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DEVELOPABLE SURFACES GENERATED BY USING MOVING FRAME

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Abstract. *In this paper, we define three different rotation-minimizing frames by rotating the moving frame around its coordinate axis vectors. Darboux vectors associated with these frames are obtained as special cases of the Darboux vector associated with the moving frame. Using these Darboux vectors and the moving frame we define six different developable surfaces. For each of these surfaces we give two invariants of curves on these surfaces to characterize their singularities. Moreover, we show that the base curves of these surfaces are contour generators with respect to an orthogonal projection or a central projection if and only if one of the invariants given for each surface is constantly equal to zero. Examples are provided to illustrate our theorems and results.*
Keywords: *Darboux vector; developable surface; moving frame; rotation minimizing frame; Legendre-Frenet frame; singularity; contour generators.*

1. INTRODUCTION

A ruled surface in three-dimensional real vector space $\mathbb{R}^3$ is a surface generated by the continuous motion of a straight line (called the ruling of the ruled surface) along a curve (called the base curve of the ruled surface). Ruled surfaces have an important role and many applications in different fields, such as Physics, Computer Aided Geometric Design (CAGD), Computer Vision Theory and the study of design problems in spatial mechanism, etc., see [1-3]. If a ruled surface has the same tangent plane at all points along a ruling, then it is called a developable surface. There are many studies interested with many properties and some characterizations of the concepts ruled and developable surfaces in $\mathbb{R}^3$, e.g. [4-9].

In [7], two types of developable surfaces are considered in $\mathbb{R}^3$; one is called the osculating developable surface, which is defined in [8], and the other is called the normal developable surface, which is defined in [5]. Osculating developable surface is generated by taken a unit speed curve (with non-zero normal curvature and geodesic torsion) on a surface in $\mathbb{R}^3$ as the base curve, and by taken the spherical osculating Darboux image along this unit speed curve as the director curve, which gives the direction of each ruling. And the normal developable surface is generated by taken a unit speed curve (with non-zero geodesic curvature and geodesic torsion) on a surface in $\mathbb{R}^3$ as the base curve, and by taken the spherical rectifying Darboux image along this unit speed curve as the director curve. Singularities of these developable surfaces are characterized. Moreover, the conditions to be a contour generator with respect to an orthogonal projection or a central projection of the base curve of osculating developable surface are given. In this paper, we generate six developable

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surfaces by taken the integral curves of the coordinate axes of the moving frame as their base curves and the spherical images of three Darboux vectors (which we have obtained from the Darboux vector of the moving frame) as their director curves. In this case; Thms 3.1, 3.3 and 4.3, Corolls. 3.2 and 4.2, and Prop. 4.1 given in [7] are obtained as particular cases of our study as stated in the application given as Subsect. 5.1.

This paper is organized as follows: In Sect. 2, we have given a brief summary of the concepts moving frame and Frenet-Serret frame. We have defined three vector fields from the Darboux vector field of moving frame. In Sect. 3, we have shown that these three vector fields are Darboux vector fields of rotation-minimizing frames (or simply RMFs), which are obtained by certain counterclockwise rotations of the moving frame around its axis vectors. In Sect. 4, we have defined six different developable surfaces by taken the integral curves of the coordinate axes of the moving frame as their base curves and the spherical images of the Darboux vector fields of RMFs as their director curves. We have characterized the singularities of these surfaces. We have also given the conditions to be contour generators of the base curves of these surfaces. The concept contour generator plays an important role in computer vision theory, see [1]. In Sect. 5, we give some applications and an example to illustrate our theorems and results.

2. PRELIMINARIES

In this section, we will recall some basic concepts of moving and Frenet-Serret frames associated with a curve in three-dimensional real vector space $\mathbb{R}^3$ to provide a background to understand the main idea of this study.

2.1. MOVING FRAME

Let $\alpha: I \subset \mathbb{R} \to \mathbb{R}^3$ be a regular space curve with arc-length parameter $s \in I$, where I denotes an open interval of the real line $\mathbb{R}$. Then, the unit vector

$$\mathrm{t}_1(s) = \alpha'(s) = \frac{d\alpha(s)}{ds}$$

is called the *unit tangent vector* of α at point $\alpha(s)$. A *moving frame* associated with the curve α is a right-handed orthonormal ordered triple-vectors $\{\mathrm{t}_1(s), \mathrm{t}_2(s), \mathrm{t}_3(s)\}$ satisfying $\mathrm{t}_1(s) = \mathrm{t}_2(s) \times \mathrm{t}_3(s)$ for all $s \in I$, where $\times$ denotes the usual cross-product on $\mathbb{R}^3$. In this text, we will call the curve α as a *spine curve*. Since $\mathrm{t}_1(s)$ is known and $\mathrm{t}_1(s) \times \mathrm{t}_2(s) = \mathrm{t}_3(s)$, a moving frame is uniquely defined by the unit vector $\mathrm{t}_2(s)$, which is called the *unit normal vector* of α at point $\alpha(s)$. Thus, $\mathrm{t}_2(s)$ is the *reference vector* of the moving frame.

Frenet-Serret type formulas of the curve α associated with moving frame can be given in matrix form as

$$\begin{bmatrix} \mathrm{t}_1 \\ \mathrm{t}_2 \\ \mathrm{t}_3 \end{bmatrix}' = \begin{bmatrix} 0 & \varkappa_1 & \varkappa_2 \\ -\varkappa_1 & 0 & \varkappa_3 \\ -\varkappa_2 & -\varkappa_3 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{t}_1 \\ \mathrm{t}_2 \\ \mathrm{t}_3 \end{bmatrix}. \tag{1}$$

Here; $\varkappa_1 = \langle \mathrm{t}_1', \mathrm{t}_2 \rangle$, $\varkappa_2 = \langle \mathrm{t}_1', \mathrm{t}_3 \rangle$, $\varkappa_3 = \langle \mathrm{t}_2', \mathrm{t}_3 \rangle$ are continuous functions, where $\mathrm{t}_i' = d\mathrm{t}_i/ds$ (for $i = 1,2$) and $\langle , \rangle$ denotes the usual inner product on $\mathbb{R}^3$. The set $\{\mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3, \varkappa_1, \varkappa_2, \varkappa_3\}$ is called the *moving frame apparatus* associated with α.

Darboux vector field (that corresponds to each point $\alpha(s)$ of curve α is the angular velocity vector of the moving frame at point $\alpha(s)$) along α associated with moving frame can be expressed as

$$\mathrm{W} = \varkappa_3 \mathrm{t}_1 - \varkappa_2 \mathrm{t}_2 + \varkappa_1 \mathrm{t}_3 \tag{2}$$

satisfying the symmetrical properties

$$W \times \mathrm{t}_1 = \mathrm{t}_1', \; W \times \mathrm{t}_2 = \mathrm{t}_2', \; W \times \mathrm{t}_3 = \mathrm{t}_3'.$$

Angular speed of the Darboux vector field W is

$$v_{ang} = \sqrt{(\varkappa_1)^2 + (\varkappa_2)^2 + (\varkappa_3)^2}.$$

The number of possible combinations of 2 objects from a set of 3 objects is $C(3,2) = P(3,2)/2! = 3$. Since the Darboux vector field W has three components (which are $\varkappa_3 \mathrm{t}_1$, $-\varkappa_2 \mathrm{t}_2$ and $\varkappa_1 \mathrm{t}_3$), we can define the three vector fields

$$W_1 = -\varkappa_2 \mathrm{t}_2 + \varkappa_1 \mathrm{t}_3, W_2 = \varkappa_3 \mathrm{t}_1 + \varkappa_1 \mathrm{t}_3, \; W_3 = \varkappa_3 \mathrm{t}_1 - \varkappa_2 \mathrm{t}_2 \tag{3}$$

with two components. *Spherical images* (i.e., *normalized forms*) of these three vector fields are

$$\begin{aligned}
\overline{W}_1 &= \frac{-\varkappa_2 \mathrm{t}_2 + \varkappa_1 \mathrm{t}_3}{\sqrt{(-\varkappa_2)^2 + (\varkappa_1)^2}} \quad \text{if} \quad (\varkappa_2, \varkappa_1) \neq (0,0) \\
\overline{W}_2 &= \frac{\varkappa_3 \mathrm{t}_1 + \varkappa_1 \mathrm{t}_3}{\sqrt{(\varkappa_3)^2 + (\varkappa_1)^2}} \quad \text{if} \quad (\varkappa_3, \varkappa_1) \neq (0,0) \\
\overline{W}_3 &= \frac{\varkappa_3 \mathrm{t}_1 - \varkappa_2 \mathrm{t}_2}{\sqrt{(\varkappa_3)^2 + (-\varkappa_2)^2}} \quad \text{if} \quad (\varkappa_3, \varkappa_2) \neq (0,0)
\end{aligned} \tag{4}$$

Non-vanishing of the function pairs $(\varkappa_2, \varkappa_1)$ (resp., $(\varkappa_3, \varkappa_1), (\varkappa_3, \varkappa_2)$) means that t_1 (resp., $\mathrm{t}_2, \mathrm{t}_3$) is a regular spherical curve, and thus $\overline{W}_1$ (resp., $\overline{W}_2, \overline{W}_3$) is a spherical dual (or the unit normal vector field) of t_1 (resp., $\mathrm{t}_2, \mathrm{t}_3$).

For further information about the concept moving frame see [10-12].

2.2. FRENET-SERRET FRAME

A well known moving frame of a spine curve α in $\mathbb{R}^3$ is the *Frenet-Serret frame*, whose three orthonormal axes vectors are defined at point $\alpha(s)$ as

$$\mathrm{t}_1(s) = \alpha'(s), \;\; \mathrm{t}_2(s) = \frac{\mathrm{t}_1'(s)}{\|\mathrm{t}_1'(s)\|} = \frac{\alpha''(s)}{\|\alpha''(s)\|}, \;\; \mathrm{t}_3(s) = \mathrm{t}_1(s) \times \mathrm{t}_2(s)$$

for all $s \in I$, where $\|\mathrm{t}_1'(s)\| = \sqrt{\langle \mathrm{t}_1'(s), \mathrm{t}_1'(s) \rangle}$ denotes the norm of the vector $\mathrm{t}_1'(s)$ and gives the length of $\mathrm{t}_1'(s)$. The vectors $\mathrm{t}_1(s)$, $\mathrm{t}_2(s)$ and $\mathrm{t}_3(s)$ are called, respectively, the *unit tangent vector*, the *principal normal vector* and the *binormal vector* of α at $\alpha(s)$. The set $\{\mathrm{t}_1(s), \mathrm{t}_2(s), \mathrm{t}_3(s)\}$ is called the *Frenet-Serret frame* of the curve α at point $\alpha(s)$, for all $s \in I$. In classical literature on differential geometry, the unit tangent vector field t_1 is generally denoted by t, the principal normal vector field t_2 by n and the binormal vector field t_3 by b, and thus the Frenet-Serret frame by $\{\mathrm{t},\mathrm{n},\mathrm{b}\}$.

