ORIGINAL PAPER

NUMERICAL SOLUTIONS OF DIFFERENTIAL-ALGEBRAIC EQUATIONS (DAEs) USING LEGENDRE POLYNOMIALS APPROXIMATIONS

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Abstract. *In this paper, the Legendre polynomials approximations are studied for the numerical solutions of differential-algebraic equations (DAEs). An algorithm of the method constructed using Legendre polynomials approximations is developed to solve DAE systems. Two test problems are answered to illustrate the Legendre polynomials approximations, and then the obtained solutions are confronted with the exact solutions of the problems. As a first step, the power series of an allowed equation system is determined, in second step it is converted into Legendre polynomials approximations structure which allows an arbitrary order. Furthermore, to get numerical solution of DAEs with Legendre polynomials approximations, a Maple algorithm is implemented. Graphs of obtained solutions are sketched, and are created tables to compare the exact solutions by using Maple programming.*

Keywords: *Legendre polynomials approximations; power series; differential-algebraic equations (DAEs).*

1. INTRODUCTION

Much consideration has as of late been given to the advancement of mathematical techniques for DAEs. The aim of this work is to take into account the numerical solutions of DAEs by using Legendre polynomials approximations.

DAEs can be considered to portray the improvement of many concept and critical frameworks. DAEs are a bunch of differential equations with supplementary algebraic restrictions in the structure:

$$G\left(t, w(t), w'(t)\right) = 0. \tag{1}$$

Eq. (1) is named a implicit DAE system. In this work, semi-explicit systems are considered, structure of algebraically restrictions differential equations

$$\begin{aligned} w'(t) &= G\left(t, w(t), v(t)\right) \\ 0 &= h\left(t, w(t), v(t)\right) \end{aligned} \tag{2}$$

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where v shows the algebraic variables, w means the differential variables [1, 2]. The mathematical strategies contrived for DAEs consider the design of the hidden DAE. Power series of the allowed DAEs framework will be figured out then, at that point change it into Legendre polynomials guess structure, which provide a subjective request for settling DAE mathematically.

2. POWER SERIES SOLUTION OF DAES

A DAE has the form

$$G(t, w, w') = 0 \tag{3}$$

with initial conditions

$$w(t_0) = w_0, w'(t_0) = w_1 \tag{4}$$

where $G \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$ are both vector capacities for which we expected adequate differentiability and the underlying conditions to be predictable, i.e.

$$G(t_0, w_0, w_0') = 0 \tag{5}$$

Suppose that solutions of Eq. (3) are

$$w = w_0 + w_1 t + e t^2 \tag{6}$$

where e is a vector function, has the same size with w_0 and w_1. Putting Eq. (6) into Eq. (3), convert the elementary functions in Eq. (3) into series in $t = 0$, disregard higher order term, linear equation of e is obtained

$$Be = A \tag{7}$$

where *B, A* are constant matrices. Figure out Eq. (7); the coefficients of t^2 in Eq. (6) can be found out. Iterating above procedure for higher order expressions, the arbitrary order power series of the solutions is obtained [3-6] for Eq. (3). The Power series given by above method can be converted into Legendre polynomials approximations, numerical solution of DAE in Eq. (3) is had.

State another type of power series in the structure

$$g(t) = g_0 + g_1 t + g_2 t^2 + \cdots + (g_n + q_1 e_1 + \cdots + q_n e_n) t^m \tag{8}$$

where n is the size of vector e, and $e_1, e_2, \ldots, e_n$ are basis whose vector e, and $q_1, q_2, \ldots, q_n$ are constants. In Eq. (6) w is a vector with n elements. In Eq. (8) every element can be indicated by the power series

$$w_j = w_{j,0} + w_{j,1} t + w_{j,2} t^2 + \cdots + e_j t^m \tag{9}$$

where w_j is the j the element of w. Putting Eq. (9) into Eq. (3), the equation is got as below:

$$g_j = \left(g_{j,n} + q_{j,1}e_1 + \cdots + q_{j,m}e_m\right)t^{n-i} + O\left(t^{n-i+1}\right) \tag{10}$$

where g_j is the j the element of $g\left(w, w', t\right)$ in Eq. (3) and i is 0 if $g\left(w, w', t\right)$ have w', 1 if does not have. Using Eqs. (10) and (7), the linear equation in Eq. (7) is determined as follows:

$$B_{j,i} = P_{j,i}, \tag{11}$$

$$A_j = -g_{j,n}. \tag{12}$$

Handling the linear equation, $e_j\left(j = 1, \ldots, n\right)$ is obtained. Putting e_j into Eq. (9), $w_j\left(j = 1, \ldots, n\right)$ with polynomials of degree m is got. From Eqs. (11) and (12), iterating this procedure, in Eq. (3) the arbitrary order power series of the solution is had for DAEs. If the above procedure is repeated, numerical solution of DAEs in the Eq. (3) is obtained [3-6].

