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THE PARALLEL TRAJECTORY RULED SURFACES

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Abstract. *The parallel curves are especially used in numerically controlled machining where a biaxial machine defines the shape of the cut made with a round cutting tool. In this study, trajectory ruled surfaces are investigated whose base curve is the parallel curve of a prescribed curve. Some results related with their developability and minimality are obtained. In addition, the conditions are investigated for the base curve to be a special curve. Parallel trajectory ruled surfaces are illustrated with examples.*
Keywords: *Ruled Surfaces; Trajectory Ruled Surfaces; Curves and Parallel Curves.*

1. INTRODUCTION

Parallel curves have an important place in the theory of curves. A parallel curve is defined as a curve with a fixed distance from a given curve. This parallelism does not mean moving parallel to a place. These curves are curves that are completely different from the original curve. Some of the studies on parallel curves are involute evolute offsets, Bertrand offsets, Mannheim offsets curves, [1-5]. Also recently, in [6], the author defined parallel curve using binormal vector field of given a curve and gave the relation between the Frenet-Serret frame of the parallel curve and the original curve. A ruled surface which is introduced by G. Monge is a surface that is constructed by the continuous movement of a line or a director along a base curve. The usage area of these surfaces is quite wide. For example, in CAD, vehicle design, architecture etc. We can see research on these surfaces in related studies, [7-11]. In this paper, we defined the trajectory ruled surfaces with parallel curve which is defined using binormal vector field of given a curve. Some results were obtained for these surfaces. Examples are presented for all surfaces.

2. MATERIALS AND METHODS

Let α (t) be a space curve with a non-vanishing second derivative. The Ferenet- Serret frame is defined as follows

$$T=\frac{\alpha'}{\|\alpha'\|},\qquad B=\frac{\alpha'\wedge\alpha''}{\|\alpha'\wedge\alpha''\|},\qquad N=B\wedge T \tag{1}$$

The curvature κ and the torsion τ of the curve $\alpha(t)$ are given by, [12]

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$$\kappa = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3} \quad , \quad \tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge \alpha''\|^2} \tag{2}$$

The well-known Frenet formulae are given by

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = v \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \quad , \tag{3}$$

where

$$v = \|\alpha'(t)\| \, . \tag{4}$$

$\{T, b, n\}$ is called the Darboux frame of the surface along the curve . In this frame T is the unit tangent vector field of the curve, n is the unit normal vector field of the surface and b is a unit vector field given by $b = n\text{x}T$. The relations between Frenet frame and Darboux frame can be given as follow

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} T \\ b \\ n \end{bmatrix}. \tag{5}$$

Derivative formulae of the Darboux frame is

$$\frac{d}{ds} \begin{bmatrix} T \\ b \\ n \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ -k_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ b \\ n \end{bmatrix} \tag{6}$$

where k_g is the geodesic curvature , k_n is the normal curvature and τ_g is the geodesic torsion of α(s).

Definition 1. The parallel curve $\bar{\alpha}(\bar{s})$ of a unit speed curve α (s) is defined as follow

$$\bar{\alpha} = \alpha(\text{s}) + \text{r}B(\text{s}), \tag{7}$$

where r ≠ 0 is a real constant, $s = s(\bar{s})$ and $\bar{s}$ are the arc length of α and $\bar{\alpha}$, respectively. B is the binormal vector to the curve α (s), [6].

Lemma 2. Let $\bar{\alpha}(\bar{s})$ be a paralel curve to a unit speed curve α (s). Then the associated Frenet- Serret frame $\{\bar{T}, \bar{N}, \bar{B}\}$ to $\bar{\alpha}$ in terms of the frame $\{T, N, B\}$ of the original curve α is given as follow, [6].

$$\begin{aligned} \bar{T}(\bar{s}) &= W\text{T} - \text{r}\tau W\text{N} \\ \bar{N}(\bar{s}) &= \left(\frac{WW' + \text{r}\text{K}\tau W^2}{\Omega}\right)\text{T} + \left(\frac{W^2\text{K} - W(r\tau W)'}{\Omega}\right)\text{N} - \frac{r\tau^2 W^2}{\Omega}\text{B} \\ \bar{B}(\bar{s}) &= \left(\frac{r\tau^3 W^3}{\Omega}\right)\text{T} + \left(\frac{r\tau^2 W^3}{\Omega}\right)\text{N} \\ &\quad + \left(\frac{W^3\text{K} - (r\tau W)' W^2 + r\tau W^2 W' + r^2 \text{K}\tau^2 W^3}{\Omega}\right)\text{B} \end{aligned} \tag{8}$$

where

$$\frac{ds}{d\bar{s}} = \frac{1}{\sqrt{1+r^2\tau^2}} = W, \tag{9}$$

and

$$\Omega = \sqrt{\begin{matrix}(WW' + \mathrm{r}K\tau W^2)^2 \\ +(W^2K - (r\tau W)'W)^2. \\ + (\mathrm{r}\tau^2 W^2)^2\end{matrix}}$$