Frenet-Serret formulas (also known as *Frenet-Serret equations*) of the curve α can be given in matrix form as

$$\begin{bmatrix}\mathrm{t}\\ \mathrm{n}\\ \mathrm{b}\end{bmatrix}' = \begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\mathrm{t}\\ \mathrm{n}\\ \mathrm{b}\end{bmatrix}, \tag{5}$$

where

$$\kappa = \langle \mathrm{t}', \mathrm{n}\rangle = \|\mathrm{t}'\| = \|\alpha''\| \ \text{ and } \ \tau = \langle \mathrm{n}', \mathrm{b}\rangle = \frac{\langle \alpha' \times \alpha'', \alpha'''\rangle}{\kappa^2}$$

are, respectively, the *curvature* and *torsion functions* of α. The set $\{\mathrm{t},\mathrm{n},\mathrm{b},\kappa,\tau\}$ is called the *Frenet-Serret frame apparatus* of α.

Darboux vector field (that corresponds to each point $\alpha(s)$ of curve α is the angular velocity vector of the Frenet-Serret frame at point $\alpha(s)$) along α associated with Frenet-Serret frame can be expressed as

$$\mathrm{w} = \tau\mathrm{t} + \kappa\mathrm{b} \tag{6}$$

satisfying the symmetrical properties

$$\mathrm{w} \times \mathrm{t} = \mathrm{t}', \mathrm{w} \times \mathrm{n} = \mathrm{n}', \mathrm{w} \times \mathrm{b} = \mathrm{b}'.$$

Angular speed of the Darboux vector field w is

$$v_{ang} = \sqrt{\tau^2 + \kappa^2}.$$

For further information about the concept Frenet-Serret frame see [12].

3. ROTATION-MINIMIZING FRAMES OBTAINED FROM MOVING FRAME

In this section, we will show that the vector fields W_1, W_2 and W_3 given in Eq. (3) are the Darboux vector fields of rotation-minimizing frames (or simply RMFs) obtained by certain counterclockwise rotations of the moving frame around its coordinate axes $\mathrm{t}_1(s)$, $\mathrm{t}_2(s)$ or $\mathrm{t}_3(s)$, for all $s \in I$. Frenet-Serret type formulas of the curves $\int \mathrm{t}_1$, $\int \mathrm{t}_2$ and $\int \mathrm{t}_3$ associated with these RMFs will also be given.

Frenet-Serret frame is not the best choice for applications in geometric modeling, because it may exhibit a strong rotation around the tangent vector of the spine curve. Another disadvantage of this frame is that it is not continously defined for a C^1 spine curve (remember that; if a function f defined on an interval I is C^n −continuous (or simply C^n), all of its derivatives of order n and less exist and are continuous on I, where $n \geq 1$). Moreover, for a C^2 spine curve the Frenet-Serret frame becomes undefined at the points where $\kappa = 0$. A moving frame that does not rotate about the instantaneous tangent vector (and thus avoid from these disadvantages) of the spine curve α is the rotation-minimizing frame of α. A RMF is characterized by the fact that the normal plane of the spine curve rotates as little as possible around the unit tangent vector along the spine curve. For further information about the concept RMF see [9].

We can define an RMF by a counterclockwise rotation of the vectors $\mathrm{t}_2(s)$ and $\mathrm{t}_3(s)$ of the moving frame around the tangent vector $\mathrm{t}_1(s)$ by an angle $\theta(s) = -\int \varkappa_3(s)ds$ (i.e., $\theta'(s) = -\varkappa_3(s)$) for all $s \in I$, namely, if we take two unit vectors $\mathrm{n}_2(s)$ and $\mathrm{n}_3(s)$ as

$$\begin{cases} \mathrm{n}_2(s) = \cos\theta(s)\mathrm{t}_2(s) + \sin\theta(s)\mathrm{t}_3(s), \\ \mathrm{n}_3(s) = -\sin\theta(s)\mathrm{t}_2(s) + \cos\theta(s)\mathrm{t}_3(s), \end{cases} \tag{7}$$

then we obtain the RMF $\{\mathrm{t}_1(s), \mathrm{n}_2(s), \mathrm{n}_3(s)\}$ associated with the spine curve $\alpha_1 = \int \mathrm{t}_1$ at point $\alpha_1(s)$. This RMF will be denoted by $\mathrm{RMF}_{\mathrm{t}_1}$.

Frenet-Serret type formulas of the curve α_1 associated with $\mathrm{RMF}_{\mathrm{t}_1}$ can be given in matrix form as

$$\begin{bmatrix} \mathrm{t}_1 \\ \mathrm{n}_2 \\ \mathrm{n}_3 \end{bmatrix}' = \begin{bmatrix} 0 & l_1 & l_2 \\ -l_1 & 0 & 0 \\ -l_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{t}_1 \\ \mathrm{n}_2 \\ \mathrm{n}_3 \end{bmatrix}, \tag{8}$$

where

$$l_1 = \langle \mathrm{t}_1', \mathrm{n}_2 \rangle = \varkappa_1 \cos\theta + \varkappa_2 \sin\theta, \quad l_2 = \langle \mathrm{t}_1', \mathrm{n}_3 \rangle = -\varkappa_1 \sin\theta + \varkappa_2 \cos\theta. \tag{9}$$

Darboux vector field (that corresponds to each point $\alpha_1(s)$ of curve α_1 is the angular velocity vector of $\mathrm{RMF}_{\mathrm{t}_1}$ at point $\alpha_1(s)$) along α_1 associated with $\mathrm{RMF}_{\mathrm{t}_1}$ can be expressed as

$$W_{\mathrm{t}_1} = -l_2 \mathrm{n}_2 + l_1 \mathrm{n}_3 \tag{10}$$

satisfying the symmetrical properties

$$W_{\mathrm{t}_1} \times \mathrm{t}_1 = \mathrm{t}_1', \quad W_{\mathrm{t}_1} \times \mathrm{n}_2 = \mathrm{n}_2', \quad W_{\mathrm{t}_1} \times \mathrm{n}_3 = \mathrm{n}_3'.$$

Spherical image of the Darboux vector field $W_{\mathbf{t}_1}$ is

$$\overline{W}_{\mathrm{t}_1} = \frac{-l_2 \mathrm{n}_2 + l_1 \mathrm{n}_3}{\sqrt{(-l_2)^2 + (l_1)^2}} \quad \text{for} \quad (l_2, l_1) \neq (0,0), \tag{11}$$

where

$$v_{ang} = \sqrt{(-l_2)^2 + (l_1)^2}$$

is the *angular speed* of $W_{\mathbf{t}_1}$.

Using Eqs. (7) and (9) in Eq. (10), we obtain

$$W_{\mathrm{t}_1} = -l_2 \mathrm{n}_2 + l_1 \mathrm{n}_3 = -\varkappa_2 \mathrm{t}_2 + \varkappa_1 \mathrm{t}_3 = W_1.$$

That means; the vector field W_1 given in Eq. (3) corresponds to the Darboux vector field of $\mathrm{RMF}_{\mathrm{t}_1}$ obtained by a counterclockwise rotation of the vector pairs $\{\mathrm{t}_2(s), \mathrm{t}_3(s)\}$ around the tangent vector $\mathrm{t}_1(s)$ by an angle $\theta(s) = -\int \varkappa_3(s)ds$.

We can also define an RMF by a counterclockwise rotation of the vectors $\mathrm{t}_3(s)$ and $\mathrm{t}_1(s)$ of the moving frame around the normal vector $\mathrm{t}_2(s)$ by an angle $\beta(s) = \int \varkappa_2(s)ds$ (i.e., $\beta'(s) = \varkappa_2(s)$) for all $s \in I$, namely, if we take two unit vectors $\mathrm{m}_3(s)$ and $\mathrm{m}_1(s)$ as

$$\begin{cases} \mathrm{m}_3(s) = \cos\beta(s)\mathrm{t}_3(s) + \sin\beta(s)\mathrm{t}_1(s), \\ \mathrm{m}_1(s) = -\sin\beta(s)\mathrm{t}_3(s) + \cos\beta(s)\mathrm{t}_1(s), \end{cases} \tag{12}$$

then we obtain the RMF $\{\mathrm{t}_2(s), \mathrm{m}_3(s), \mathrm{m}_1(s)\}$ associated with the spine curve $\alpha_2 = \int \mathrm{t}_2$ at point $\alpha_2(s)$. This RMF will be denoted by c.

