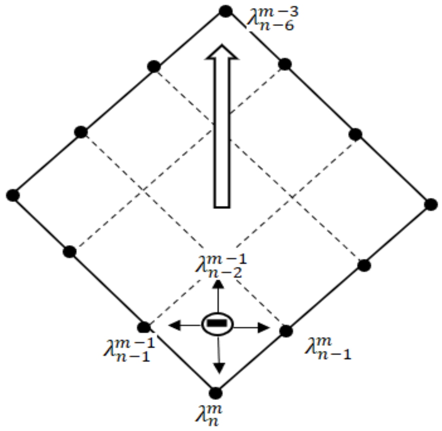

Figure 6. Planar view of $\lambda_{n-6}^{m-3} = \lambda_{n-2}^{m-1}-\lambda_{n-1}^{m-1} - \lambda_{n-1}^{m} + \lambda_n^m$

3. Using (7) and Fig. 2, we obtain Fig. 6.
We obtain

$$\begin{aligned}\lambda_{n-6}^{m-3} &= \lambda_{n-2}^{m-1}-\lambda_{n-1}^{m-1} - \lambda_{n-1}^{m} + \lambda_n^m,\\ &= N_{n-m-1}N_m - N_{n-m}N_m - N_{n-m-1}N_{m+1} + N_{n-m}N_{m+1}\\ &= N_m(N_{n-m-1} - N_{n-m}) - N_{m+1}(N_{n-m-1} - N_{n-m})\\ &= -N_{n-m-3}(N_m - N_{m+1}) = N_{n-m-3}N_{m-2}.\end{aligned}$$

4. Using (7) and Fig.2, we have the Fig. 7.

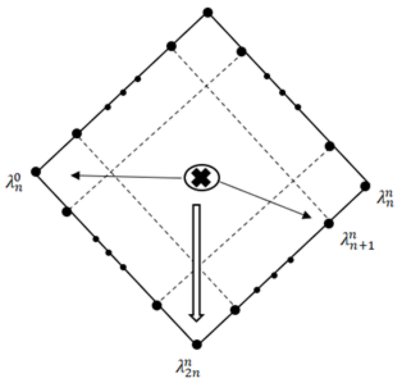

Figure 7. Planar view of $\lambda_{2n}^n = \lambda_n^0\lambda_{n+1}^n$

We obtain

$$\lambda_{2n}^{n} = \lambda_{n}^{0}\lambda_{n+1}^{n} = N_n N_1 N_1 N_{n+1} = N_n N_{n+1}.$$

CONCLUSION

In this study, the new Narayana triangle was created by utilizing the visual representation of Fibonacci numbers in a Pscal-like triangle and some identities belonging to this triangle have been made more concrete and expressed visually. This study can be compared with the results obtained by studying Tribonacci, Perrin, Leonardo number sequences and even number sequences with more than 3 recurrences.

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REFERENCES

[1] Allouche, J.P., Johnson, J., *Articles of 3rd Computer Music Conference* JIM96, 1996.
[2] Ramirez, J.L., Sirvent, V.F., *Annales Mathematicae et Informaticae*. **45,** 91, 2015.
[3] Özkan, E., Kuloğlu, B., *Asian-European Journal of Mathematics*, **14(**6), 2150100, 2021.
[4] Özkan, E., Kuloğlu, B., and Peters, J.F., *Chaos Solitons and Fractals*, **15(**5), 111473, 2021.
[5] Petersen, T.K., *Eulerian Numbers*, Springer, New York, pp. 19-36, 2015.
[6] Shannon, A.G., Bernstein, L., *Bulletin of the Australian Mathematical Society*, **8**(2), 261, 1973.
[7] Ramaswamy, S., *Journal of Advanced Research in Dynamical and Control Systems*, **12(**SP7), 2558, 2020.
[8] Guo, J.W., Lian, X., *Journal of Algebraic Combinatorics*, **55,** 4, 2022.
[9] Soykan, Y., *Earthline Journal of Mathematical Sciences*, **7**(1), 77, 2021.
[10] Goy, T., *Annales Mathematicae et Informaticae*, **49**, 75, 2018.
[11] Carlitz, L., Riordan, J., *Duke Mathematical Journal*, **31**(3), 371, 1964.
[12] Deveci, Ö., Shannon, A.G., *Mathematica Montisnigri*, **L**, 36, 2021.
[13] Sloane, N.J.A., *A Handbook of Integer Sequences*, Academic Press, New York, 1973.
[14] Dişkaya, O., Menken, H., *Journal of Contemporary Applied Mathematics*, **10**, 2, 2020.
[15] Hosoya, H., *Fibonacci Quaterly*, **14**, 173, 1976.
[16] Özkan, E., *Applied Mathematics and Computation*, **144**(2-3), 517, 2003.
[17] Özkan, E., Aydın, H., Dikici, R., *Applied Mathematics and Computation*, **141**, 2, 2003.
[18] Marin, M., *Rendiconti di Matematica Applicata*, **17**(7), 103, 1997.
[19] Marin, M., *Archivum Mathematicum*, **33**(3), 301, 1997.