Remark 1. Consider the following initial value problem

$$G\left(t_0, w_0, w_0'\right) = 0, w\left(t_0\right) = w_0. \tag{13}$$

The solution of this problem can be supposed that

$$w = w_0 + et \tag{14}$$

and handling above procedure, the solutions of Eq. (13) are got.

3. LEGENDRE POLYNOMIALS APPROXIMATIONS

The Legendre polynomials have numerous uncommon properties and they are generally utilized in mathematical investigation and applied science. Define the Legendre polynomials as follows.

$$\begin{aligned} R_0\left(t\right) &= 1 \\ R_n\left(t\right) &= \frac{1}{n!2^n}\frac{d^n}{dt^n}\left[\left(t^2 - 1\right)^n\right] \end{aligned} \tag{15}$$

For example,

$$\begin{aligned} R_0\left(t\right) &= 1 \\ R_1\left(t\right) &= t \\ R_2\left(t\right) &= \frac{1}{2}\left(3t^2 - 1\right) \end{aligned} \tag{16}$$

$$R_3(t)=\frac{1}{2}\left(5t^3-3t\right)$$
$$R_4(t)=\frac{1}{8}\left(35t^4-30t^2+3\right)$$
$$\vdots$$

(Fig. 1). For all n, the general expressions are

$$R_k(t)=\frac{1}{2^k}\sum_{n=0}^{k}\binom{k}{n}(t-1)^{k-n}(t+1)^n \tag{17}$$

[7, 8]. The inverse relations are as follows:

$$1=R_0(t)$$
$$t=R_1(t)$$
$$t^2=\frac{1}{3}\left[R_0(t)+2R_2(t)\right]$$
$$t^3=\frac{1}{5}\left[3R_1(t)+2R_3(t)\right]$$
$$t^4=\frac{1}{35}\left[7R_0(t)+20R_2(t)+8R_4(t)\right]$$
$$\vdots$$

The overall relations for the powers of t as far as polynomials $R_k(t)$ are [7, 8]

$$t_k=\sum_{n=k,k-2,\dots}\frac{(2n+1)n!}{2^{(k-n)/2}\left(\frac{1}{2}(k-n)!\right)(n+k+1)!}R_n(t).$$

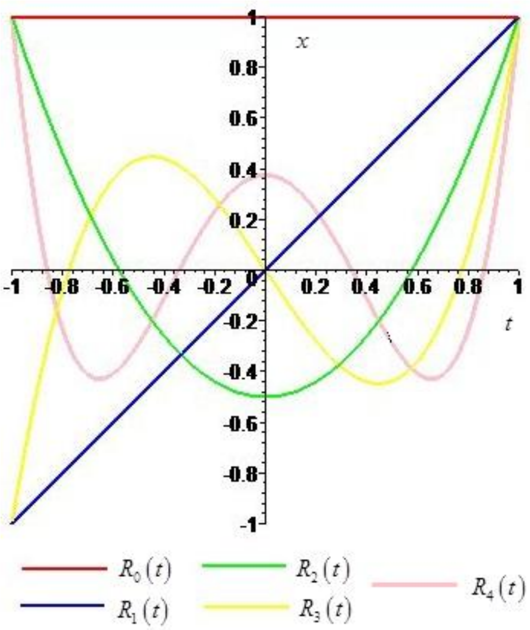

Figure 1. Legendre polynomials with degrees 0, 1, 2, 3, 4.

3.1. LEGENDRE POLYNOMIALS PROPERTIES

- Degree and normalization:

$$degR_k = k, R_k(1) = 1, k \geq 0 \tag{18}$$

- Triple recurrence relation: For $k \geq 1$,

$$R_{k+1}(t) = \frac{2k+1}{k+1} t R_k(t) - \frac{k}{k+1} R_{k-1}(t) \tag{19}$$

- Size and orthogonality:

$$\langle R_i, R_j \rangle = \begin{cases} \dfrac{2}{2j+1}, & i = j \\ 0, & i \neq j \end{cases} \tag{20}$$

- Roots: All roots of $R_k(t)$ are found in $[-1,1]$ [7, 8].

