ORIGINAL PAPER

NEIMARK-SACKER BIFURCATION AND CONTROL OF CHAOTIC BEHAVIOR IN A DISCRETE-TIME PLANT-HERBIVORE SYSTEM

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Abstract. *In this study, the dynamics of a discrete-time plant-herbivore model obtained using the forward Euler method are discussed. The existence of fixed points is investigated. A topological classification is made to examine the behavior of the positive fixed point where the plant and the herbivore coexist. In addition, the existence conditions and direction of Neimark-Sacker bifurcation of the model are investigated using bifurcation theory. Hybrid control method is applied to control the chaos caused by Neimark-Sacker bifurcation. Examples including time series figures, bifurcation figures, phase portraits and maximum Lyapunov exponent are provided to support our theoretical results.*

Keywords: *stability; plant herbivore system; Neimark-Sacker bifurcation; chaos control.*

1. INTRODUCTION

Prey-predator models, one of the building blocks of ecosystems, are among important study subjects in mathematical biology [1-4]. The Lotka–Volterra prey–predator model is well-known as one of the fundamental population models [5, 6]. In 1965, Holling [7] introduced three types of functional responses in order to simulate predation processes, following the pioneering theoretical research of Lotka [5] and Volterra [6] in the last century. The forms are presented based on the Michaelis-Menten or (Michaelis-Menten-Holling type II) function by authors in [8]. In [8, 9], the function $p(x) = \beta x / \sigma + x$ is proposed as one the predator response functional. Depending on this function, the following continuous-time plant-herbivore model with a constant carrying capacity of the plant is given as:

$$\begin{aligned} x'(t) &= \eta x(1-x) - \frac{\beta xy}{\sigma + x} \\ y'(t) &= y\left(\frac{\alpha x}{\sigma + x} - \delta\right) \end{aligned} \tag{1}$$

where, α, β, η, σ and δ are conversion rate, capturing rate; plant intrinsic growth parameter, half saturation constant, herbivore's death rates, respectively. $x(t)$ represents densities of plant species at time t and $y(t)$ is population densities of herbivore at time t.

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The continuous-time prey-predator version has been analyzed by researchers [8, 10-12]. On the other hand, some researchers analyze similar continuous models by discretization [13-17]. Another way to understand the behavior of species involving competitive interactions is to use discretization methods [18]. Discrete-time models allow random time step units for non-overlapping generations. These models are more realistic for a description of processes with different characteristic times which can retain the essential features of the corresponding continuous-time models and include richer applications [13, 19-31].

It is important to understand changes in the nature of stability in a dynamic system ([32, 33] and references therein). As a parameter changes in system; the stability of the systems may change, new stable points may appear, stable points may disappear or vice versa. Bifurcation theory is applied to investigate the changes that occur in the qualitative or topological structure of a continuous-time or discrete-time system (These reviews can be viewed in detail through the references cited here.)

In this study, we will examine the dynamics of the discrete-time plant-herbivore model obtained by discretization of system (1):

$$\begin{aligned} x_{n+1} &= x_n + h\left(\eta x_n(1-x_n) - \frac{\beta x_n y_n}{\sigma + x_n}\right) \\ y_{n+1} &= y_n + h\left(\frac{\alpha x_n y_n}{\sigma + x_n} - \delta y_n\right). \end{aligned} \tag{2}$$

This paper is organized as follows: in Section 2, we detect the existence of fixed points, and give the topological classification of the coexistence fixed point. In section 3, bifurcation analysis of the coexistence fixed point of system (2) is discussed. Section 4 includes control of complex behaviors of system (2). The numerical simulations which confirm the results obtained for system (2) is carried out in Section 5. Finally, in Section 6, the conclusions are presented.

2. EXISTENCE AND TOPOLOGICAL CLASIFICATIONS OF THE FIXED POINT *E_2* OF MODEL (2)

The fixed points of system (2) are solutions of the following system:

$$\begin{aligned} x^* &= x^* + h\left(\eta x^*(1-x^*) - \frac{\beta x^* y^*}{\sigma + x^*}\right) \\ y^* &= y^* + h\left(\frac{\alpha x^* y^*}{\sigma + x^*} - \delta y^*\right). \end{aligned}$$

From there, we get

$$E_0 = (0,0),\ E_1 = (1,0)\ and\ E_2 = (x^*, y^*) = \left(\frac{\delta\sigma}{\alpha-\delta}, \frac{\alpha\eta\sigma(\alpha-\delta-\delta\sigma)}{\beta(\alpha-\delta)^2}\right).$$

Also, the Jacobian matrix J of system (2) evaluated at the fixed point E_2 is as follows:

$$J(x^*,y^*)=\begin{bmatrix} \dfrac{\alpha^2-\alpha\delta\left(1+h\eta\left(-1+\sigma\right)\right)-h\delta^2\eta\left(1+\sigma\right)}{\alpha\left(\alpha-\delta\right)} & -\dfrac{h\beta\delta}{\alpha} \\ \dfrac{h\eta\left(\alpha-\delta\left(1+\sigma\right)\right)}{\beta} & 1 \end{bmatrix}$$

where $x^*=\dfrac{\delta\sigma}{\alpha-\delta}, y^*=\dfrac{\alpha\eta\sigma\left(\alpha-\delta-\delta\sigma\right)}{\beta\left(\alpha-\delta\right)^2}$; and the characteristic equation of the Jacobian matrix J can be found as

$$\lambda^2+\frac{-2\alpha-h\delta\eta+\dfrac{h\delta\left(\alpha+\delta\right)\eta\sigma}{\alpha-\delta}}{\alpha}\lambda+\frac{\left(\alpha-\delta\right)\left(\alpha+h\left(1+h\left(\alpha-\delta\right)\right)\delta\eta\right)-h\delta\left(\alpha+\delta+h\alpha\delta-h\delta^2\right)\eta\sigma}{\alpha\left(\alpha-\delta\right)}=0.$$