Frenet-Serret type formulas of the curve α_2 associated with $\mathrm{RMF}_{\mathrm{t}_2}$ can be given in matrix form as

$$\begin{bmatrix} \mathrm{t}_2 \\ \mathrm{m}_3 \\ \mathrm{m}_1 \end{bmatrix}' = \begin{bmatrix} 0 & \lambda_1 & \lambda_2 \\ -\lambda_1 & 0 & 0 \\ -\lambda_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{t}_2 \\ \mathrm{m}_3 \\ \mathrm{m}_1 \end{bmatrix}, \tag{13}$$

where

$$\lambda_1 = \langle \mathrm{t}_2', \mathrm{m}_3 \rangle = \varkappa_3 \cos\beta - \varkappa_1 \sin\beta, \;\; \lambda_2 = \langle \mathrm{t}_2', \mathrm{m}_1 \rangle = -\varkappa_3 \sin\beta - \varkappa_1 \cos\beta. \tag{14}$$

Darboux vector field (that corresponds to each point $\alpha_2(s)$ of curve α_2 is the angular velocity vector of $\mathrm{RMF}_{\mathrm{t}_2}$ at point $\alpha_2(s)$) along α_2 associated with $\mathrm{RMF}_{\mathrm{t}_2}$ can be expressed as

$$W_{\mathrm{t}_2} = -\lambda_2 \mathrm{m}_3 + \lambda_1 \mathrm{m}_1 \tag{15}$$

satisfying the symmetrical properties

$$W_{\mathrm{t}_2} \times \mathrm{t}_2 = \mathrm{t}_2', \;\; W_{\mathrm{t}_2} \times \mathrm{m}_2 = \mathrm{m}_2', \; W_{\mathrm{t}_2} \times \mathrm{m}_3 = \mathrm{m}_3'.$$

Spherical image of the Darboux vector field $W_{\mathbf{t}_2}$ is

$$\overline{W}_{\mathrm{t}_2} = \frac{-\lambda_2 \mathrm{m}_3 + \lambda_1 \mathrm{m}_1}{\sqrt{(-\lambda_2)^2 + (\lambda_1)^2}} \;\; \text{for} \;\; (\lambda_2, \lambda_1) \neq (0,0), \tag{16}$$

where

$$v_{ang} = \sqrt{(-\lambda_2)^2 + (\lambda_1)^2}$$

is the *angular speed* of W_{t_2}.

Using Eqs. (12) and (14) in Eq. (15), we obtain

$$W_{\mathrm{t}_2} = -\lambda_2 \mathrm{m}_3 + \lambda_1 \mathrm{m}_1 = \varkappa_3 \mathrm{t}_1 + \varkappa_1 \mathrm{t}_3 = W_2.$$

That means; the vector field W_2 given in Eq. (3) corresponds to the Darboux vector field of $\mathrm{RMF}_{\mathrm{t}_2}$ obtained by a counterclockwise rotation of the vector pairs $\{\mathrm{t}_3(s), \mathrm{t}_1(s)\}$ around the normal vector $\mathrm{t}_2(s)$ by an angle $\beta(s) = \int \varkappa_2(s) ds$.

The last RMF we will define in this study is obtained by a counterclockwise rotation of the vectors $\mathrm{t}_1(s)$ and $\mathrm{t}_2(s)$ of the moving frame around the vector $\mathrm{t}_3(s)$ by an angle $\gamma(s) = -\int \varkappa_1(s) ds$ (i.e., $\gamma'(s) = -\varkappa_1(s)$) for all $s \in I$, namely, if we take two unit vectors $\mathrm{r}_1(s)$ and $\mathrm{r}_2(s)$ as

$$\begin{cases} \mathrm{r}_1(s) = \cos\gamma(s)\mathrm{t}_1(s) + \sin\gamma(s)\mathrm{t}_2(s), \\ \mathrm{r}_2(s) = -\sin\gamma(s)\mathrm{t}_1(s) + \cos\gamma(s)\mathrm{t}_2(s), \end{cases} \tag{17}$$

then we obtain the RMF $\{\mathrm{t}_3(s), \mathrm{r}_1(s), \mathrm{r}_2(s)\}$ associated with the spine curve α_3 at point $\alpha_3(s)$. This RMF will be denoted by $\mathrm{RMF}_{\mathrm{t}_3}$.

Frenet-Serret type formulas of the curve α_3 associated with $\mathrm{RMF}_{\mathrm{t}_3}$ can be given in matrix form as

$$\begin{bmatrix} \mathrm{t}_3 \\ \mathrm{r}_1 \\ \mathrm{r}_2 \end{bmatrix}' = \begin{bmatrix} 0 & \mu_1 & \mu_2 \\ -\mu_1 & 0 & 0 \\ -\mu_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{t}_3 \\ \mathrm{r}_1 \\ \mathrm{r}_2 \end{bmatrix}, \tag{18}$$

where

$$\mu_1 = \langle t_3', r_1 \rangle = -\varkappa_2 \cos\gamma - \varkappa_3 \sin\gamma, \ \ \mu_2 = \langle t_3', r_2 \rangle = \varkappa_2 \sin\gamma - \varkappa_3 \cos\gamma. \tag{19}$$

Darboux vector field (that corresponds to each point $\alpha_3(s)$ of curve α_3 is the angular velocity vector of RMF_{t_3} at point $\alpha_3(s)$) along α_3 associated with RMF_{t_3} can be expressed as

$$W_{t_3} = -\mu_2 r_1 + \mu_1 r_2 \tag{20}$$

satisfying the symmetrical properties

$$W_{t_3} \times t_3 = t_3', \ W_{t_3} \times r_1 = r_1', \ W_{t_3} \times r_2 = r_2'.$$

Spherical image of the Darboux vector field $W_{\mathbf{t}_3}$ is

$$\overline{W}_{t_3} = \frac{-\mu_2 r_1 + \mu_1 r_2}{\sqrt{(-\mu_2)^2 + (\mu_1)^2}} \ \text{ for } \ (\mu_2, \mu_1) \neq (0,0), \tag{21}$$

where

$$v_{ang} = \sqrt{(-\mu_2)^2 + (\mu_1)^2}$$

is the *angular speed* of $W_{\mathbf{t}_3}$.

Using Eqs. (17) and (19) in Eq. (20), we obtain

$$W_{t_3} = -\mu_2 r_1 + \mu_1 r_2 = \varkappa_3 t_1 - \varkappa_2 t_2 = W_3.$$

That means; the vector field W_3 given in Eq. (3) corresponds to the Darboux vector field of RMF_{t_3} obtained by a counterclockwise rotation of the vector pairs $\{t_1(s), t_2(s)\}$ around the vector $t_3(s)$ by an angle $\gamma(s) = -\int \varkappa_1(s) ds$.

4. DEVELOPABLE SURFACES WITH DIRECTORS $\overline{W}_1$, $\overline{W}_2$ AND $\overline{W}_3$

In this section, we will define six different developable surfaces by using the spherical images of the Darboux vector fields W_1, W_2, W_3 and the moving frame $\{t_1, t_2, t_3\}$. For each of these surfaces we will introduce two invariants to characterize their singularities. We also give the conditions to be contour generators of the base curves of these surfaces by using these invariants.

4.1. DEVELOPABLE SURFACES WITH DIRECTOR $\overline{W}_1$

Using the unit vectors $t_1(s)$ and $\overline{W}_1(s)$ we can define the unit vector $u_1(s) = \overline{W}_1(s) \times t_1(s)$ at point $\alpha_1(s)$, for all $s \in I$. In this case, we obtain the rigth-handed orthonormal frame $\{t_1(s), u_1(s), \overline{W}_1(s)\}$ of the spine curve $\alpha_1 = \int t_1$ at point $\alpha_1(s)$. Note that $\{t_1, u_1, \overline{W}_1\}$ is the Frenet-Serret frame of the curve α_1. *Frenet-Serret formulas* of the curve α_1 can be given in matrix form as

$$\begin{bmatrix} \mathrm{t}_1 \\ u_1 \\ \overline{W}_1 \end{bmatrix}' = \begin{bmatrix} 0 & \kappa_1 & 0 \\ -\kappa_1 & 0 & \tau_1 \\ 0 & -\tau_1 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{t}_1 \\ u_1 \\ \overline{W}_1 \end{bmatrix}, \tag{22}$$

where

$$\kappa_1 = \langle \mathrm{t}_1', u_1 \rangle = \sqrt{(l_1)^2 + (l_2)^2} = \kappa,\ \tau_1 = \langle u_1', \overline{W}_1 \rangle = \frac{(l_1)^2}{(l_1)^2 + (l_2)^2} \left(\frac{l_2}{l_1} \right)' = \tau$$

are, respectively, the *curvature* and *torsion functions* of α_1.

We know that a *ruled surface* in $\mathbb{R}^3$ is (locally) the image of a map

$$f: I \times \mathbb{R} \to \mathbb{R}^3$$

defined by

$$f(s,u) = \rho(s) + uv(s), \tag{23}$$

where $\rho: I \to \mathbb{R}^3$, $v: I \to \mathbb{S}^2 = \{\mathrm{x} \in \mathbb{R}^3 : \|\mathrm{x}\| = 1\}$ are smooth mappings and I is an open interval of $\mathbb{R}$. The curves ρ and v are called the *base curve* and the *director curve* of the ruled surface f, respectively. The straight lines $u \to \rho(s) + uv(s)$ are the *rulings* of f. *Striction curve* of f is defined to be

$$f^*(s) = \rho(s) - \frac{\langle \rho'(s), v'(s) \rangle}{\langle v'(s), v'(s) \rangle} v(s).$$

In the case $\langle \rho'(s), v'(s) \rangle = 0$, base curve ρ is the striction curve. Ruled surface f is said to be *developable* if

$$\det\left(\rho'(s), v(s), v'(s)\right) = 0,$$

for all $s \in I$, see [14].

Using the unit vector fields t_2, t_3 and $\overline{W}_1$, we can define the following two ruled surfaces:

1. By the mapping

$$\phi_{\mathrm{t}_2}: I \times \mathbb{R} \to \mathbb{R}^3$$

defined by

$$\phi_{\mathbf{t}_2}(s,u) = \int \mathbf{t}_2(s) ds + u\overline{W}_1(s). \tag{24}$$

2. By the mapping

$$\phi_{\mathbf{t}_3}: I \times \mathbb{R} \to \mathbb{R}^3$$

defined by

$$\phi_{\mathbf{t}_3}(s,u) = \int \mathbf{t}_3(s) ds + u\overline{W}_1(s). \tag{25}$$

Proposition 1. *Ruled surfaces* $\phi_{\boldsymbol{t}_2}(s,u) = \int \boldsymbol{t}_2(s) ds + u\overline{W}_1(s)$ *and* $\phi_{\boldsymbol{t}_3}(s,u) = \int \boldsymbol{t}_3(s) ds + u\overline{W}_1(s)$ *are developable.*

Proof: From Eqs. (4) and (22), we know that

$$\overline{W}_1(s) = \frac{-\varkappa_2(s)\mathrm{t}_2(s) + \varkappa_1(s)\mathrm{t}_3(s)}{\sqrt{\left(-\varkappa_2(s)\right)^2 + \left(\varkappa_1(s)\right)^2}} \text{ for } (\varkappa_2(s), \varkappa_1(s)) \neq (0,0)$$

and

$$\overline{W}_1'(s) = -\tau_1(s)u_1(s),$$

respectively, where

$$u_1(s) = \overline{W}_1(s) \times \mathrm{t}_1(s) = \frac{\varkappa_1(s)\mathrm{t}_2(s) + \varkappa_2(s)\mathrm{t}_3(s)}{\sqrt{(\varkappa_1(s))^2 + (\varkappa_2(s))^2}}$$

for all $s \in I$. Since $\det(\mathrm{t}_2(s), \overline{W}_1(s), \overline{W}_1'(s)) = 0$ and $\det(\mathrm{t}_3(s), \overline{W}_1(s), \overline{W}_1'(s)) = 0$, the surfaces are ϕ_{t_2} and ϕ_{t_3} are developable.