3.2. MAPLE ALGORITHM

When the test problem is solved, after $w_n(t)$ is obtained, we'll use the following algorithm to find $w_n^*(t)$.

```
> legpol:=proc(g,n) local d1,d2,sd,c,y,i:
> for k from 1 to 9 do
> denk(k):=p[k]=R(k,t):
> od:
> d1:=seq(denk(l),l=1..9):
> d2:=seq(t**j,j=1..9):
> sd:=solve({d1},{d2}):
> c:=evalf(subs(sd,w[n](t))):
> v:=subs(d1,c):
> print(w[n](t)**(`*`)=y):
> end:
```

4. NUMERICAL EXPERIMENTS

In this part, two DAEs are handled by Legendre Polynomials Approximations.

Example 1.

Consider the DAEs

$$
\begin{aligned}
w_2(t) &= -2e^{-t} - w_1'(t) \\
w_1(t) &= w_2'(t) \\
e^{-t} &= w_3(t)
\end{aligned}
\tag{21}
$$

with initial conditions

$$
w_1(1) = \frac{1}{e}, \quad w_2(1) = -\frac{1}{e}, \quad w_3(1) = \frac{1}{e} \tag{22}
$$

The system of equations (21) has exact solution

$$
w_1(t) = \frac{1}{e^t}, \quad w_2(t) = -\frac{1}{e^t}, \quad w_3(t) = \frac{1}{e^t} \tag{23}
$$

[9]. After the method is applied to Eq. (21) for power series solutions in section 2, if Legendre polynomials approximations is used in section 3, the following numerical solutions are obtained:

```
The Solutions of DAEs with Power Series and Legendre Polynomials
Approximations:
```

$$
\begin{aligned}
w_1(t) = {} & 0.3678794412 + 0.3678794412t + 0.9999999999t^2 - 0.06131324021t^3 \\
& - 0.513868434610^{-25} t^4 - 0.01666666667t^5 + 0.002777777778t^6 \\
& + 0.298721320910^{-23} t^7 - 0.785770011310^{-23} t^8 - 0.551146384610^{-5} t^9
\end{aligned}
$$

$$
\begin{aligned}
w_2(t) = {} & -0.3678794412 + 0.3678794412t + 0.1839397206t^2 + 0.3333333334t^3 \\
& - 0.01532831005t^4 - 0.102773687010^{-25} t^5 - 0.002777777778t^6 \\
& + 0.0003968253968t^7 + 0.373401651210^{-24} t^8 - 0.873077790710^{-24} t^9
\end{aligned}
$$

$$
\begin{aligned}
w_3(t) = {} & 0.3678794412 - 0.9999999999t + 0.5000000000t^2 - 0.1666666667t^3 \\
& + 0.04166666669t^4 - 0.008333333334t^5 + 0.0014t^6 \\
& - 0.0002t^7 + 0.000025t^8 - 0.275573192310^{-5} t^9
\end{aligned}
$$

$$
\begin{aligned}
w_1(t)^* = {} & 0.3678794411 + 0.3678794413t + 1.000000000t^2 - 0.06131324018t^3 \\
& - 0.846213858410^{-23} t^4 - 0.01666666666t^5 + 0.002777777778t^6 \\
& - 0.785770011410^{-23} - 0.551146384610^{-5} t^9
\end{aligned}
$$

$$
\begin{aligned}
w_2(t)^* = {} & -0.3678794412 + 0.3678794411t + 0.1839397206t^2 + 0.3333333336t^3 \\
& - 0.01532831005t^4 - 0.129420943110^{-23} t^5 - 0.002777777778t^6 \\
& + 0.0003968253968t^7 + 0.373401651210^{-24} t^8 - 0.873077790810^{-24} t^9
\end{aligned}
$$

$$
\begin{aligned}
w_3(t)^* = {} & 0.3678794412 - 0.9999999998t + 0.4999999999t^2 - 0.1666666668t^3 \\
& + 0.04166666668t^4 - 0.008333333334t^5 + 0.00139t^6 \\
& - 0.000198t^7 + 0.0000248t^8 - 0.275573192310^{-5} t^9
\end{aligned}
$$

where $w_3(t)$, $w_2(t)$, and $w_1(t)$ are the power series solutions of DAE, $w_3^*(t)$, $w_2^*(t)$, and $w_1^*(t)$ are the Legendre Polynomials Approximations of $w_3(t)$, $w_2(t)$, and $w_1(t)$.

Example 2.