Definition 3. The curve α (s) lying on a regular surface is an asymptotic line if its normal curvature k_n vanishes, [12].

The trace of $\vec{P}$ oriented line an along α (s) is generally a ruled surface. A parametric equation of this ruled surface is given by

$$\psi(s,v) = \alpha(s) + v\vec{P}(s), \tag{10}$$

where $\vec{P}$ (s) is the director vector and α (s) is the directix.

The Gaussian curvature and the mean curvature of the surface $\psi(s,v)$ are given by

$$K = (s,v) = \frac{LN - M^2}{EG - F^2}, \quad H = (s,v) = \frac{EN + GL - 2FM}{2(EG - F^2)}, \tag{11}$$

where the elements of fundamental forms on the surface ψ are defined by, [12]

$$E = \|\psi_v\|^2, \quad F = \langle \psi_s, \psi_v \rangle, \quad G = \|\psi_v\|^2 \tag{12}$$

and

$$L = \langle \psi_{ss}, \psi_s \times \psi_v \rangle, \quad N = \langle \psi_{tt}, \psi_s \times \psi_v \rangle, \quad M = \langle \psi_{st}, \psi_s \times \psi_v \rangle \tag{13}$$

respectively.

3. RESULTS

Case 1.

Let $\bar{\alpha}(\bar{s})$ be the parallel curve of a regular unit curve α (s) . The parametric representation of the ruled surface formed by the tangent of the curve $\bar{\alpha}(\bar{s})$ are follow as,

$$\bar{\psi}(\bar{s},v) = \bar{\alpha}(\bar{s}) + v\,\overline{T(\bar{s})}\,. \tag{14}$$

If definition 1 and Eq. (8) are used in Eq.(14), then the parametric representation of the ruled surfaces in Eq.(14) according to the s parameter are

$$\begin{aligned}\bar{\psi}(\bar{s},v) = {} & \alpha(s) + \mathrm{r}B(s) \\ & + vWT - \mathrm{r}v\tau W N.\end{aligned} \tag{15}$$

We take the first derivative of both sides of the ruled surface $\bar{\psi}\ (\bar{s}, v)$ in (15) with respect to s

$$\frac{\partial \bar{\psi}}{\partial \bar{s}} \cdot \frac{d\bar{s}}{ds} = (1+vW' + \mathrm{r}vW\,\kappa\,)\mathrm{T} + (-\mathrm{r}\tau + vW\,\kappa + \mathrm{r}vW\tau' - \mathrm{r}vW'\tau\,)\mathrm{N} + (-\,\mathrm{r}vW\tau^2\,)\,\mathrm{B} \qquad (16)$$

Using Eq. (9) in Eq. (16), we have

$$\bar{\psi}_{\bar{s}} = \frac{\partial \bar{\psi}}{\partial \bar{s}} = (W + vW\,W' + \mathrm{r}vW^2\tau\,\kappa\,)\mathrm{T} + (-\mathrm{r}\,W\,\tau + vW^2\,\kappa\; + \mathrm{r}vW^2\tau' - \mathrm{r}v\,W\,W'\tau\,)\mathrm{N} + (-\,\mathrm{r}vW^2\tau^2\,)\,\mathrm{B}. \qquad (17)$$