To investigate the dynamics of the coexistence fixed point of the system (2), we can give the definition and the lemma as follows:

Definition 1. The following situaitons are valid for the fixed point $\left(x^*,y^*\right)$ of any system

(i) If $|\lambda_1|<1$ and $|\lambda_2|<1$, then $\left(x^*,y^*\right)$ is a sink point, and it is locally asymptotically stable;

(ii) If $|\lambda_1|>1$ and $|\lambda_2|>1$, then $\left(x^*,y^*\right)$ is a source point, and it is locally unstable;

(iii) If $|\lambda_1|<1$ and $|\lambda_2|>1$ (or $|\lambda_1|>1$ and $|\lambda_2|<1$), then $\left(x^*,y^*\right)$ is a saddle point;

(iv) If either $|\lambda_1|=1$ or $|\lambda_2|=1$, then $\left(x^*,y^*\right)$ is non-hyperbolic point.

Lemma 1. [28, 34, 35] Let $F\left(x\right)=x^2+Bx+C$. Suppose that $F\left(1\right)>0$, λ_1 and λ_2 are two roots of $F(x)=0$. Then;

(i) $F\left(-1\right)>0$ and $C<1$ if and only if $|\lambda_1|<1$ and $|\lambda_2|<1$;

(ii) $F\left(-1\right)<0$ if and only if $|\lambda_1|>1$ and $|\lambda_2|<1$(or $|\lambda_1|<1$ and $|\lambda_2|>1$;

(iii) $F\left(-1\right)>0$ and $C>1$ if and only if $|\lambda_1|>1$ and $|\lambda_2|>1$;

(iv) $F\left(-1\right)=0$ and $B\neq 0,2$ if and only if $|\lambda_1|=-1$ and $|\lambda_2|\neq 1$;

(v) $B^2-4C<0$ and $C=1$ if and only if λ_1 and λ_2 are a pair of conjugate complex roots and $|\lambda_1|=|\lambda_2|=1$.

For the dynamics of the system (2), if the Lemma 1 is used, then we have the following theorem.

Theorem 1. Assume that $\delta<\dfrac{\alpha}{1+\sigma}$. The coexistence fixed point E_2 is a

i) sink point if

$$\frac{\alpha-\delta}{\alpha+\delta}<\sigma<\frac{\alpha-\delta}{\delta}, \alpha>\delta,\ 0<\eta\leq\frac{4\alpha(\alpha-\delta)^2(\alpha-\delta(1+\sigma))}{\delta(-\alpha+\delta+(\alpha+\delta)\sigma)^2}, 0<h<\frac{-\alpha+\delta+(\alpha+\delta)\sigma}{(\alpha-\delta)(\alpha-\delta(1+\sigma))},$$

ii) source point if

$$0<\eta\leq\frac{4\alpha(\alpha-\delta)^2(\alpha-\delta(1+\sigma))}{\delta(-\alpha+\delta+(\alpha+\delta)\sigma)^2},\quad h>\frac{-\alpha+\delta+(\alpha+\delta)\sigma}{(\alpha-\delta)(\alpha-\delta(1+\sigma))},\quad \alpha>\delta,\frac{\alpha-\delta}{\alpha+\delta}<\sigma<\frac{\alpha-\delta}{\delta},$$

iii) saddle point if

$$\frac{\alpha-\delta}{\alpha+\delta}<\sigma<\frac{\alpha-\delta}{\delta}, \alpha>\delta, \eta>\frac{4\alpha(\alpha-\delta)^2(\alpha-\delta(1+\sigma))}{\delta(-\alpha+\delta+(\alpha+\delta)\sigma)^2}\ \textit{and}\ h^-<h<h^+ \ni$$

$$h^-=\frac{2}{-\alpha+\delta}-\sqrt{\frac{(\alpha-\delta)^2(-4\alpha^2+4\alpha\delta+\delta\eta)+2(\alpha-\delta)\delta(2\alpha(\alpha-\delta)-(\alpha+\delta)\eta)\sigma+\delta(\alpha+\delta)^2\eta\sigma^2}{(\alpha-\delta)^2\delta\eta(-\alpha+\delta+\delta\sigma)^2}}+\frac{1+\sigma}{\alpha-\delta(1+\sigma)}$$

$$h^+=\frac{2}{-\alpha+\delta}+\sqrt{\frac{(\alpha-\delta)^2(-4\alpha^2+4\alpha\delta+\delta\eta)+2(\alpha-\delta)\delta(2\alpha(\alpha-\delta)-(\alpha+\delta)\eta)\sigma+\delta(\alpha+\delta)^2\eta\sigma^2}{(\alpha-\delta)^2\delta\eta(-\alpha+\delta+\delta\sigma)^2}}+\frac{1+\sigma}{\alpha-\delta(1+\sigma)},$$

iv) flip bifurcation point if

$$h=h_{FB}=\frac{\alpha\delta\eta(-1+\sigma)+\delta^2\eta(1+\sigma)\pm\sqrt{\delta\eta(\delta\eta(-\alpha+\delta+(\alpha+\delta)\sigma)^2-4\alpha(\alpha-\delta)^2(\alpha-\delta(1+\sigma)))}}{(\alpha-\delta)\delta\eta(\alpha-\delta(1+\sigma))}$$

such that $\frac{\left(-2\alpha^2+\alpha\delta(2+h\eta(-1+\sigma))+h\delta^2\eta(1+\sigma)\right)}{\alpha(\alpha-\delta)}\neq 0,-2.$

3. NEIMARK-SACKER BIFURCATION

In this section, we obtain the following bifurcation conditions for the coexistence fixed point E_2 of the system (2):

$$E_2^{NSB}=\left\{\eta,\beta,\alpha,\sigma,\delta,h\in\mathbb{R}_+:\alpha>\delta,\ \frac{\alpha-\delta}{\alpha+\delta}<\sigma<\frac{\alpha-\delta}{\delta},\quad 0<\eta<\frac{4\alpha(\alpha-\delta)^2(\alpha-\delta(1+\sigma))}{\delta(-\alpha+\delta+(\alpha+\delta)\sigma)^2}, \textit{and}\ h=h_{NSB}\right\}$$

where $h_{NSB}=\frac{-\alpha+\delta+(\alpha+\delta)\sigma}{(\alpha-\delta)(\alpha-\delta(1+\sigma))}$.