We consider the DAE

$$\begin{aligned} w_1(t) &= tw_2'(t) - w_1'(t) + w_2(t) + tw_2(t) \\ \sin t &= w_2(t) \end{aligned} \tag{24}$$

with initial conditions

$$w_1'(0) = -1, \quad w_1(0) = 1, \quad w_2'(0) = 1, \quad w_2(0) = 0 \tag{25}$$

The system of equations (24) has exact solution

$$w_1(t) = (\sin t)t + \frac{1}{e^t}, \quad \sin t = w_2(t) \tag{26}$$

[9]. Again, the method is applied to Eq. (24) for power series solutions in section 2, then if Legendre polynomials approximations is used in section 3, the following numerical solutions are obtained:

The Solutions of DAEs with Power Series and Legendre Polynomials Approximations:

$$\begin{aligned} w_1(t) = {} & 1 - t + 1.500000000t^2 - 0.1667t^3 \\ & - 0.125t^4 - 0.0083t^5 + 0.00972t^6 \\ & - 0.0001984t^7 - 0.0001736t^8 - 0.27557319210^{-5}t^9 \end{aligned}$$

$$\begin{aligned} w_2(t) = {} & t - 0.167t^3 + 0.0083t^5 - 0.0001984t^7 \\ & + 0.275573192210^{-5}t^9 \end{aligned}$$

$$\begin{aligned} w_1(t)^* = {} & 1.000000001 - 1.000000001t + 1.500000001t^2 - 0.1666666668t^3 \\ & - 0.1250000000t^4 - 0.008333333334t^5 + 0.009722222222t^6 \\ & - 0.0001984126984t^7 - 0.0001736111111t^8 - 0.275573192210^{-5}t^9 \end{aligned}$$

$$\begin{aligned} w_2(t)^* = {} & 1.0000001t - 0.166666667t^3 + 0.00833334t^5 - 0.00019841279t^7 \\ & + 0.27557310^{-5}t^9 \end{aligned}$$

where $w_2(t)$, and $w_1(t)$ are the power series solutions of DAE, $w_2^*(t)$, and $w_1^*(t)$ are the Legendre Polynomials Approximations of $w_2(t)$, and $w_1(t)$.

Table 1 and table 2 indicate the values of exact solutions and the numerical solutions of $w_2(t)$, and $w_1(t)$. They are had by the following Maple commands:

```
> print(t,`  `,w[1](t),`  `,w[1](t)**(`*`),`  `,
abs(w[1](t)**(`*`)-w[1](t)));
> print(`--------------------------------------------------`);
> for t from 0 to 1 by 0.1 do
```

```
> print([t], [(sin(t))*t+1/exp(t)], [w[1](t)**(`*`)],
[abs(w[1](t)**(`*`)-(sin(t))*t+1/exp(t))]);
> end do;
```

$$t, \quad w_1(t), \quad , w_1(t)^*, \quad , \left|w_1(t)^* - w_1(t)\right|$$

$[0],[1],[1.000000001],\left[0.1\ \ 10^{-8}\right]$

$[0.1],[0.9148207597],[0.9148207606],\left[0.7105087454\,10^{-9}\right]$

$[0.2],[0.8584646193],[0.8584646201],\left[0.2760172809\,10^{-9}\right]$

$[0.3],[0.8294742827],[0.8294742834],\left[0.7582767886\,10^{-9}\right]$

$[0.4],[0.8260873829],[0.8260873834],\left[0.8293385110\,10^{-9}\right]$

$[0.5],[0.8462434290],[0.8462434267],\left[0.2744397610\,10^{-8}\right]$

$[0.6],[0.8875971201],[0.8875971027],\left[0.1768724257\,10^{-7}\right]$

$[0.7],[0.9475376848],[0.9475376011],\left[0.844986830\,10^{-7}\right]$

$[0.8],[1.023213837],[1.023213516],\left[0.3213829175\,10^{-6}\right]$

$[0.9],[1.111563878],[1.111563836],\left[0.1041296199\,10^{-5}\right]$

$[1.0],[1.209350426],[1.209347443],\left[0.2982453422\,10^{-5}\right]$

```
> print(t,`   `,w[2](t),`  `,w[2](t)**(`*`),`
`,abs(w[2](t)**(`*`)-w[2](t)));
> print(`---------------------------------------------------`);
> for t from 0 to 1 by 0.1 do
> print([t], [sin(t)],  [w[2](t)**(`*`)], [abs(w[2](t)**(`*`)-
sin(t))]);
> end do;
```