The first and the second derivatives of both sides of the ruled surfaces in Eq. (15) with respect to v are

$$\bar{\psi}_v = W\mathrm{T} - \mathrm{r}\,W\,\tau\mathrm{N}\;, \qquad (18)$$

and

$$\bar{\psi}_{vv} = 0\;. \qquad (19)$$

The second derivative of both sides of Eq. (15) with respect to s is

$$\frac{\partial^2 \bar{\psi}}{\partial \bar{s}^2}\left(\frac{d\bar{s}}{ds}\right)^2 + \frac{d\bar{\psi}}{d\bar{s}}\,\frac{d^2\bar{s}}{ds^2} = [\,\mathrm{r}\tau\,\kappa - vW\kappa^2 + v\,W'' + \mathrm{r}\,v\,\tau\kappa'\,W + 2\mathrm{r}vW'\tau\,\kappa + 2\mathrm{r}W\tau'\,\kappa\,v]\,\mathrm{T} + [\kappa - \mathrm{r}\tau' + vW'\,\kappa + \mathrm{r}\,v\,\mathrm{W}\tau\kappa^2 + \mathrm{r}\,v\,\mathrm{W}\tau^3 - \mathrm{r}vW\tau'' - \mathrm{r}vW''\tau + v\,\kappa W' + vW\kappa' - 2\mathrm{r}v\tau'\,W'\,]\,\mathrm{N} + [\,-\mathrm{r}\tau^2 + v\,\tau\,\kappa W - 2\mathrm{r}vW'\tau^2 - 3\mathrm{r}\,v\,\tau\tau' W\,]\,\mathrm{B}\,.$$

Using Eq. (17), we can write

$$\psi_{\bar{s}\bar{s}} = \frac{\partial^2 \bar{\psi}}{\partial \bar{s}^2} = (rW^2\,\tau\,\kappa - vW^3\kappa^2 + vW^2W'' + 2\,\mathrm{r}vW^3\tau'\,\kappa + \mathrm{r}vW^3\tau\kappa' + 3\,\mathrm{r}v\tau\kappa W^2W' + WW' + v\,W(W')^2\,)\,\mathrm{T} + (W^2\,\kappa - \mathrm{r}W^2\tau' + 3\,vW^2W'\,\kappa + vW^3\kappa' + \mathrm{r}vW^3\tau\kappa^2 + \mathrm{r}vW^3\tau^3 - 3\mathrm{r}vW^2W'\tau' - \mathrm{r}vW^3\tau'' - \mathrm{r}WW'\tau - \mathrm{r}v\tau W^2W'' - \mathrm{r}\,v\,\tau W(W')^2)\mathrm{N} + (-rW^2\tau^2 + vW^3\,\kappa\,\tau - 3\mathrm{r}vW^3\tau\tau' - 3\mathrm{r}vW^2W'\tau^2\,)\mathrm{B}\,. \qquad (20)$$

If derivative of both sides of Eq. (17) with respect to v is, then we have

$$\bar{\psi}_{\overline{sv}} = (W\,W' + rW^2\tau\,\kappa\,)\mathrm{T} + (W^2\,\kappa - rW^2\tau' - \mathrm{r}\,W\,W'\tau\,) + (-rW^2\tau^2\,)\mathrm{B}. \qquad (21)$$

The elements of fundamental forms of the ruled surfaces in (15) are as follows

$$E = \langle \bar{\psi}_{\bar{s}}, \bar{\psi}_{\bar{s}} \rangle = (\,W + v\,W\,W' + \mathrm{r}v\tau\,\kappa W^2\,)^2 + (-\mathrm{r}\,\tau\,W + v\kappa W^2\; -+\,\mathrm{r}v\tau' W^2 - \mathrm{r}v\,\tau W\,W'\,)^2 + (\,\mathrm{r}v\tau^2\,W^2)^2$$

$$F = W^2 + vW^2W' + r\tau^2W^2 + r^2v\,\tau\tau'W^3 + r^2v\tau^2W^2W',$$
$$G = W^2\,\tau^2 + r^2v^2\tau^2 \quad , \tag{22}$$

and $\quad M = 0, \quad N = 0,$

$$L = v^2\begin{bmatrix} -2r^3\kappa\,\tau^3\tau'W^6 - r^3\kappa\,'\tau^4W^6 - 3r^3\kappa\,\tau^4W^5\,W' \\ -r\kappa\,'\tau^2W^6 - r^2\tau^5W^6 + r^2\tau^2W^6\tau'' - \tau\kappa\,^2W^6 \\ +2r\kappa\,\tau\,\tau'\,W^6 + 3r^2\tau^2\tau'W^5\,W' + 3r^2\tau(\,\tau'\,)^2\,W^6 \end{bmatrix} \tag{23}$$
$$+v\begin{bmatrix} -r^3W^5\,\tau^4\kappa\, + 2r^2W^5\tau^2\tau' + 3\,r^2W^5\,W' \\ -r^2W^4\,W'\tau^3 - r^2\tau^3\,W^6\kappa\,^2 \\ +3r^3\tau'\tau^3\,\kappa\,W^6 + 3r^3\tau^4\kappa\,W'W^5 \end{bmatrix} + [r^3\kappa\,\tau^4W^5].$$

Remark 4. Since the ruled surface formed by the tangent vector on the curve is developable, Gaussian curvature of the ruled surface $\bar{\psi}$ is zero.