Although E_2 is locally asymptotic stable when $0<h<\frac{-\alpha+\delta+(\alpha+\delta)\sigma}{(\alpha-\delta)(\alpha-\delta(1+\sigma))}$, E_2 loses stability for $h=h_{NSB}$. Moreover, for $h=\frac{-\alpha+\delta+(\alpha+\delta)\sigma}{(\alpha-\delta)(\alpha-\delta(1+\sigma))}$, the characteristic polynomial of the matrix J_{E_2} is

$$F(\lambda)=\lambda^2+\frac{\lambda\left(-2\alpha^4+2\alpha\delta^2\left(\delta+\eta\left(-1+\sigma\right)\right)\left(1+\sigma\right)+\delta^3\eta\left(1+\sigma\right)^2+2\alpha^3\delta\left(3+\sigma\right)+\alpha^2\delta\left(\eta\left(-1+\sigma\right)^2-2\delta\left(3+2\sigma\right)\right)\right)}{\alpha\left(\alpha-\delta\right)^2\left(\alpha-\delta\left(1+\sigma\right)\right)}+1.$$

From $F(\lambda)=0$, we can get

$$\lambda_{1,2}=\frac{\eta+\alpha\left(2+\frac{\eta}{\delta}\right)+\frac{\left(\alpha+\delta\right)^2\eta\sigma}{\left(\alpha-\delta\right)^2}+\frac{\alpha^2\eta}{\delta\left(-\alpha+\delta+\delta\sigma\right)}}{2\alpha}$$

$$\mp i\frac{\sqrt{-\beta^2\delta\eta\left(-\alpha+\delta+\left(\alpha+\delta\right)\sigma\right)^2\left(\left(\alpha-\delta\right)^2\left(-4\alpha^2+4\alpha\delta+\delta\eta\right)+2\left(\alpha-\delta\right)\delta\left(2\alpha\left(\alpha-\delta\right)-\left(\alpha+\delta\right)\eta\right)\sigma+\delta\left(\alpha+\delta\right)^2\eta\sigma^2\right)}}{2\alpha\beta\left(\alpha-\delta\right)^2\left(\alpha-\delta\left(1+\sigma\right)\right)}$$

such that $|\lambda|=|\bar{\lambda}|=1$. For, $h\in E_2^{NSB}$, we obtain

$$\left.\frac{\partial|\lambda(h)|}{\partial h}\right|_{h=\frac{-\alpha+\delta+(\alpha+\delta)\sigma}{(\alpha-\delta)(\alpha-\delta(1+\sigma))}}\neq 0.$$

Also, if $trJ(x^*,y^*)\neq 0,-1$ or equivalently

$$\frac{\left(2\alpha+h\delta\eta-\frac{h\delta\left(\alpha+\delta\right)\eta\sigma}{\alpha-\delta}\right)}{\alpha}\neq 0,-1, \tag{3}$$

then it is confirmed that

$$\lambda^k(h)\neq 1 \text{ for } k=1,2,3,4.$$

By simple calculation, we can find eigenvectors $q,p\in\mathbb{C}^2$ which correspond to the eigenvalue λ of the matrix $J(E_2^{NSB})$ and the eigenvalue $\bar{\lambda}$ of the matrix $J(E_2^{NSB})^T$, respectively. Let us take the scalar product in $\mathbb{C}^2:\langle p,q\rangle=\bar{p}_1q_1+\bar{p}_2q_2$. To obtain the normalization $\langle p,q\rangle=1$, it is normalized the vector p according to q. Using the transformation $u=x-\frac{\delta\sigma}{\alpha-\delta}, v=y-\frac{\alpha\eta\sigma\left(\alpha-\delta-\delta\sigma\right)}{\beta\left(\alpha-\delta\right)^2}$, the fixed point E_2 is shifted to the origin. So, we obtain

$$\begin{pmatrix}u\\v\end{pmatrix}\to J_{E_2}\begin{pmatrix}u\\v\end{pmatrix}+\begin{pmatrix}F(u,v)\\G(u,v)\end{pmatrix} \tag{4}$$

where

$$F(u,v)=-\frac{h\eta\left(\alpha^2\left(-1+\sigma\right)-\delta^2\left(1+\sigma\right)+\alpha\delta\left(2+\sigma\right)\right)}{\alpha^2\sigma}u^2-\frac{h\beta\left(\alpha-\delta\right)^2}{\alpha^2\sigma}uv$$
$$-\frac{h\left(\alpha-\delta\right)^2\eta\left(\alpha-\delta\left(1+\sigma\right)\right)}{\alpha^3\sigma^2}u^3+\frac{h\beta\left(\alpha-\delta\right)^3}{\alpha^3\sigma^2}u^2v$$

and

$$G(u,v)=-\frac{h(\alpha-\delta)\eta(\alpha-\delta(1+\sigma))}{\alpha\beta\sigma}u^2+\frac{h(\alpha-\delta)^2}{\alpha\sigma}uv$$
$$+\frac{h(\alpha-\delta)^2\eta(\alpha-\delta(1+\sigma))}{\alpha^2\beta\sigma^2}u^3-\frac{h(\alpha-\delta)^3}{\alpha^2\sigma^2}u^2v.$$