$$t, \quad w_2(t), \quad , w_2(t)^*, \quad , \left|w_2(t)^* - w_2(t)\right|$$

$[0],[0],[0],[0]$

$[0.1],[0.09983341665],[0.09983342664],\left[0.9996561477\,10^{-8}\right]$

$[0.2],[0.1986693308],[0.19866993509],\left[0.1999452722\,10^{-7}\right]$

$[0.3],[0.2955202067],[0.2955202367],\left[0.2996856386\,10^{-7}\right]$

$[0.4],[0.3894183423],[0.3894183823],\left[0.4005448299\,10^{-7}\right]$

$[0.5],[0.4794255386],[0.4794255888],\left[0.5017736316\,10^{-7}\right]$

$[0.6],[0.5646424734],[0.5646425339],\left[0.6053153120\,10^{-7}\right]$

$[0.7],[0.6442176872],[0.6442177588],\left[0.71531215\,410^{-7}\right]$

$[0.8],[0.7173560909],[0.7173561751],\left[0.841408796\,10^{-7}\right]$

$[0.9],[0.7833269096],[0.7833270111],\left[0.101540884\,10^{-6}\right]$

$[1.0],[0.8414709848],[0.8414711159],\left[0.131140000\,10^{-6}\right]$

The figures of exact solutions and Legendre Polynomials Approximations of $w_1(t)$ and $w_2(t)$ are sketched using the following.

```
> with(plots):
> q1:=plot(t*sin(t)+exp(-t), t=-4..4, color=red):
> q2:=plot(w[1](t)**(`*`), t=-4..4, color=navy):
> display(q1,q2);
```

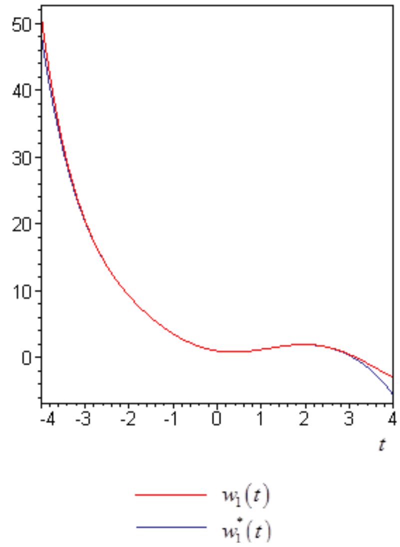

Figure 2. The graphs of $w_1(t)$ **and** $w_1^*(t)$**.**

```
> with(plots):
> s1:=plot(sin(t), t=-4..4, color=red):
> s2:=plot(w[2](t)**(`*`), t=-4..4, color=navy):
> display(s1,s2);
```

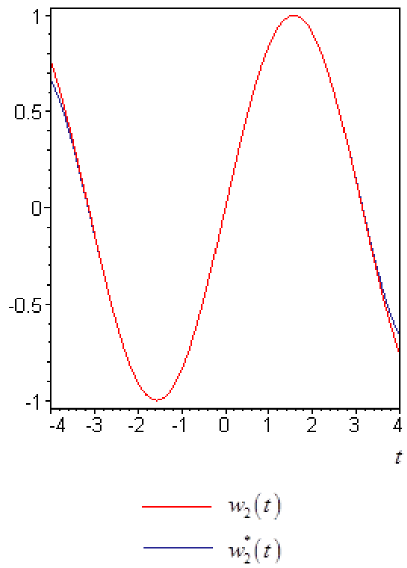

Figure 3. The graphs of $w_2(t)$ **and** $w_2^*(t)$**.**

5. CONCLUSION

In this paper, Legendre Polynomials Approximations has been recommended for solving DAEs. The calculations related with the model talked about above were performed by utilizing Maple 17. Results demonstrate the benefits of the technique.

REFERENCES

[1] Brenan, K. E., Campbell, S. L., Petzold, L. R., *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*, Nort-Holland, 1989.

[2] Ascher, U. M., Petzold, L. R., *Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations*, Society for Industrial and Applied Mathematics, Philadelphia, 1998.

[3] Celik, E., Karaduman, E., Bayram, M., *International Journal of Quantum Chemistry*, **89**, 447, 2002.

[4] Celik, E., Bayram, M., *Applied Mathematics and Computations*, **137**, 151, 2003.

[5] Celik, E, Bayram, M., *Journal of the Franklin Institute*, **342**, 1 2005.

[6] Celik, E., Yeloglu, T., *International Journal of Computer Mathematics*, **83**, 651, 2006.

[7] Attar, R.E., *Legendre Polynomials and Functions*, Create Space Independent Publishing Platform, 2009.

[8] Koepf, W., *Orthogonal Polynomials and Special Functions*, **1817**, 1, 2003.

[9] Wang, W., *Applied Mathematics and Computations*, **167**, 1350, 2005.