Corollary 5. The mean curvature of the surface $\bar{\psi}$ along the parallel curve is

$$H(s,0) = r^3\,\mathrm{K}\tau^4W^7.$$

The necessary and sufficient condition for the ruled surface to be minimal along the parallel curve, the given curve α (s) must be planar.

Case 2.

The parametric representation of the ruled surfaces formed by the normal $\bar{N}$ of the curve $\bar{\alpha}\,(\bar{s})$ are as follows,

$$\bar{\psi}_{\,\bar{N}}(\bar{s},\mathrm{v}) = \bar{\alpha}\,(\bar{s}) + \mathrm{v}\,\bar{N}(\mathrm{s})\,. \tag{24}$$

where $\bar{\alpha}\,(\bar{s})$ are the parallel curves of a regular unit speed curve $\alpha(s)$.

Using definition 1 and Eqs. (8), (24), the parametric representation of the ruled surfaces in Eq. (24) according to the s parameter are

$$\bar{\psi}_{\,\bar{N}}(\bar{s},\mathrm{v}) = \;= \alpha(s) + rB(s) \;\; + v(f_1T + f_2N + f_3B)\,, \tag{25}$$

where

$$f_1 = \frac{WW' + r\kappa\tau W^2}{\Omega} \quad , \quad f_2 = \frac{W^2\kappa - W\,(r\tau W)'}{\Omega}\,, \quad f_3 = -\frac{r\tau^2W^2}{\Omega}\,. \tag{26}$$

The first derivative of both sides of the ruled surface $\bar{\psi}\,(\bar{s},\mathrm{v})$ in Eq. (25) with respect to s is

$$\frac{\partial\bar{\psi}}{\partial\bar{s}}\,.\frac{d\bar{s}}{ds} = (1 + vf_1{}' - vf_2\kappa\,)T$$
$$+ (-r\tau + vf_1\kappa + vf_2{}' + \; vf_3\tau\,)N + (vf_2\tau + vf_3{}')B \tag{27}$$

Using Eq. (9) and Eq. (26), we have

$$\bar{\psi}_{\bar{s}} = (\,W + vf_1{}'W - vf_2\kappa\,W)T$$

$$+ (-r\tau W + vf_1\kappa W + vf_2{}'W + vf_3\tau W) \mathrm{N} + (vf_2\tau W + vf_3{}'W)\mathrm{B}. \quad (28)$$

The first and the second derivatives of both sides of the ruled surfaces in Eq. (25) with respect to v are

$$\bar{\psi}_v = f_1T + f_2N + f_3B \quad (29)$$

and

$$\bar{\psi}_{vv} = 0. \quad (30)$$

The second derivative of both sides of Eq. (27) with respect to s is

$$\frac{\partial^2\psi}{\partial\bar{s}^2}\left(\frac{d\bar{s}}{ds}\right)^2 + \frac{d\psi}{d\bar{s}}\frac{d^2\bar{s}}{ds^2} = (r\tau\kappa + vf_1{}'' - vf_1\kappa^2 - 2vf_2{}'\kappa - vf_2\kappa' + vf_3\tau\kappa)T + (\kappa - r\tau' + 2vf_1{}'\kappa + vf_1\kappa' + vf_2{}'' - vf_2\mathrm{K}^2 - vf_2\tau^2 - 2vf_3{}'\tau - vf_3\tau') N + (-r\tau^2 + vf_1\tau\kappa + 2vf_2{}'\tau + vf_2\tau' + vf_3{}'' - vf_3\tau^2)B \quad (31)$$

Using Eqs. (27), (9) ,we give

$$\psi_{\bar{s}\bar{s}} = (rW^2\tau\mathrm{K} + vW^2f_1{}'' - vW^2f_1\kappa^2 - 2v\kappa W^2f_2{}' - vW^2f_2\kappa' + vW^2f_3\tau\kappa + WW'+vf1'WW'- vWW'\tau\kappa\, T+ (W^2\kappa - rW^2\tau' + 2vW^2f_1{}'\kappa + vW^2f_1\kappa' + vW^2f_2{}'' - vW^2f_2\kappa^2 - vW^2f_2\tau^2 - 2vW2f3'\tau - rWW'\tau - vf3\tau'W2 + vf1\kappa WW' + vf2'\kappa WW' - vf3WW'\tau\; N \quad (32)$$
$$+ (-rW^2\tau^2 + vW^2f_1\tau\kappa + 2vW^2f_2{}'\tau + vW^2f_2\tau' + vW^2f_3{}'' - vW^2f_3\tau^2 + v\tau f2WW' + vf3'WW'\; B.$$