From there, we get that

$$B_1(u,v)=\sum_{j,k=1}^{2}\frac{\partial^2 F}{\partial\xi_j\partial\xi_k}\Big|_{\xi=0}=-\frac{2h\eta\left[\alpha^2(-1+\sigma)+\alpha\delta(2+\sigma)-\delta^2(1+\sigma)\right]}{\alpha^2\sigma}u_1v_1$$
$$-\frac{h\beta(\alpha-\delta)^2}{\alpha^2\sigma}(u_2v_1+u_1v_2),$$
$$B_2(u,v)=\sum_{j,k=1}^{2}\frac{\partial^2 G}{\partial\xi_j\partial\xi_k}\Big|_{\xi=0}=-\frac{2h[\alpha-\delta]\eta[\alpha-\delta(1+\sigma)]}{\alpha\beta\sigma}u_1v_1$$
$$-\frac{2h(\alpha-\delta)^3}{\alpha^2\sigma^2}(u_2v_1+u_1v_2),$$

$$C_1(u,v,w)=\sum_{j,k,l=1}^{2}\frac{\partial^3 F}{\partial\xi_j\partial\xi_k\partial\xi_l}\Big|_{\xi=0}=-\frac{6h(\alpha-\delta)^2\eta(\alpha-\delta(1+\sigma))}{\alpha^3\sigma^2}u_1v_1w_1+\frac{2h\beta(\alpha-\delta)^3}{\alpha^3\sigma^2}u_1v_1w_2$$
$$+\frac{2h\beta(\alpha-\delta)^3}{\alpha^3\sigma^2}u_1v_2w_1+\frac{2h\beta(\alpha-\delta)^3}{\alpha^3\sigma^2}u_2v_1w_1,$$

and

$$C_2(u,v,w)=\sum_{j,k,l=1}^{2}\frac{\partial^3 G}{\partial\xi_j\partial\xi_k\xi_l}\Big|_{\xi=0}=\frac{6h(\alpha-\delta)^2\eta[\alpha-\delta(1+\sigma)]}{\alpha^2\beta\sigma^2}u_1v_1w_1-\frac{2h(\alpha-\delta)^3}{\alpha^2\sigma^2}u_1v_1w_2$$
$$-\frac{2h(\alpha-\delta)^3}{\alpha^2\sigma^2}u_1v_2w_1-\frac{2h(\alpha-\delta)^3}{\alpha^2\sigma^2}u_2v_1w_1.$$

$\forall U\in\mathbb{R}^2$ can be uniquely represented as

$$U=zq+\overline{zq}$$

for some $z\in\mathbb{C}$. Also, $\overline{z}$ is the conjugate of that complex number $z,$ and $z=\langle p,U\rangle.$ For all sufficiently small $|h|$ about $h_{NSB},$ we can transform the system (2) as follows:

$$z\to\lambda(h)z+g(z,\overline{z},h),$$

where $\lambda(h)=(1+w(h))e^{i\theta(h)}$ with $w(h_{NSB})=0$ and $g(z,\overline{z},h)$ is a complex valued smooth function of z and $\overline{z}$. Taylor expression of g with respect to $g(z,\overline{z})$ is

$$g(z,\overline{z},h)=\sum_{k+l>2}\frac{1}{k!l!}g_{kl}(h)z^k\overline{z}^l$$

and the Taylor coefficients g_{kl} calculated through vector functions are expressed by the formulas

$$g_{20}(h_{NSB})=\langle p,B(q,q)\rangle \tag{5}$$

$$g_{11}(h_{NSB})=\langle p,B(q,\overline{q})\rangle \tag{6}$$

$$g_{02}(h_{NSB})=\langle p,B(\overline{q},\overline{q})\rangle \tag{7}$$

$$g_{21}(h_{NSB})=\langle p,C(q,q,\overline{q})\rangle. \tag{8}$$

For the system (4) which exhibits Neimark-Sacker bifurcation, the coefficient $\varphi(h_{NSB})$ determining the direction of the appearance of the invariant curve can be calculated as

$$\varphi(h_{NSB})=\operatorname{Re}\left(\frac{e^{i\theta(h_{NSB})}}{2}g_{21}\right)-\operatorname{Re}\left(\frac{(1-2e^{i\theta(h_{NSB})})e^{-2i\theta(h_{NSB})}}{2(1-e^{i\theta(h_{NSB})})}g_{20}g_{11}\right)-\frac{1}{2}|g_{11}|^2-\frac{1}{4}|g_{02}|^2 \tag{9}$$

where $e^{i\theta(h_{NSB})}=\lambda(h_{NSB})$.

Theorem 2. If (3) holds, $\varphi(h_{NSB})\neq 0$ and the parameter changes its value in a small vicinity of E_2^{NSB}, then the system (2) passes through a Neimark-Sacker bifurcation at the only fixed point E_2. Moreover if $\varphi(h_{NSB})<0$ $(\varphi(h_{NSB})>0)$, then there exists a unique attracting (repelling) invariant closed curve which bifurcates from E_2.

4. CHAOS CONTROL

Although there are many control methods [23-27, 36-40] for the stabilization of a dynamic system that exhibits chaotic behavior, it is desired to stabilize the solutions by choosing the appropriate method for the system. Thus, the effect of chaos is reduced or completely eliminated by trying to pull chaotic orbits into a fixed orbit. In this section, it is aimed to control the chaos behavior of the system with the hybrid control method.