The parametrization of the *striction curve* on the ruled surface $\phi_{\mathbf{t}_2}$ (resp., $\phi_{\mathbf{t}_3}$) can be given by

$$\begin{aligned} \phi_{\mathrm{t}_2}^*(s) &= \int \mathrm{t}_2(s)ds - \frac{\left\langle \mathrm{t}_2(s), \overline{W}_1'(s) \right\rangle}{\left\langle \overline{W}_1'(s), \overline{W}_1'(s) \right\rangle} \overline{W}_1(s) \\ &= \int \mathrm{t}_2(s)ds + \frac{\varkappa_1(s)}{\tau_1(s)\sqrt{\left(\varkappa_1(s)\right)^2 + \left(\varkappa_2(s)\right)^2}} \overline{W}_1(s) \end{aligned} \tag{26}$$

$$\begin{aligned} \text{(resp., } \phi_{\mathrm{t}_3}^*(s) &= \int \mathrm{t}_3(s)ds - \frac{\left\langle \mathrm{t}_3(s), \overline{W}_1'(s) \right\rangle}{\left\langle \overline{W}_1'(s), \overline{W}_1'(s) \right\rangle} \overline{W}_1(s)) \\ &= \int \mathrm{t}_3(s)ds + \frac{\varkappa_2(s)}{\tau_1(s)\sqrt{\left(\varkappa_1(s)\right)^2 + \left(\varkappa_2(s)\right)^2}} \overline{W}_1(s)). \end{aligned} \tag{27}$$

Differentiating Eq. (26) with respect to s, we get

$$\phi_{\mathrm{t}_2}^{*\prime}(s) = \delta_{\mathrm{t}_2}(s)\overline{W}_1(s),$$

where $\tau_1(s) \neq 0$ and

$$\delta_{\mathbf{t}_2}(s) = -\frac{\varkappa_2(s)}{\sqrt{\left(\varkappa_1(s)\right)^2 + \left(\varkappa_2(s)\right)^2}} + \left(\frac{\varkappa_1(s)}{\tau_1(s)\sqrt{\left(\varkappa_1(s)\right)^2 + \left(\varkappa_2(s)\right)^2}} \right)'. \tag{28}$$

Differentiating Eq. (27) with respect to s, we also get

$$\phi_{\mathbf{t}_3}^{*\prime}(s) = \delta_{\mathbf{t}_3}(s)\overline{W}_1(s),$$

where

$$\delta_{\mathbf{t}_3}(s) = \frac{\varkappa_1(s)}{\sqrt{\left(\varkappa_1(s)\right)^2 + \left(\varkappa_2(s)\right)^2}} + \left(\frac{\varkappa_2(s)}{\tau_1(s)\sqrt{\left(\varkappa_1(s)\right)^2 + \left(\varkappa_2(s)\right)^2}}\right)'. \tag{29}$$

The *unit normal vector* of the ruled surface $\phi_{\mathbf{t}_2}$, given by Eq. (24), at point $\phi_{\mathbf{t}_2}(s,u)$ is defined to be

$$n_{\phi_{\mathbf{t}_2}}(s,u) = \frac{\left(\partial\phi_{\mathbf{t}_2}(s,u)/\partial s\right) \times \left(\partial\phi_{\mathbf{t}_2}(s,u)/\partial u\right)}{\left\|\left(\partial\phi_{\mathbf{t}_2}(s,u)/\partial s\right) \times \left(\partial\phi_{\mathbf{t}_2}(s,u)/\partial u\right)\right\|}.$$

We know that a surface is singular at a point when the unit normal vector vanishes at that point. Thus, to find the *singular points* of the surface $\phi_{\mathbf{t}_2}$, we have to solve

$$\frac{\partial\phi_{\mathrm{t}_2}}{\partial s} \times \frac{\partial\phi_{\mathrm{t}_2}}{\partial u} = (0,0,0) = 0.$$

Since $\phi_{\mathrm{t}_2}(s,u) = \int \mathrm{t}_2(s)ds + u\overline{W}_1(s)$, we get

$$\frac{\partial\phi_{\mathrm{t}_2}}{\partial s} \times \frac{\partial\phi_{\mathrm{t}_2}}{\partial u} = \left(\mathrm{t}_2 + u\overline{W}_1'\right) \times \overline{W}_1 = \left(-\tau_1 u + \frac{\varkappa_1}{\sqrt{(\varkappa_1)^2 + (\varkappa_2)^2}}\right)\mathrm{t}_1,$$

where

$$\tau_1 = \frac{(l_1)^2}{(l_1)^2 + (l_2)^2}\left(\frac{l_2}{l_1}\right)' = \varkappa_3 - \frac{\varkappa_1'\varkappa_2 - \varkappa_1\varkappa_2'}{(\varkappa_1)^2 + (\varkappa_2)^2}. \tag{30}$$

Therefore, $(s_0,u_0) \in I \times \mathbb{R}$ is a singular point of $\phi_{\mathbf{t}_2}$ if and only if $\tau_1(s_0) \neq 0$ and

$$u_0 = \frac{\varkappa_1(s_0)}{\tau_1(s_0)\sqrt{(\varkappa_1(s_0))^2 + (\varkappa_2(s_0))^2}}.$$

Also to find the *singular points* of the surface $\phi_{\mathbf{t}_3}$, given by Eq. (25), we have to solve

$$\frac{\partial\phi_{\mathrm{t}_3}}{\partial s} \times \frac{\partial\phi_{\mathrm{t}_3}}{\partial u} = 0.$$

Since $\phi_{\mathrm{t}_3}(s,u) = \int \mathrm{t}_3(s)ds + u\overline{W}_1(s)$, we get

$$\frac{\partial\phi_{\mathrm{t}_3}}{\partial s} \times \frac{\partial\phi_{\mathrm{t}_3}}{\partial u} = \left(-\tau_1 u + \frac{\varkappa_2}{\sqrt{(\varkappa_1)^2 + (\varkappa_2)^2}}\right)\mathrm{t}_1.$$

Therefore, $(s_0,u_0) \in I \times \mathbb{R}$ is a singular point of $\phi_{\mathbf{t}_3}$ if and only if $\tau_1(s_0) \neq 0$ and

$$u_0 = \frac{\varkappa_2(s_0)}{\tau_1(s_0)\sqrt{(\varkappa_1(s_0))^2 + (\varkappa_2(s_0))^2}}.$$

In [15], it is stated that the locus of singular points of a developable surface is precisely its striction curve.

We now give the definition of the concept *contour generator* which plays an important role in the theory of computer vision: The *contour generator* Γ is the set of points X on a smooth surface $\mathbb{M}$ at which rays are tangent to the surface, see Fig. 1. In other words: Let $\mathbb{M} \subset \mathbb{R}^3$ be a surface and n be a unit normal vector field on $\mathbb{M}$. For a unit vector $x \in \mathbb{S}^2$ (where $\mathbb{S}^2$ denotes the Euclidean 2-sphere), the *contour generator of the orthogonal projection* with the direction x is defined to be

$$\{p \in \mathbb{M} : \langle n, x(p) \rangle = 0\},$$

which is in fact the singular set of the orthogonal projection with direction x. And for a poind $c \in \mathbb{R}^3$, the *contour generator of the central projection* with the center c is defined to be

$$\{p \in \mathbb{M} : \langle p - c, n(p) \rangle = 0\},$$

which is in fact the singular set of the central projection with center c, see [1, 7].

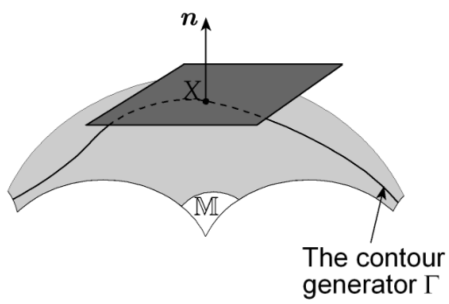

Figure 1. Contour generator Γ on a smooth surface $\mathbb{M}$.

It is importhant to emphasize that the invariants $\delta_{t_2}(s)$, $\delta_{t_3}(s)$ and $\tau_1(s)$ given, respectively, by the Eqs. (28), (29) and (30) characterize singularities of the developable surfaces ϕ_{t_2} and ϕ_{t_3}. Moreover, the regular base curve $\int t_2$ of ϕ_{t_2} is a contour generator with respect to an orthogonal projection (resp., central projection) if and only if $\tau_1(s)$ (resp., $\delta_{t_2}(s)$) is constantly equal to zero, while the regular base curve $\int t_3$ of ϕ_{t_3} is a contour generator with respect to an orthogonal projection (resp., central projection) if and only if $\tau_1(s)$ (resp., $\delta_{t_3}(s)$) is constantly equal to zero.

By the following theorem we can classify the developable surfaces ϕ_{t_2} and ϕ_{t_3} into cylindrical or conical surfaces:

Theorem 2. *For the developable surface $\phi_{t_2}(s,u) = \int t_2(s)ds + u\overline{W}_1(s)$ (resp., $\phi_{t_3}(s,u) = \int t_3(s)ds + u\overline{W}_1(s)$), given by Eq. (24) (resp., Eq. (25)), we have the following two conditions:*

1. The following are equivalent:
   (a) $\phi_{\mathbf{t}_2}$ (resp., $\phi_{\mathbf{t}_3}$) is a cylindrical surface.
   (b) $\tau_1(s) = 0$, for all $s \in I$.
   (c) Curve $\int \mathbf{t}_2(s)ds$ (resp., $\int \mathbf{t}_3(s)ds$) is a contour generator with respect to an orthogonal projection.