Derivative of both sides of Eq. (28) with respect to v is

$$\bar{\psi}_{\bar{s}v} = (Wf_1{}' - Wf_2\kappa)T + (Wf_1\kappa + Wf_2{}' - Wf_3\tau)N + (Wf_2\tau + Wf_3{}')B \;. \quad (33)$$

The first and second elements of fundamental forms of the ruled surfaces $\bar{\psi}_{\bar{N}}$ are

$$\bar{E} = (W + vWf_1{}' - vWf_2\kappa)^2 + (-rW\tau + vWf_1\kappa + vWf_2{}' + vWf_3\tau)^2 + (vWf_2\tau + vWf_3{}')^2$$
$$\bar{F} = Wf_1 + vWf_1f_1{}' + vWf_2f_2{}' + vWf_3f_3{}' , \quad (34)$$
$$\bar{G} = f_1{}^2 + f_2{}^2 + f_3{}^2 ,$$

and

$$\bar{M}(s,0) = (Wf_1{}' - Wf_2\mathrm{K})(-rWf_3\tau) - Wf_3(W\mathrm{K}f_1 + Wf_2' - Wf_3\tau) + (Wf_2\tau + Wf_3{}')(Wf_2 + rf_1W\tau)$$
$$\bar{N}(s,0) = 0 , \quad (35)$$
$$\bar{L}(s,0) = -r^2W^3f_3\tau^2\kappa - W^3f_3\kappa + rW^3f_3\tau' - rW^3f_2\tau^2 - r^2W^3f_1\tau^3 - 2r\tau W^2f_3W'$$

Lemma 6. The ruled surfaces $\bar{\psi}_{\bar{N}}$ is developable or minimal along parallel curves $\bar{\alpha}$ if and only if

$$f_3 = -\frac{r\tau^2W^2}{\Omega} = 0 \quad \text{or} \quad \tau = 0 .$$

Corollary 7. The ruled surface is developable or minimal along the parallel curve if and only if the given curve α (s) is planar.

Using Eq. (8) , we have

$$\bar{\alpha}_{\bar{s}\bar{s}} = (rW^2\tau\kappa + WW')T + (\kappa W^2 - rW^2\tau' - rWW'\tau)N + (-rW^2\tau^2)B \quad .$$

The normal of ruled surface along the curve in Eq.(24) is

$$\bar{N}(s,0) = (-rWf_3\tau, -Wf_3, Wf_2 + rWf_1\tau).$$
$$\langle \bar{\alpha}_{\bar{s}\bar{s}}, \bar{N} \rangle = k_n = -W^3 r\tau^2 \left[(rf_3\kappa + f_2 + rf_1\tau) - \frac{W^2}{\Omega}(r\tau' - \kappa) \right]$$

Using $f_3 = -\frac{r\tau^2 W^2}{\Omega}$, we can give the following corollory.

Corollary 8. The base curves are asymptotic curves on the ruled surface in Eq.(24) if and only if $\kappa = r\tau'$ or the given curve α (s) is planar.

Case 3.

The ruled surfaces formed by the binormal of the curve $\bar{\alpha}(\bar{s})$ are

$$\bar{\psi}(\bar{s},\mathrm{v}) = \bar{\alpha}(\bar{s}) + vB(\bar{s}) \,. \tag{36}$$

If definition 1 and Eq. (8) are used in Eq. (36), then the parametric representation according to the s parameter is

$$\bar{\psi}(\bar{s},\mathrm{v}) = \alpha(s) + rB(s) + v(g_1T + g_2N + g_3B) \tag{37}$$

where,

$$g_1 = \left(\frac{r^2\tau^3W^3}{\Omega}\right), \quad g_2 = \left(\frac{r\tau^2W^3}{\Omega}\right), \quad g_3 = \frac{W^3\kappa - (r\tau W)'W^2 + r\tau W^2W' + r^2\kappa\tau^2W^3}{\Omega} \,.$$