To control the system (2) subject to bifurcation, we consider the following controlled system

$$\begin{aligned} x_{n+1}&=\phi\left[x_n+h\left(\eta x_n(1-x_n)-\frac{\beta x_n y_n}{\sigma+x_n}\right)\right]+(1-\phi)x_n \\ y_{n+1}&=\phi\left[y_n+h\left(\frac{\alpha x_n y_n}{\sigma+x_n}-\delta y_n\right)\right]+(1-\phi)y_n \end{aligned} \tag{10}$$

where $\phi \in (0,1)$ is control parameter. The Jacobian matrix of controlled system (10) is given by

$$\begin{bmatrix} 1+h\left(\left(1-x^*\right)\eta - x^*\eta + \dfrac{x^*y^*\beta}{\left(x^*+\sigma\right)^2} - \dfrac{y^*\beta}{x^*+\sigma}\right)\phi & -\dfrac{hx^*\beta\phi}{x^*+\sigma} \\ h\left(-\dfrac{x^*y^*\alpha}{\left(x^*+\sigma\right)^2} + \dfrac{y^*\alpha}{x^*+\sigma}\right)\phi & 1+h\left(-\delta+\dfrac{x^*\alpha}{x^*+\sigma}\right)\phi \end{bmatrix}.$$

If

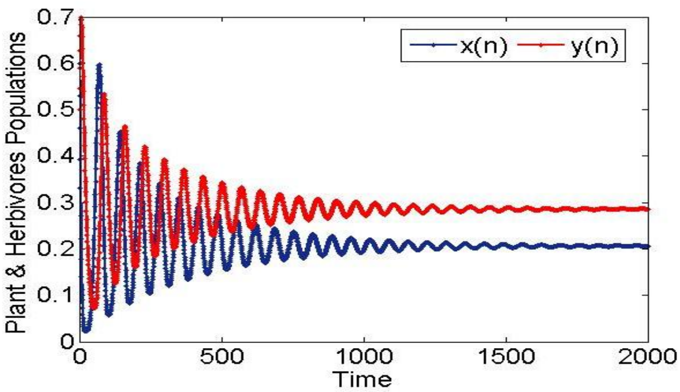
$$\left|\frac{\left(-2\alpha^2 + h\delta^2\eta\left(1+\sigma\right)\phi + \alpha\delta\left(2+h\eta\left(-1+\sigma\right)\phi\right)\right)}{\alpha\left(\alpha-\delta\right)}\right|$$
$$< 1 + \frac{h\delta^2\eta\left(1+\sigma\right)\phi\left(-1+h\delta\phi\right) + \alpha^2\left(1+h^2\delta\eta\phi^2\right) - \alpha\delta\left(1+h\eta\left(-1+\sigma\right)\phi + h^2\delta\eta\left(2+\sigma\right)\phi^2\right)}{\alpha\left(\alpha-\delta\right)} < 2,$$

then the the fixed point (x^*, y^*) of the controlled system (10) is locally asymptotic stable.

5. NUMERICAL SIMULATIONS

Example 1. By considering the condition (i) in Theorem 1, we choose the coefficients $\delta = 0.15$, $h = 0.5$, $\eta = 0.35$, $\beta = 1.27$, $\alpha = 0.95$ and $\sigma = 1.1$. Then the system (2) can be written as follows:

$$x_{n+1} = x_n + 0.5\left(0.35x_n(1-x_n) - \frac{1.27x_ny_n}{1.1+x_n}\right)$$
$$y_{n+1} = y_n + 0.5\left(\frac{0.95x_ny_n}{1.1+x_n} - 0.15y_n\right).$$



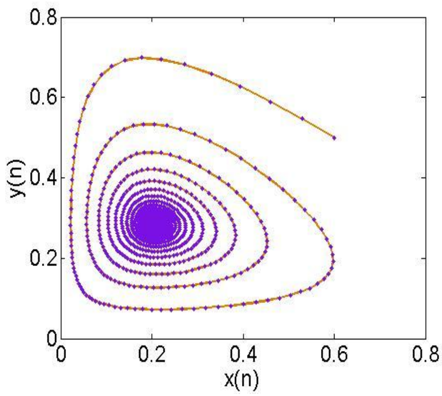


Figure 1. Phase Portrait and Time Series plot of System (2) when $\delta = 0.15$, $h = 0.5$ $\eta = 0.35$, $\beta = 1.27$, $\alpha = 0.95$ **and** $\sigma = 1.1$ **with initial point** $(0.6, 0.5)$.

The coexistence fixed point E_2 of system (2) is locally asymptotically stable since $\delta = 0.15 < \alpha/(1+\sigma) = 0.45$ and $h = 0.5 < 0.807$ (see Theorem 1). For these parameter values, the positive fixed point occurs at the points $x^* \cong 0.21$ $y^* \cong 0.29$. The eigenvalues of Jacobian matrix of the system (2) are $\lambda_\pm = 0.9928 \pm i0.0934$. These are the complex conjugate eigenvalues with $|\lambda_\pm| = 0.9971 < 1$. We say that E_2 is in the stable region for the parameter values $\delta = 0.15,\ h = 0.5$, $\eta = 0.35, \beta = 1.27, \alpha = 0.95, \sigma = 1.1$. According to these values, the time series and phase portrait plots of system (2) are displayed in Figure 1.

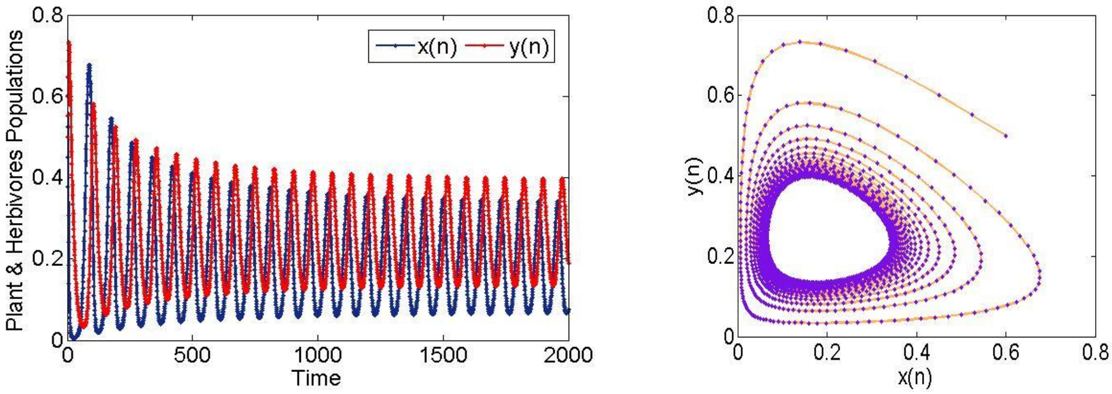


Figure 2. Phase Portrait and Time Series plot of System (2) when $\delta = 0.15,\ h = 0.45$ $\eta = 0.35, \beta = 1.25, \alpha = 0.95, \sigma = 0.9$ **with initial point** $(0.6, 0.5)$.