2. If $\tau_1(s) \neq 0$, then the following are equivalent:
   (a) $\phi_{\mathbf{t}_2}$ (resp., $\phi_{\mathbf{t}_3}$) is a conical surface.
   (b) $\delta_{\mathbf{t}_2}(s) = 0$ (resp., $\delta_{\mathbf{t}_3}(s) = 0$), for all $s \in I$.
   (c) Curve $\int \mathbf{t}_2(s)ds$ (resp., $\int \mathbf{t}_3(s)ds$) is a contour generator with respect to a central projection.

Proof: 1. We know that if all rulings of a ruled surface are parallel, then the ruled surface is cylindrical. This means that the ruled surface $f(s,u)=\rho(s)+uv(s)$, given by Eq. (23), is cylindrical if and only if

$$v(s)\times v'(s)=0,$$

for all $s\in I$. Thus, since

$$\overline{W}_1(s)\times\overline{W}_1'(s)=-\tau_1(s)u\mathbf{t}_1(s),$$

the surfaces $\phi_{\mathrm{t}_2}(s,u)=\int \mathrm{t}_2(s)ds+u\overline{W}_1(s)$ and $\phi_{\mathrm{t}_3}(s,u)=\int \mathrm{t}_3(s)ds+u\overline{W}_1(s))$ are cylindrical if and only if $\tau_1(s)=0$, where

$$\tau_1(s)=\varkappa_3(s)-\frac{\varkappa_1'(s)\varkappa_2(s)-\varkappa_1(s)\varkappa_2'(s)}{(\varkappa_1(s))^2+(\varkappa_2(s))^2},$$

for all $s\in I$. Therefore, first case become equivalent to the second case. Now suppose that the third case holds. Then there exists a vector $x\in\mathbb{S}^2$ such that $\langle\mathbf{t}_1,x\rangle=0$. So there are $a,b\in\mathbb{R}$ such that $x=au_1+b\overline{W}_1$. From $\langle\mathbf{t}_1,x\rangle=0$, we get $\langle\mathbf{t}_1',x\rangle=0$, and thus we have $a=0$, so that $x=\pm\overline{W}_1(s)$. Namely, the first case holds. Conversely, suppose that the first case holds. This means that $\overline{W}_1(s)$ is a constant. Then, we choose $x=\overline{W}_1(s)\in\mathbb{S}^2$. By the definition of $\overline{W}_1(s)$, we have $\langle x,\mathbf{t}_1\rangle=0$. Therefore, the first case entails the third case, and this completes the proof of Condition 1.

2. We also know that if all rulings of a ruled surface intersect in a point, then the ruled surface is conical. This means that if the derivative of the striction curve on a ruled surface vanishes, then the ruled surface becomes a cone. Striction curve of the surface $\phi_{\mathbf{t}_2}$ (resp., $\phi_{\mathbf{t}_3}$) is

$$\phi_{\mathrm{t}_2}^*(s)=\int \mathrm{t}_2(s)ds+\frac{\varkappa_1(s)}{\tau_1(s)\sqrt{(\varkappa_1(s))^2+(\varkappa_2(s))^2}}\overline{W}_1(s))$$

$$(\text{resp., }\ \phi_{\mathrm{t}_3}^*(s)=\int \mathrm{t}_3(s)ds+\frac{\varkappa_2(s)}{\tau_1(s)\sqrt{(\varkappa_1(s))^2+(\varkappa_2(s))^2}}\overline{W}_1(s)).$$

The first derivative of $\phi_{\mathbf{t}_2}^*$ (resp., $\phi_{\mathbf{t}_3}^*$) becomes $\phi_{\mathbf{t}_2}^{*'}(s)=\delta_{\mathbf{t}_2}(s)\overline{W}_1(s)$ (resp., $\phi_{\mathbf{t}_3}^{*'}(s)=\delta_{\mathbf{t}_3}(s)\overline{W}_1(s)$), where

$$\delta_{\mathbf{t}_2}(s)=-\frac{\varkappa_2(s)}{\sqrt{(\varkappa_1(s))^2+(\varkappa_2(s))^2}}+\left(\frac{\varkappa_1(s)}{\tau_1(s)\sqrt{(\varkappa_1(s))^2+(\varkappa_2(s))^2}}\right)'$$

$$(\text{resp.}, \delta_{\mathbf{t}_3}(s)=\frac{\varkappa_1(s)}{\sqrt{(\varkappa_1(s))^2+(\varkappa_2(s))^2}}+\left(\frac{\varkappa_2(s)}{\tau_1(s)\sqrt{(\varkappa_1(s))^2+(\varkappa_2(s))^2}}\right)').$$

Thus, the derivative of the striction curve $\phi_{\mathrm{t}_2}^*$ (resp., $\phi_{\mathrm{t}_3}^*$) vanishes if and only if $\delta_{\mathrm{t}_2}(s)=0$ (resp., $\delta_{\mathrm{t}_3}(s)=0$), and this means that the first and second cases are equivalent. From the definition of contour generator with respect to a central projection, third case means that there exists $c\in\mathbb{R}^3$ such that $\langle\int \mathrm{t}_2(s)ds-c,\mathrm{t}_1\rangle=0$ (resp., $\langle\int \mathrm{t}_3(s)ds-c,\mathrm{t}_1\rangle=0$). If the first case holds, then the striction curve

$$\phi^*_{\mathrm{t}_2}(s)=\int \mathrm{t}_2(s)ds+\frac{\varkappa_1(s)}{\tau_1(s)\sqrt{(\varkappa_1(s))^2+(\varkappa_2(s))^2}}\overline{W}_1(s)$$

$$(resp., \phi^*_{\mathrm{t}_3}(s)=\int \mathrm{t}_3(s)ds+\frac{\varkappa_2(s)}{\tau_1(s)\sqrt{(\varkappa_1(s))^2+(\varkappa_2(s))^2}}\overline{W}_1(s))$$

is constant. For the constant point $c=\phi^*_{\mathbf{t}_2}(s)$ (resp., $c=\phi^*_{\mathbf{t}_3}(s)$), we have

$$\langle\int \mathrm{t}_2(s)ds-c,\mathrm{t}_1(s)\rangle=\langle\int \mathrm{t}_2(s)ds-\phi^*_{\mathrm{t}_2}(s),\mathrm{t}_1(s)\rangle=0$$

(resp., $\langle\int \mathrm{t}_3(s)ds-c,\mathrm{t}_1(s)\rangle=0$). This means that the third case holds. For the converse, by the third case, there exists a point $c\in\mathbb{R}^3$ such that

$$\langle\int \mathrm{t}_2(s)ds-c,\mathrm{t}_1(s)\rangle=0$$

(resp., $\langle\int \mathrm{t}_3(s)ds-c,\mathrm{t}_1(s)\rangle=0$). Taking the derivative of the both side, we have

$$\langle\int \mathrm{t}_2(s)ds-c,\mathrm{t}_1(s)\rangle'=\langle\int \mathrm{t}_2(s)ds-c,\kappa_1(s)u_1(s)\rangle=0$$

(resp., $\langle\int \mathrm{t}_3(s)ds-c,\mathrm{t}_1(s)\rangle'=0$). That means there exist differentiable functions f_1 and g_1 (resp., f_2 and g_2) such that $\int \mathrm{t}_2(s)ds-c=f_1(s)\mathrm{t}_1(s)+g_1(s)\overline{W}_1(s)$ (resp., $\int \mathrm{t}_3(s)ds-c=f_2(s)\mathrm{t}_1(s)+g_2(s)\overline{W}_1(s)$). From third case we know that $\langle\int \mathrm{t}_2(s)ds-c,\mathrm{t}_1(s)\rangle=0$ (resp., $\langle\int \mathrm{t}_3(s)ds-c,\mathrm{t}_1(s)\rangle=0$), and this yields $\int \mathrm{t}_2(s)ds-c=g_1(s)\overline{W}_1(s)$ (resp., $\int \mathrm{t}_3(s)ds-c=g_2(s)\overline{W}_1(s)$). Taken again the derivative of $\langle\int \mathrm{t}_2(s)ds-c,\mathrm{t}_1(s)\rangle=0$ (resp., $\langle\int \mathrm{t}_3(s)ds-c,\mathrm{t}_1(s)\rangle=0$), we get

$$\langle\int \mathrm{t}_2(s)ds-c,\mathrm{t}_1(s)\rangle''=\frac{\kappa_1(s)\varkappa_1(s)}{\sqrt{(\varkappa_1(s))^2+(\varkappa_2(s))^2}}+g_1(s)\kappa_1(s)\tau_1(s)=0$$

(resp., $\langle\int \mathrm{t}_3(s)ds-c,\mathrm{t}_1(s)\rangle''=0$) which yields

$$g_1(s)=-\frac{\varkappa_1(s)}{\tau_1(s)\sqrt{(\varkappa_1(s))^2+(\varkappa_2(s))^2}}$$

$$(resp., g_2(s)=-\frac{\varkappa_2(s)}{\tau_1(s)\sqrt{(\varkappa_1(s))^2+(\varkappa_2(s))^2}}).$$

It follows that

$$\begin{aligned} c &=\int \mathrm{t}_2(s)ds-g_1(s)\overline{W}_1(s)\\ &=\int \mathrm{t}_2(s)ds+\frac{\varkappa_1(s)}{\tau_1(s)\sqrt{(\varkappa_1(s))^2+(\varkappa_2(s))^2}}\overline{W}_1(s)\\ &=\phi^*_{\mathrm{t}_2}(s)\end{aligned}$$

(resp., $c = \phi^*_{\mathbf{t}_3}(s)$). Therefore, $\phi^*_{\mathbf{t}_2}(s)$ (resp., $\phi^*_{\mathbf{t}_3}(s)$) is constant, so that the first case holds. This completes the proof of Condition 2.