The first derivative of both sides of the ruled surface $\bar{\psi}(\bar{s},\mathrm{v})$ in Eq. (37) with respect to s

$$\frac{\partial\bar{\psi}}{\partial\bar{s}}\cdot\frac{d\bar{s}}{ds} = (1 + vg_1' - vg_2\kappa)T + (-r\tau + vg_1\kappa + vg_2' + vg_3\tau)N + (vg_2\tau + vg_3')B \tag{38}$$

Using Eqs. (9) , (38), we give

$$\bar{\psi}_{\bar{s}} = (W + vWg_1' - vWg_2\kappa)T + (-\mathrm{r}W\tau + vWg_1\kappa + vWg_2' - vWg_3\tau)N + (vWg_2\tau + vWg_3')B \quad . \tag{39}$$

The first and the second derivatives of both sides of the ruled surfaces in Eq. (37) with respect to v are

$$\bar{\psi}_v = g_1T + g_2N + g_3B \tag{40}$$

and

$$\bar{\psi}_{vv} = 0 \ . \tag{41}$$

The second derivative of both sides of Eq. (38) with respect to s is

$$\frac{\partial^2\psi}{\partial\bar{s}^2}\left(\frac{d\bar{s}}{ds}\right)^2 + \frac{d\psi}{d\bar{s}}\frac{d^2\bar{s}}{ds^2} = (r\tau\kappa + vg_1'' - vg_1\kappa^2 - 2vg_2'\kappa - vg_2\kappa + vg_3\tau\kappa)T(\kappa - r\tau' + 2vg1'\kappa + vg1\kappa' + vg2'' - vg2\kappa2 - vg2\tau2 - 2vg3'\tau - vg3\tau' \ N + -r\tau2 + vg1\tau\kappa + 2vg2'\tau + vg2\tau' + vg3'' - vg3\tau2 \ B \tag{42}$$

Differentiating Eq. (9) and using Eq. (39), we can write

$$\begin{aligned}\psi_{\bar{s}\bar{s}} = {} & (r\tau W^2\kappa + vW^2 g_1'' - vg_1W^2\kappa^2 - 2vg_2'W^2\kappa - vg_2W^2\kappa' + vg_3W^2\tau\kappa + WW' + vg_1'WW' - vWW'\tau\kappa)T \\ & + (W^2\kappa - rW^2\tau' + 2vg_1'W^2\kappa + vg_1W^2\kappa' + vg_2''W^2 - vW^2g_2\kappa^2 - vW^2g_2\tau^2 \\ & - 2vg_3'W^2\tau - rWW'\tau - vg_3\tau'W^2 + vg_1\kappa WW' + vg_2'\kappa WW' - vg_3\tau WW')N \\ & + \left(\begin{matrix}-rW^2\tau^2 + vW^2g_1\tau\kappa + 2vg_2'W^2\tau + vg_2W^2\tau' + vW^2g_3'' - vg_3W^2\tau^2 + v\tau g_2WW' + \\ vg_3'WW'\end{matrix}\right)B.\end{aligned} \tag{43}$$

Differentiating both sides of Eq. (39) with respect to v, we have

$$\begin{aligned}\bar{\psi}_{\bar{s}v} = {} & (Wg_1' - Wg_2\kappa)T \\ & + (Wg_1\kappa + Wg_2' - Wg_3\tau)N + (Wg_2\tau + Wg_3')B \ .\end{aligned} \tag{44}$$

The coefficients of the first and the second fundamental forms of the ruled surfaces in Eq.(36) are

$$\begin{aligned}\bar{E} &= (W + v\mathrm{W}g_1' - vWg_2\kappa)^2 + (-\mathrm{r}W\tau + vWg_1\kappa + vWg_2' - vWg_3\tau)^2 + (vWg_2\tau + vWg_3')^2, \\ \bar{F} &= g_1W + vg_1g_1'W + vg_2g_2'W + g_3g_3'W, \\ \bar{G} &= g_1^2 + g_2^2 + g_3^2 \ .\end{aligned} \tag{45}$$

and

$$\begin{aligned}\bar{M}(s,0) &= (Wg_1' - Wg_2\kappa)(-rWg_3\tau) - Wg_3(W\kappa g_1 + Wg_2' - Wg_3\tau) \\ &\quad + (Wg_2\tau + Wg_3')(Wg_2 + rg_1\tau W), \\ \bar{N}(s,0) &= 0 \ , \\ \bar{L}(s,0) &= -r^2g_3\tau^2\kappa W^3 - g_3\kappa W^3 + rW^3g_3\tau' - rg_2\tau^2W^3 - r^2g_1\tau^3W^3 - 2r\tau g_3W'W^2\end{aligned} \tag{46}$$