Let us take $\delta = 0.15,\ h = 0.45$ $\eta = 0.35, \beta = 1.25, \alpha = 0.95, \sigma = 0.9$. The coexistence fixed point E_2 of system (2) is unstable since $\delta = 0.15 < \alpha/(1+\sigma) = 0.5$ and $h = 0.45 > 0.357$ (see Theorem 1). For these parameter values, the coexistence fixed point occurs at the points $x^* \cong 0.1687, y^* \cong 0.2487$; and $\lambda_\pm = 0.9970 \pm i0.0862$ are the complex conjugate eigenvalues with $|\lambda_\pm| = 1.0008 > 1$. We say that E_2 is in the unstable region for the parameter values $\delta = 0.15,\ h = 0.45$, $\eta = 0.35, \beta = 1.25, \alpha = 0.95, \sigma = 0.9$ (see Figure 2).

Example 2. By considering the parameter values $\delta = 0.9,\ \eta = 4.1, \beta = 3, \alpha = 3.5, \sigma = 0.9$ and initial point $(0.2, 0.1)$, we obtain the following system

$$x_{n+1} = x + h[4.1x(1-x) - \frac{3xy}{0.9+x}]$$
$$y_{n+1} = y + h[\frac{3.5xy}{0.9+x} - 0.9y].$$

For $h_{NSB} = 0.292222$, the Neimark-Sacker bifurcation emerges at the fixed point $x^* = 0.311538461538461 54$ and $y^* = 1.1399334319526626$ produced by calculations. The eigenvalues of the Jacobian matrix $J(x^*, y^*) = \begin{bmatrix} 0.8388477358481655 & -0.22542820308183437 \\ 0.7148717948717949 & 1 \end{bmatrix}$ are

$$\lambda_1 = 0.9194238679240827 + 0.393268039753446 93i$$

and

$$\lambda_2 = 0.9194238679240827 - 0.393268039753446 93i$$

such that $\left|\lambda_{1,2}\right| = 1$. So, we get the complex eigenvectors correspondig to eigenvalues

$$q \sim (-0.09827863529054795 + 0.4796686717841284i, 0.8719284805270887)^T$$

and

$$p \sim (0.8719284805270887, 0.09827863529054795 - 0.4796686717841284i)^T.$$

To get $\langle p, q \rangle = 1$, we calculate as

$$p \sim (-1.0423861915773012i, -0.5734415277933639 - 0.11749162304229392i)^T.$$

By considering the formulas (5-8), the normal coefficients of the system (2) can be obtained as follow:

$$g_{20}(h_{NSB}) = 1.072359684111499 + 1.0320391716580672i \quad (11)$$

$$g_{11}(h_{NSB}) = 0.5019341926520112 + 0.5354694635345718i \quad (12)$$

$$g_{02}(h_{NSB}) = 0.7528982576646169 - 0.5271566561738714i \quad (13)$$

$$g_{21}(h_{NSB}) = -0.43939695753008834 + 0.018314528713435696i \quad (14)$$

such that

$$F(u,v) = -0.692345382494331u^2 - 0.41744990651261554u^3$$
$$-0.5375257687074829uv + 0.4436720630601448u^2v + o(\|U\|^4)$$

and

$$G(u,v) = -0.590049387089947u^2 + 0.4870248909313848u^3$$
$$+0.6271133968253968uv - 0.5176174069035022u^2v + o(\|U\|^4).$$

From (9), we get $\varphi(h_{NSB}) = -2.20423 < 0$. So, a supercritical Neimark-Sacker bifurcation emerges at $h_{NSB} = 0.292222$.

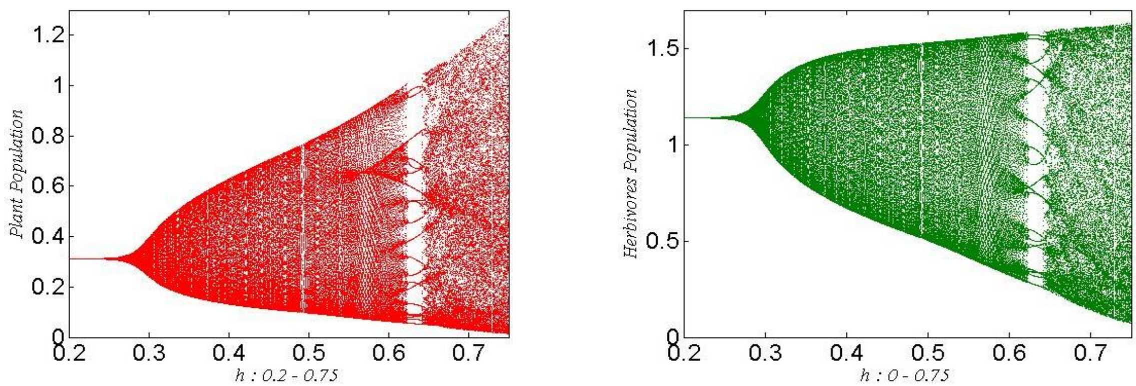


Figure 3. Neimark-Sacker bifurcation plot of System (2).