By the following theorem we give the non-singular points of the developable surfaces $\phi_{\mathbf{t}_2}$ and $\phi_{\mathbf{t}_3}$. For these surfaces we also give the local diffeomorphism conditions to cuspidal edge $C \times \mathbb{R}$ or to swallowtail SW:

Theorem 3. *For the developable surface $\phi_{t_2}(s,u) = \int t_2(s)ds + u\overline{W}_1(s)$ (resp., $\phi_{t_3}(s,u) = \int t_3(s)ds + u\overline{W}_1(s)$), given by Eq. (24) (resp., Eq. (25)), we have the following three conditions:*

1. $\phi_{\mathbf{t}_2}$ (resp., $\phi_{\mathbf{t}_3}$) is non-singular at points (s_0, u_0) if and only if $\tau_1(s_0) \neq 0$ and

$$u_0 \neq \frac{\varkappa_1(s_0)}{\tau_1(s_0)\sqrt{(\varkappa_1(s_0))^2 + (\varkappa_2(s_0))^2}}$$

$$(\text{resp.}, \ u_0 \neq \frac{\varkappa_2(s_0)}{\tau_1(s_0)\sqrt{(\varkappa_1(s_0))^2 + (\varkappa_2(s_0))^2}}).$$

2. $\phi_{\mathbf{t}_2}$ (resp., $\phi_{\mathbf{t}_3}$) is locally diffeomorphic to cuspidal edge $C \times \mathbb{R}$ at points (s_0, u_0) if and only if $\tau_1(s_0) \neq 0$, $\delta_{\mathbf{t}_2}(s_0) \neq 0$ (resp., $\delta_{\mathbf{t}_3}(s_0) \neq 0$) and

$$u_0 = \frac{\varkappa_1(s_0)}{\tau_1(s_0)\sqrt{(\varkappa_1(s_0))^2 + (\varkappa_2(s_0))^2}}$$
$$(\text{resp.}, \ u_0 = \frac{\varkappa_2(s_0)}{\tau_1(s_0)\sqrt{(\varkappa_1(s_0))^2 + (\varkappa_2(s_0))^2}}).$$

3. $\phi_{\mathbf{t}_2}$ (resp., $\phi_{\mathbf{t}_3}$) is locally diffeomorphic to swallowtail SW at points (s_0, u_0) if and only if $\tau_1(s_0) \neq 0$, $\delta_{\mathbf{t}_2}(s_0) = 0$ (resp., $\delta_{\mathbf{t}_3}(s_0) = 0$), $\delta'_{\mathbf{t}_2}(s_0) \neq 0$ (resp., $\delta'_{\mathbf{t}_3}(s_0) \neq 0$) and

$$u_0 = \frac{\varkappa_1(s_0)}{\tau_1(s_0)\sqrt{(\varkappa_1(s_0))^2 + (\varkappa_2(s_0))^2}}$$
$$(\text{resp.}, \ u_0 = \frac{\varkappa_2(s_0)}{\tau_1(s_0)\sqrt{(\varkappa_1(s_0))^2 + (\varkappa_2(s_0))^2}}).$$

here; $C \times \mathbb{R} = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = u, a_2 = v^2, a_3 = v^3\}$ is the *cuspidaledge* and $SW = (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = u, \ a_2 = 3v^2 + uv^2, a_3 = 4v^3 + 2uv\}$ is the *swallowtail*, see Fig. 2.

Figure 2. Left figure represents the cuspidaledge $C \times \mathbb{R}$, while the right figure represents the swallowtail SW.

The proof of Theorem 3 can be easily given by using a similar method given in the proof of Theorem3.3 in [7].

4.2. DEVELOPABLE SURFACES WITH DIRECTOR $\overline{W}_2$

Using the unit vectors $\mathrm{t}_2(s)$ and $\overline{W}_2(s)$ we can define the unit vector $u_2(s) = \overline{W}_2(s) \times \mathrm{t}_2(s)$ at point $\alpha_2(s)$, for all $s \in I$. In this case, we obtain the rigth-handed orthonormal frame $\{\mathrm{t}_2(s), u_2(s), \overline{W}_2(s)\}$ of the spine curve $\alpha_2 = \int \mathrm{t}_2$ at point $\alpha_2(s)$. Note that $\{\mathrm{t}_2, u_2, \overline{W}_2\}$ is the Frenet-Serret frame of the curve α_2. *Frenet-Serret formulas* of the curve α_2 can be given in matrix form as

$$\begin{bmatrix} \mathrm{t}_2 \\ u_2 \\ \overline{W}_2 \end{bmatrix}' = \begin{bmatrix} 0 & \kappa_2 & 0 \\ -\kappa_2 & 0 & \tau_2 \\ 0 & -\tau_2 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{t}_2 \\ u_2 \\ \overline{W}_2 \end{bmatrix}, \tag{31}$$

where

$$\kappa_2 = \langle \mathrm{t}_2', u_2 \rangle = \sqrt{(\lambda_1)^2 + (\lambda_2)^2}, \;\; \tau_2 = \langle u_2', \overline{W}_2 \rangle = \frac{(\lambda_1)^2}{(\lambda_1)^2 + (\lambda_2)^2} \left(\frac{\lambda_2}{\lambda_1} \right)'$$

are, respectively, the *curvature* and *torsion functions* of α_2.

Using the unit vector fields t_1, t_3 and $\overline{W}_2$, we can define the following two ruled surfaces:

1. By the mapping

$$\varphi_{\mathbf{t}_1} : I \times \mathbb{R} \to \mathbb{R}^3$$

defined by

$$\varphi_{\mathrm{t}_1}(s,u) = \int \mathrm{t}_1(s) ds + u\overline{W}_2(s). \tag{32}$$

2. By the mapping

$$\varphi_{\mathbf{t}_3} : I \times \mathbb{R} \to \mathbb{R}^3$$

defined by

$$\varphi_{\mathbf{t}_3}(s,u) = \int \mathbf{t}_3(s) ds + u\overline{W}_2(s). \tag{33}$$

Proposition 4. *Ruled surfaces $\varphi_{t_1}(s,u) = \int t_1(s)ds + u\overline{W}_2(s)$ and $\varphi_{t_3}(s,u) = \int t_3(s)ds + u\overline{W}_2(s)$ are developable.*

The proof of Prop. 2 can be easily given by using a similar method given in the proof of Prop. 1. By using a similar method used in Subsect. 4.1, the invariants characterizing the singularities of the developable surfaces $\varphi_{\mathbf{t}_1}$ and $\varphi_{\mathbf{t}_3}$ will be obtained as

$$\eta_{\mathbf{t}_1}(s) = \frac{\varkappa_3(s)}{\sqrt{\left(\varkappa_1(s)\right)^2 + \left(\varkappa_3(s)\right)^2}} - \left(\frac{\varkappa_1(s)}{\tau_2(s)\sqrt{\left(\varkappa_1(s)\right)^2 + \left(\varkappa_3(s)\right)^2}} \right)', \tag{34}$$

$$\eta_{\mathbf{t}_3}(s) = \frac{\varkappa_1(s)}{\sqrt{\left(\varkappa_1(s)\right)^2 + \left(\varkappa_3(s)\right)^2}} + \left(\frac{\varkappa_3(s)}{\tau_2(s)\sqrt{\left(\varkappa_1(s)\right)^2 + \left(\varkappa_3(s)\right)^2}}\right)', \tag{35}$$

$$\tau_2(s) = -\varkappa_2(s) + \frac{\varkappa_1(s)\varkappa_3'(s) - \varkappa_1'(s)\varkappa_3(s)}{\left(\varkappa_1(s)\right)^2 + \left(\varkappa_3(s)\right)^2}, \tag{36}$$

where $\tau_2(s) \neq 0$. In addition, these invariants also determine the contour generator types of the base curves of the surfaces $\varphi_{\mathbf{t}_1}$ and $\varphi_{\mathbf{t}_3}$.

By the following theorem we can classify the developable surfaces $\varphi_{\mathbf{t}_1}$ and $\varphi_{\mathbf{t}_3}$ into cylindrical or conical surfaces:

Theorem 5. *For the developable surface $\varphi_{t_1}(s,u) = \int t_1(s)ds + u\overline{W}_2(s)$ (resp., $\varphi_{t_3}(s,u) = \int t_3(s)ds + u\overline{W}_2(s)$), given by Eq. (32) (resp., Eq. (33)), we have the following two conditions:*

1. The following are equivalent:
   (a) $\varphi_{\mathbf{t}_1}$ (resp., $\varphi_{\mathbf{t}_3}$) is a cylindrical surface.
   (b) $\tau_2(s) = 0$, for all $s \in I$.
   (c) Curve $\int \mathrm{t}_1(s)ds$ (resp., $\int \mathrm{t}_3(s)ds$) is a contour generator with respect to an orthogonal projection.

2. If $\tau_2(s) \neq 0$, then the following are equivalent:
   (a) $\varphi_{\mathbf{t}_1}$ (resp., $\varphi_{\mathbf{t}_3}$) is a conical surface.
   (b) $\eta_{\mathbf{t}_1}(s) = 0$ (resp., $\eta_{\mathbf{t}_3}(s) = 0$), for all $s \in I$.
   (c) Curve $\int \mathrm{t}_1(s)ds$ (resp., $\int \mathrm{t}_3(s)ds$) is a contour generator with respect to a central projection.

The proof of Theorem 4 can be easily given by using a similar method given in the proof of Theorem 2.