Corollary 9. The ruled surfaces $\bar{\psi}_{\bar{B}}$ is developable along parallel curves $\bar{\alpha}$ if and only if

$$r\tau'W^2 - \kappa = 0.$$

Examples

Let us take the unit speed space curve

$$\alpha(s) = (\frac{\sqrt{3}}{2} sin\ s, \frac{s}{2}, \frac{\sqrt{3}}{2} cos\ s).$$

Then, its is easy to show that,

$$T(s) = \left(\frac{\sqrt{3}}{2} cos\ s, \frac{1}{2}, -\frac{\sqrt{3}}{2} sin\ s\right),$$
$$N(s) = (-sin\ s, 0, -cos\ s),$$
$$B(s) = \left(-\frac{1}{2} cos\ s, \frac{\sqrt{3}}{2}, \frac{1}{2} sin\ s\right),$$

where $\kappa = \frac{\sqrt{3}}{2}$, $\tau = \frac{1}{2}$.

One of the parallel curves of $\alpha(s)$ is

$$\bar{\alpha} = (\frac{\sqrt{3}}{2} sins - 2coss\ \ \frac{s}{2} + 2\sqrt{3},\ 2sins + \frac{\sqrt{3}}{2} coss),$$

where r=4. Using Eqs. (8), (9), (26), and (37), we give Frenet frame of paralel curve

$$\bar{T}(\bar{s}) = (\frac{4\sqrt{17}}{17} sins + \frac{\sqrt{51}}{34} coss, \frac{\sqrt{17}}{34}, \frac{4\sqrt{17}}{17} coss - \frac{\sqrt{51}}{34} sins)$$
$$\bar{N}(\bar{s}) = (\frac{4\sqrt{19}}{19} coss - \frac{\sqrt{57}}{19} sins,\ \ 0, -\frac{4\sqrt{19}}{19} sins - \frac{\sqrt{57}}{19} coss)$$
$$\bar{B}(\bar{s}) = (-\frac{2\sqrt{323}}{323} sins - \frac{\sqrt{969}}{646} coss, \frac{\sqrt{323}}{34}, \frac{\sqrt{969}}{646} sins - \frac{2\sqrt{323}}{323} coss),$$

where

$$W = \frac{1}{\sqrt{17}},\quad \Omega = \frac{\sqrt{19}}{34},$$
$$f_1 = 2\frac{\sqrt{57}}{19},\quad f_2 = \frac{\sqrt{57}}{19},\quad f_3 = \frac{2\sqrt{19}}{19}$$
$$g_1 = \frac{4\sqrt{323}}{323},\quad g_2 = \frac{2\sqrt{323}}{323},\quad g_3 = \frac{5\sqrt{969}}{323}.$$

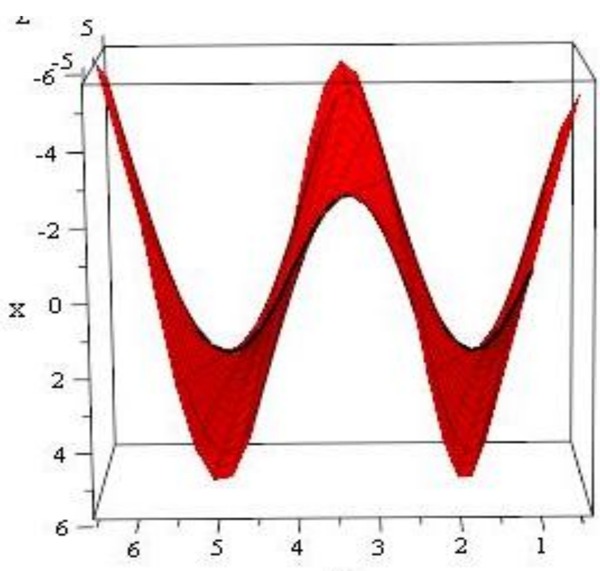

Figure 1. The ruled surface in Eq.(14)