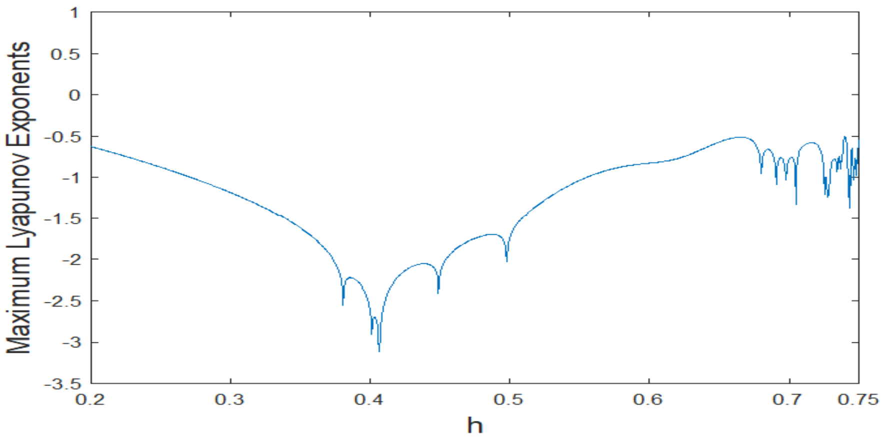


Figure 4. Lypanouv Exponent of the positive fixed point E_2 of system (2).

The positive fixed point E_2 of system (2) has Neimark-Sacker bifurcation at $h=0.292222$ with the parameter values $\delta=0.9$, $\eta=4.1, \beta=3, \alpha=3.5, \sigma=0.9$ and initial point $(0.2, 0.1)$. In Fig. 3, Neimark-Sacker bifurcation diagrams of the fixed point E_2 of the model (2) in the planes (h,x) and (h,y) are exhibited respectively. It is clearly observed that for $h<2.9222$, the interior (coexistence) fixed point of the model (2) is locally asymptotically stable. At $h=2.9222$, model loses its stability and move a stable invariant cycle for $h>2.9222$. Also, when $h>2.9222$, the invariant cycle moves to a quasi-periodic orbit, these orbits occur in the period windows. Finally, the orbits tend to chaos when the bifurcation parameter h increases.

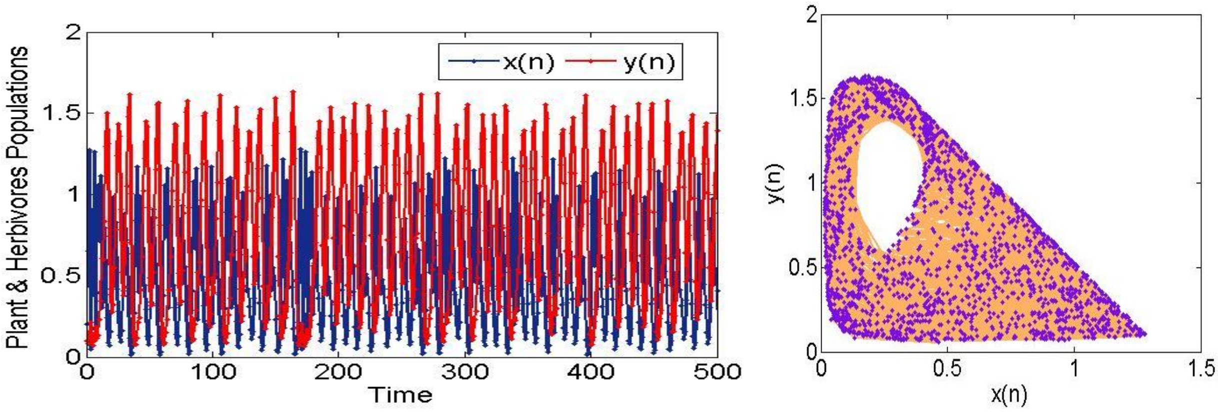


Figure 5. Phase Portrait and Time Series plot of System (2) when $h=0.75$**,** $\delta=0.9$, $\eta=4.1, \beta=3, \alpha=3.5, \sigma=0.9$ **and initial point** $(0.2, 0.1)$.

Fig.5 exhibits that the positive fixed point E_2 of system (2) has chaos with the parameter values $h=0.75$, $\delta=0.9$, $\eta=4.1, \beta=3, \alpha=3.5, \sigma=0.9$ and initial point $(0.2, 0.1)$.

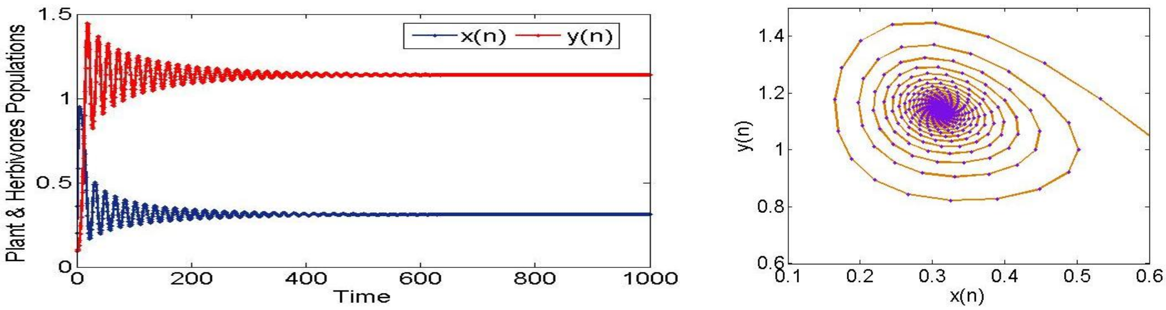


Figure 6. Phase Portrait and Time Series plot of System (2) when $h = 0.74$, $\delta = 0.9$, $\eta = 4.1$, $\beta = 3$, $\alpha = 3.5$, $\sigma = 0.9$, $\phi = 0.35$ **and initial point** $(0.2, 0.1)$

In Fig. 6, it is seen that the positive fixed point E_2 of system (2) has chaos control with the parameter values h = 0.74, $\delta = 0.9$, $\eta = 4.1$, $\beta = 3$, $\alpha = 3.5$, $\sigma = 0.9$, $\phi = 0.35$ and initial point $(0.2, 0.1)$.