By the following theorem we give the non-singular points of the developable surfaces $\varphi_{\mathbf{t}_1}$ and $\varphi_{\mathbf{t}_3}$. For these surfaces we also give the local diffeomorphism conditions to cuspidal edge $C \times \mathbb{R}$ or to swallowtail SW using the invariants $\eta_{\mathbf{t}_1}(s), \eta_{\mathbf{t}_3}(s)$ and $\tau_2(s)$:

Theorem 6. *For the developable surface $\varphi_{t_1}(s,u) = \int t_1(s)ds + u\overline{W}_2(s)$ (resp., $\varphi_{t_3}(s,u) = \int t_3(s)ds + u\overline{W}_2(s)$), given by Eq. (32) (resp., Eq. (33)), we have the following three conditions:*

1. $\varphi_{\mathbf{t}_1}$ (resp., $\varphi_{\mathbf{t}_3}$) is non-singular at points (s_0, u_0) if and only if $\tau_2(s_0) \neq 0$ and

$$u_0 \neq -\frac{\varkappa_1(s_0)}{\tau_2(s_0)\sqrt{(\varkappa_1(s_0))^2 + (\varkappa_3(s_0))^2}}$$

$$\text{(resp.,}\quad u_0 \neq \frac{\varkappa_3(s_0)}{\tau_2(s_0)\sqrt{(\varkappa_1(s_0))^2 + (\varkappa_3(s_0))^2}}\text{).}$$

2. $\varphi_{\mathbf{t}_1}$ (resp., $\varphi_{\mathbf{t}_3}$) is locally diffeomorphic to cuspidal edge $C \times \mathbb{R}$ at points (s_0, u_0) if and only if $\tau_2(s_0) \neq 0$, $\eta_{\mathbf{t}_1}(s_0) \neq 0$ (resp., $\eta_{\mathbf{t}_3}(s_0) \neq 0$) and

$$u_0 = -\frac{\varkappa_1(s_0)}{\tau_2(s_0)\sqrt{(\varkappa_1(s_0))^2 + (\varkappa_3(s_0))^2}}$$

$$(\text{resp.},\ u_0 = \frac{\varkappa_3(s_0)}{\tau_2(s_0)\sqrt{(\varkappa_1(s_0))^2 + (\varkappa_3(s_0))^2}}).$$

3. $\varphi_{\mathbf{t}_1}$ (resp., $\varphi_{\mathbf{t}_3}$) is locally diffeomorphic to swallowtail SW at points (s_0, u_0) if and only if $\tau_2(s_0) \neq 0$, $\eta_{\mathbf{t}_1}(s_0) = 0$ (resp., $\eta_{\mathbf{t}_3}(s_0) = 0$), $\eta'_{\mathbf{t}_1}(s_0) \neq 0$ (resp., $\eta'_{\mathbf{t}_3}(s_0) \neq 0$) and

$$u_0 = -\frac{\varkappa_1(s_0)}{\tau_2(s_0)\sqrt{(\varkappa_1(s_0))^2 + (\varkappa_3(s_0))^2}}$$

$$(\text{resp.},\ u_0 = \frac{\varkappa_3(s_0)}{\tau_2(s_0)\sqrt{(\varkappa_1(s_0))^2 + (\varkappa_3(s_0))^2}}).$$

The proof of Theorem 6 can be easily given by using a similar method given in the proof of Theorem 3.3 in [7].

4.3. DEVELOPABLE SURFACES WITH DIRECTOR $\overline{W}_3$

Using the unit vectors $\mathrm{t}_3(s)$ and $\overline{W}_3(s)$ we can define the unit vector $u_3(s) = \overline{W}_3(s) \times \mathrm{t}_3(s)$ at point $\alpha_3(s)$, for all $s \in I$. In this case, we obtain the rigth-handed orthonormal frame $\{\mathrm{t}_3(s), u_3(s), \overline{W}_3(s)\}$ of the spine curve $\alpha_3 = \int \mathrm{t}_3$ at point $\alpha_3(s)$. Note that $\{\mathrm{t}_3, u_3, \overline{W}_3\}$ is the Frenet-Serret frame of the curve α_3. *Frenet-Serret formulas* of the curve α_3 can be given in matrix form as

$$\begin{bmatrix} \mathrm{t}_3 \\ u_3 \\ \overline{W}_3 \end{bmatrix}' = \begin{bmatrix} 0 & \kappa_3 & 0 \\ -\kappa_3 & 0 & \tau_3 \\ 0 & -\tau_3 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{t}_3 \\ u_3 \\ \overline{W}_3 \end{bmatrix}, \tag{37}$$

where

$$\kappa_3 = \langle \mathrm{t}'_3, u_3 \rangle = \sqrt{(\mu_1)^2 + (\mu_2)^2},\ \ \tau_3 = \langle u'_3, \overline{W}_3 \rangle = \frac{(\mu_1)^2}{(\mu_1)^2 + (\mu_2)^2}\left(\frac{\mu_2}{\mu_1}\right)'$$

are, respectively, the *curvature* and *torsion functions* of α_3.

Using the unit vector fields t_1, t_2 and $\overline{W}_3$, we can define the following two ruled surfaces:

1. By the mapping

$$\psi_{\mathrm{t}_1} : \mathrm{I} \times \mathbb{R} \to \mathbb{R}^3$$

defined by

$$\psi_{\mathrm{t}_1}(\mathrm{s}, \mathrm{u}) = \int \mathrm{t}_1(\mathrm{s})\mathrm{ds} + \mathrm{u}\overline{\mathrm{W}}_3(\mathrm{s}). \tag{38}$$

2. By the mapping

$$\psi_{t_2}: I \times \mathbb{R} \to \mathbb{R}^3$$

defined by

$$\psi_{t_2}(s,u) = \int t_2(s)ds + u\overline{W}_3(s). \tag{39}$$

Proposition 7. *Ruled surfaces $\psi_{t_1}(s,u) = \int t_1(s)ds + u\overline{W}_3(s)$ and $\psi_{t_2}(s,u) = \int t_2(s)ds + u\overline{W}_3(s)$ are developable.*

The proof of Prop. 3 can be easily given by using a similar method given in the proof of Prop. 1.

Using again a similar method used in Subsect. 4.1, the invariants characterizing the singularities of the developable surfaces $\psi_{\mathbf{t}_1}$ and $\psi_{\mathbf{t}_2}$ will be obtained as

$$\zeta_{\mathbf{t}_1}(s) = \frac{\varkappa_3(s)}{\sqrt{\left(\varkappa_2(s)\right)^2 + \left(\varkappa_3(s)\right)^2}} - \left(\frac{\varkappa_2(s)}{\tau_3(s)\sqrt{\left(\varkappa_2(s)\right)^2 + \left(\varkappa_3(s)\right)^2}}\right)', \tag{40}$$

$$\zeta_{\mathbf{t}_2}(s) = \frac{-\varkappa_2(s)}{\sqrt{\left(\varkappa_2(s)\right)^2 + \left(\varkappa_3(s)\right)^2}} - \left(\frac{\varkappa_3(s)}{\tau_3(s)\sqrt{\left(\varkappa_2(s)\right)^2 + \left(\varkappa_3(s)\right)^2}}\right)', \tag{41}$$

$$\tau_3(s) = \varkappa_1(s) + \frac{\varkappa_2(s)\varkappa_3'(s) - \varkappa_2'(s)\varkappa_3(s)}{\left(\varkappa_2(s)\right)^2 + \left(\varkappa_3(s)\right)^2}, \tag{42}$$

where $\tau_3(s) \neq 0$. In addition, these invariants also determine the contour generator types of the base curves of the surfaces $\psi_{\mathbf{t}_1}$ and $\psi_{\mathbf{t}_2}$.

By the following theorem we can classify the developable surfaces $\psi_{\mathbf{t}_1}$ and $\psi_{\mathbf{t}_2}$ into cylindrical or conical surfaces:

Theorem 8. *For the developable surface $\psi_{t_1}(s,u) = \int t_1(s)ds + u\overline{W}_3(s)$ (resp., $\psi_{t_2}(s,u) = \int t_2(s)ds + u\overline{W}_3(s)$), given by Eq. (38) (resp., Eq. (39)), we have the following two conditions:*

1. The following are equivalent:
   (a) ψ_{t_1} (resp., ψ_{t_2}) is a cylindrical surface.
   (b) $\tau_3(s) = 0$, for all $s \in I$.
   (c) Curve $\int t_1(s)ds$ (resp., $\int t_2(s)ds$) is a contour generator with respect to an orthogonal projection.

2. If $\tau_3(s) \neq 0$, then the following are equivalent:
   (a) ψ_{t_1} (resp., ψ_{t_2}) is a conical surface.
   (b) $\zeta_{t_1}(s) = 0$ (resp., $\zeta_{t_2}(s) = 0$), for all $s \in I$.
   (c) Curve $\int t_1(s)ds$ (resp., $\int t_2(s)ds$) is a contour generator with respect to a central projection.

The proof of Theorem 6 can be easily given by using a similar method given in the proof of Theorem 2.

By the following theorem we give the non-singular points of the developable surfaces $\psi_{\mathbf{t}_1}$ and $\psi_{\mathbf{t}_2}$. For these surfaces we also give the local diffeomorphism conditions to cuspidal edge $C \times \mathbb{R}$ or to swallowtail SW using the invariants $\zeta_{\mathbf{t}_1}(s), \zeta_{\mathbf{t}_2}(s)$ and $\tau_3(s)$:

Theorem 9. *For the developable surface $\psi_{t_1}(s,u) = \int t_1(s)ds + u\overline{W}_3(s)$ (resp., $\psi_{t_2}(s,u) = \int t_2(s)ds + u\overline{W}_3(s)$), given by Eq. (38) (resp., Eq. (39)), we have the following three conditions:*

1. $\psi_{\mathbf{t}_1}$ (resp., $\psi_{\mathbf{t}_2}$) is non-singular at points (s_0, u_0) if and only if $\tau_3(s_0) \neq 0$ and

$$u_0 \neq -\frac{\varkappa_2(s_0)}{\tau_3(s_0)\sqrt{(\varkappa_2(s_0))^2 + (\varkappa_3(s_0))^2}}$$

$$(\text{resp.,} \quad u_0 \neq -\frac{\varkappa_3(s_0)}{\tau_3(s_0)\sqrt{(\varkappa_2(s_0))^2 + (\varkappa_3(s_0))^2}}).$$

2. $\psi_{\mathbf{t}_1}$ (resp., $\psi_{\mathbf{t}_2}$) is locally diffeomorphic to cuspidal edge $C \times \mathbb{R}$ at points (s_0, u_0) if and only if $\tau_3(s_0) \neq 0$, $\zeta_{\mathbf{t}_1}(s_0) \neq 0$ (resp., $\zeta_{\mathbf{t}_2}(s_0) \neq 0$) and

$$u_0 = -\frac{\varkappa_2(s_0)}{\tau_3(s_0)\sqrt{(\varkappa_2(s_0))^2 + (\varkappa_3(s_0))^2}}$$

$$(\text{resp.,} \quad u_0 = -\frac{\varkappa_3(s_0)}{\tau_3(s_0)\sqrt{(\varkappa_2(s_0))^2 + (\varkappa_3(s_0))^2}}).$$

3. $\psi_{\mathbf{t}_1}$ (resp., $\psi_{\mathbf{t}_2}$) is locally diffeomorphic to swallowtail SW at points (s_0, u_0) if and only if $\tau_3(s_0) \neq 0$, $\zeta_{\mathbf{t}_1}(s_0) = 0$ (resp., $\zeta_{\mathbf{t}_2}(s_0) = 0$), $\zeta'_{\mathbf{t}_1}(s_0) \neq 0$ (resp., $\zeta'_{\mathbf{t}_2}(s_0) \neq 0$) and

$$u_0 = -\frac{\varkappa_2(s_0)}{\tau_3(s_0)\sqrt{(\varkappa_2(s_0))^2 + (\varkappa_3(s_0))^2}}$$

$$(\text{resp.,} \quad u_0 = -\frac{\varkappa_3(s_0)}{\tau_3(s_0)\sqrt{(\varkappa_2(s_0))^2 + (\varkappa_3(s_0))^2}}).$$

The proof of Theorem 9 can be easily given by using a similar method given in the proof of Theorem. 3.3 in [7].