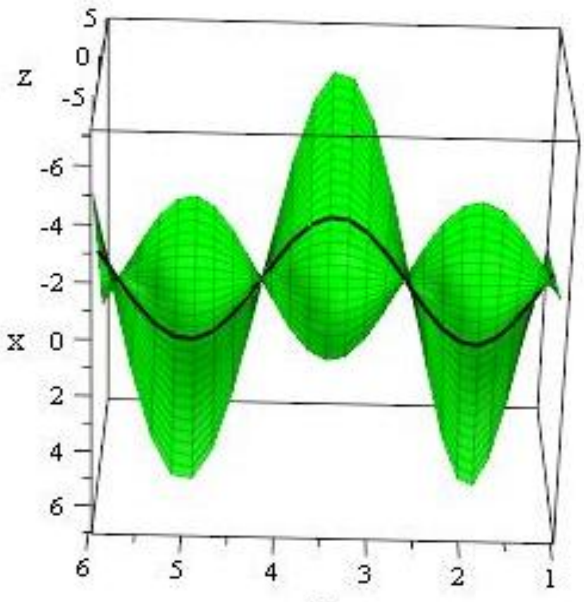

Figure 2. The ruled surface in Eq.(25)

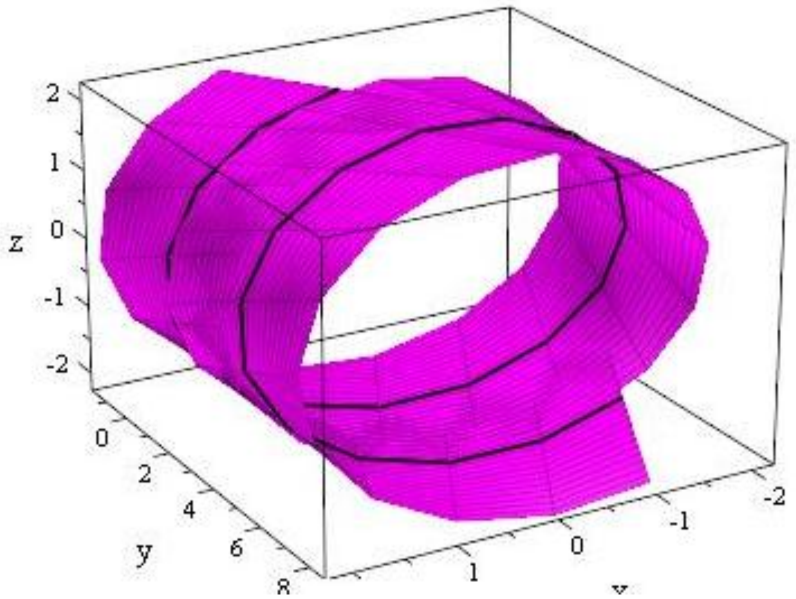

Figure 3. The ruled surface in Eq.(36)

4. CONCLUSION

The trajectory ruled surface formed by the parallel curve of a curve and its properties were investigated. The condition of the parallel curve being a special curve was examined. An example of ruled surfaces is given.

REFERENCES

[1] Wang, F., Liu, H., *Mathematics in Practice and Theory*, **37**(1), 141, 2007.

[2] Liu, H., Wang, F., *Journal of Geometry*, **88**(1-2), 120, 2008.

[3] Ravani, B., Ku, T.S., *Computer Aided Geometric Design*, **23**(2), 145, 1991.

[4] Orbay, K., Kasap, E., *International Journal of Physical Sciences*, **4**(5), 261, 2009.

[5] Kazaz, M., Uğurlu, H.H., Önder, M., Seda, O.R.A.L., *Afyon Kocatepe University Journal of Science and Engineering*, **16**(1), 76, 2016.

[6] Aldossary, M.T., Gazwani, M.A., *Global Journal of Advanced Research on Classical and Modern Geometries*, **9**(1), 43, 2020.

[7] Küçük, A., Gürsoy, O., *Applied mathematics and computation*, **151**(3), 763, 2004.

[8] Yıldız, Ö.G., Akyiğit, M., Tosun, M., *Mathematical Methods in the Applied Sciences*, **44**(9), 7463, 2021.

[9] Güler, F., *International Journal of Computational Methods*, **18**(10), 2150050, 2021.

[10] Unluturk, Y., Cimdiker, M., Ekici, C., *Communication in Mathematical Modeling and Applications*, **1**(1), 26, 2016.

[11] Güler, F., Bayram, E., Kasap, E., *International Journal of Geometric Methods in Modern Physics*, **15**(11), 1850195, 2018.

[12] Do Carmo, M. P., *Differential Geometry of Curves and Surfaces*, Prentice Hall, Englewood Cliffs, 1976.