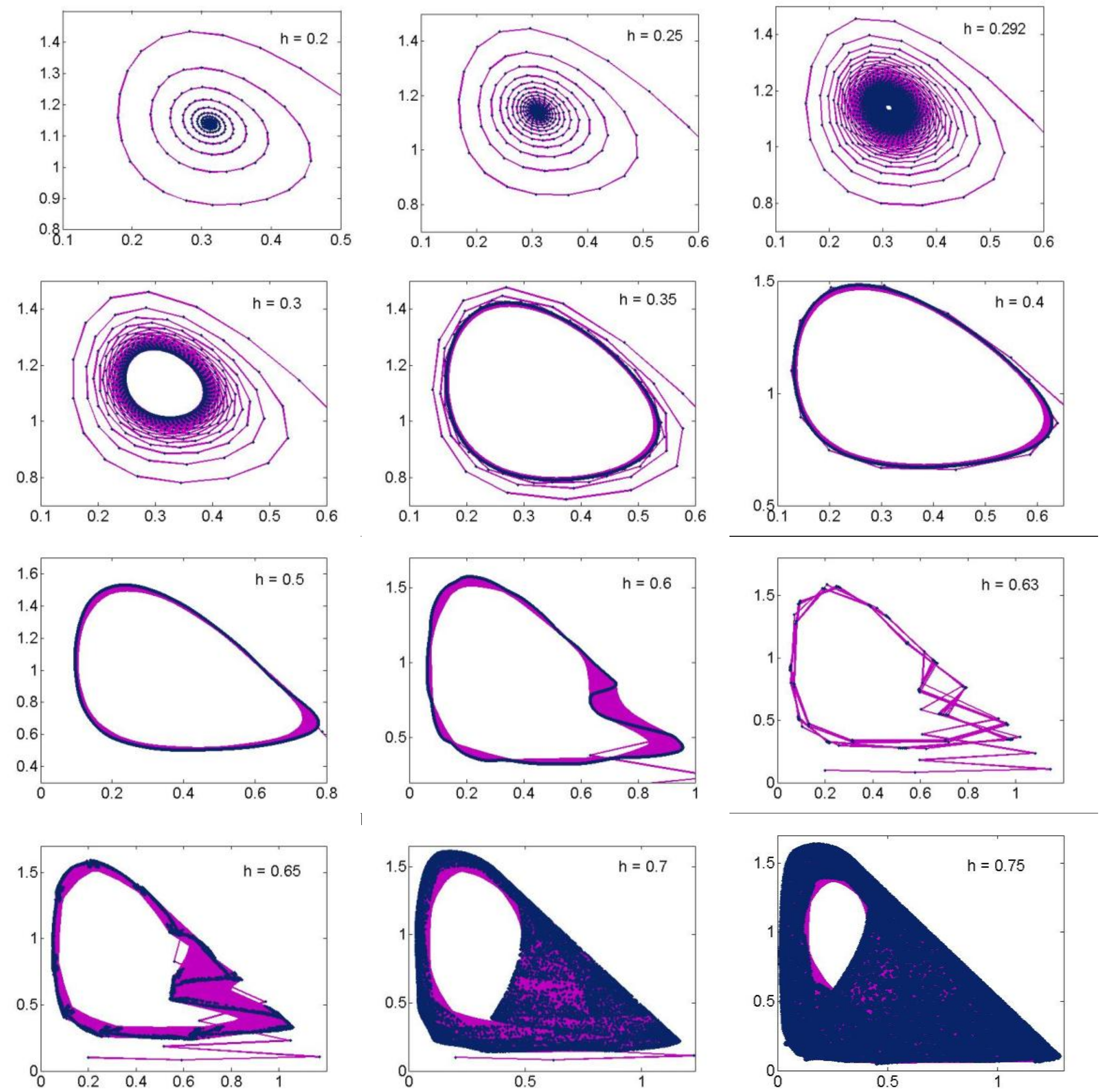


Figure 7. Phase planes of system (2) for various *h* with the parameter values $\delta = 0.9$, $\eta = 4.1$, $\beta = 3$, $\alpha = 3.5$, $\sigma = 0.9$ **and initial point** $(0.2, 0.1)$.

In Fig. 7, model (2) has various phase planes for different values of h. The solution curve settles down as stability after spirals inwards for $h < 0.292$. For h from 0.3 to 0.4, the curve settles down as a limit cycle from spirally inwards and indicating instability. For $h = 0.5 - 0.6$, the solution curve spirals inwards but does not converge to a point. Finally for $h = 0.65 - 0.75$, the circle disappears, and chaotic attractors appear. From the bifurcation and phase plane diagrams, we can conclude our justifications.

6. CONCLUSIONS

Depending on the characteristics of herbivores and their plant hosts, the interaction of these two species yield different results. For many years, plants have sought to create a versatile defense mechanism against herbivores to deter attackers and undermine the health of pests. Herbivores, on the other hand, try to overcome plant defenses with various strategies to provide the necessary nutrients. A harmonious interaction occurs when herbivore is undetectable by the plant or when herbivore develops its ability to defeat the plant's defenses. Thus, the herbivore can develop and reproduce successfully. Otherwise, there will be no interaction between the host plant and herbivores.

The behavior of plant-herbivore species was analyzed with the model discussed in this study. The existence of Neimark-Sacker bifurcation at the coexistence fixed point was investigated. A chaos control strategy was presented to control the chaos.

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