5. APPLICATIONS

In this section, some applications are given by using Darboux, Alternative and Legendre Frenet frames instead of the moving frame.

5.1. DARBOUX FRAME

Let us consider a surface $\mathbb{M} = X(U)$ locally embedded by $X: U \subset \mathbb{R}^2 \to \mathbb{R}^3$, and let $\overline{\alpha}: I \subset \mathbb{R} \to U$ be a regular plane curve, where U is an open set of $\mathbb{R}^2$, $\overline{\alpha}(t) = (u(t), v(t))$ and I is an open interval of $\mathbb{R}$. Then we have a regular space curve $\alpha = X \circ \overline{\alpha}: I \to \mathbb{M} \subset \mathbb{R}^3$ on the surface $\mathbb{M}$. The *unit normal vector field* on $\mathbb{M}$ is

$$n(u,v) = \frac{(\partial X(u,v)/\partial u) \times (\partial X(u,v)/\partial v)}{\|(\partial X(u,v)/\partial u) \times (\partial X(u,v)/\partial v)\|}.$$

Since α is a regular space curve, we adopt the arc-length parameter and denote $\alpha(s) = X(u(s), v(s))$. Then, we have the *unit tangent vector field* $\mathrm{t}(s) = \alpha'(s)$, where $\alpha'(s) = (d\alpha/ds)(s)$. We also have $n_\alpha(s) = n \circ \overline{\alpha}(s)$, which is the *unit normal vector field* of $\mathbb{M}$ along $\alpha(s)$. Moreover, we define $b(s) = n_\alpha(s) \times \mathrm{t}(s)$. Then, we have an *orthonormal frame* $\{\mathrm{t}(s), b(s), n_\alpha(s)\}$ along $\alpha(s)$, which is called the *Darboux frame* along $\alpha(s)$.

Frenet-Serret type formulas of the spine curve α can be given in matrix form as

$$\begin{bmatrix} \mathrm{t} \\ b \\ n_\alpha \end{bmatrix}' = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} \mathrm{t} \\ b \\ n_\alpha \end{bmatrix},$$

where $\kappa_g = \langle \mathrm{t}', b \rangle$, $\kappa_n = \langle \mathrm{t}', n_\alpha \rangle$, $\tau_g = \langle b', n_\alpha \rangle$ are the *geodesic curvature*, the *normal curvature* and the *geodesic torsion functions* of α, respectively. For further information about the concept Darboux frame see [5, 8].

If we take the Darboux frame $\{\mathrm{t}(s), b(s), n_\alpha(s)\}$ as the moving frame $\{\mathrm{t}_1(s), \mathrm{t}_2(s), \mathrm{t}_3(s)\}$ (i.e., $\mathrm{t} = \mathrm{t}_1$, $b = \mathrm{t}_2$, $n_\alpha = \mathrm{t}_3$), then we can give the following two conditions:

1. Developable surface $\psi_{\mathrm{t}_1}(s,u) = \int \mathrm{t}_1(s)ds + u\overline{W}_3(s)$, given by Eq. (38), corresponds to the *osculating developable surface* of $\mathbb{M}$ along α, which is given in [8] by

$$OD_\alpha(s,u) = \alpha(s) + u\overline{D}_0(s) = \alpha(s) + u\frac{\tau_g(s)\mathbf{t}(s) - \kappa_n(s)\, b(s)}{\sqrt{(\tau_g(s))^2 + (\kappa_n(s))^2}}.$$

The following equations can be given:

$$\alpha(s) = \int \mathrm{t}_1(s)ds, \overline{W}_3(s) = \overline{D}_0(s) = \frac{\tau_g(s)\mathrm{t}(s) - \kappa_n(s)\, b(s)}{\sqrt{(\tau_g(s))^2 + (\kappa_n(s))^2}},$$

$$\tau_g(s) = \varkappa_3(s), \ \ \kappa_n(s) = \varkappa_2(s), \ \ \mathrm{t}(s) = \mathrm{t}_1(s), \ \ b(s) = \mathrm{t}_2(s),$$

where $\overline{D}_0(s)$ is the *normalized osculating Darboux vector* (i.e., spherical image of osculating Darboux vector) along α. The cylindrical and conical cases for OD_α can be given by Theorem 8, and the conditions to being locally diffeomorphic to cuspidal edge $C \times \mathbb{R}$ or to swallowtail SW for OD_α can be given by Theorem 9. The contour generator cases for the base curve α of OD_α can also be given by Theorem 8.

2. Developable surface $\varphi_{\mathrm{t}_1}(s,u) = \int \mathrm{t}_1(s)ds + u\overline{W}_2(s)$, given by Eq. (32), corresponds to the *normal developable surface* of $\mathbb{M}$ along $\alpha(s)$, which is given in [5] by

$$ND_\alpha(s,u) = \alpha(s) + u\overline{D}_r(s) = \alpha(s) + u\frac{\tau_g(s)\mathrm{t}(s) + \kappa_g(s)n_\alpha(s)}{\sqrt{(\tau_g(s))^2 + (\kappa_g(s))^2}}.$$

The following equations can be given:

$$\alpha(s) = \int \mathrm{t}_1(s)ds, \overline{W}_2(s) = \overline{D}_r(s) = \frac{\tau_g(s)\mathrm{t}(s) + \kappa_g(s)n_\alpha(s)}{\sqrt{(\tau_g(s))^2 + (\kappa_g(s))^2}},$$

$$\tau_g(s) = \varkappa_3(s), \kappa_g(s) = \varkappa_1(s), \mathrm{t}(s) = \mathrm{t}_1(s), n_\alpha(s) = \mathrm{t}_3(s),$$

where $\overline{D}_r(s)$ is the *normalized normal Darboux vector* (i.e., spherical image of normal Darboux vector) along α. The cylindrical and conical cases for ND_α can be given by Theorem 5, and the conditions to being locally diffeomorphic to cuspidal edge $C \times \mathbb{R}$ or to swallowtail SW for ND_α can be given by Theorem 6. The contour generator cases for the base curve α of ND_α can also be given by Theorem 5.

5.2. ALTERNATIVE FRAME

Similar to Frenet-Serret frame $\{\mathrm{t}(s), \mathrm{n}(s), \mathrm{b}(s)\}$ along a spine curve $\alpha(s)$ in $\mathbb{R}^3$, we can define an orthonormal frame whose three orthonormal axis vectors are defined at point $\alpha(s)$ as

$$\mathrm{n}(s) = \frac{\mathrm{t}'(s)}{\|\mathrm{t}'(s)\|},\ \ \mathrm{c}(s) = \frac{\mathrm{n}'(s)}{\|\mathrm{n}'(s)\|},\ \ \overline{\mathrm{w}}(s) = \mathrm{n}(s) \times \mathrm{c}(s) = \frac{\tau(s)\mathrm{t}(s) + \kappa(s)\mathrm{b}(s)}{\sqrt{(\tau(s))^2 + (\kappa(s))^2}}$$

for all $s \in I$. The vectors $\mathrm{n}(s)$, $\mathrm{c}(s)$ and $\overline{\mathrm{w}}(s)$ are called, respectively, the *principal normal vector*, the *derivative of the principal normal vector* and the *normalized Darboux vector* of α at $\alpha(s)$. The normalized Darboux vector $\overline{\mathrm{w}}$ is the spherical image of the Darboux vector given by Eq. (6). The set $\{\mathrm{n}(s), \mathrm{c}(s), \overline{\mathrm{w}}(s)\}$ is called the *alternative frame* of the curve α at point $\alpha(s)$, for all $s \in I$.

Frenet-Serret formulas of the spine curve $\varsigma = \int \mathrm{n}$ can be given in matrix form as

$$\begin{bmatrix} \mathrm{n} \\ \mathrm{c} \\ \overline{\mathrm{w}} \end{bmatrix}' = \begin{bmatrix} 0 & f & 0 \\ -f & 0 & g \\ 0 & -g & 0 \end{bmatrix} \begin{bmatrix} \mathrm{n} \\ \mathrm{c} \\ \overline{\mathrm{w}} \end{bmatrix},$$

where $f = \sqrt{\kappa^2 + \tau^2}$, $g = (\kappa^2/((\kappa^2 + \tau^2))(\tau/\kappa)'$ are, respectively, the *curvature* and *torsion functions* of ς, see [16].

If we take the alternative frame $\{\mathrm{n}(s), \mathrm{c}(s), \overline{\mathrm{w}}(s)\}$ as the moving frame $\{\mathrm{t}_1(s)$, $\mathrm{t}_2(s)$, $\mathrm{t}_3(s)\}$, then the developable surface $\varphi_{\mathrm{t}_1}(s,u) = \int \mathrm{t}_1(s)ds + u\overline{W}_2(s)$, given by Eq. (32), can be represented using the alternative frame as

$$\varphi_{\mathrm{t}_1}(s,u) = \int \mathrm{n}(s)ds + u\overline{D}_1(s)$$

while the developable surface $\varphi_{\mathrm{t}_3}(s,u)=\int \mathrm{t}_3(s)ds+u\overline{W}_2(s)$, given by Eq. (33), can be represented using the alternative frame as

$$\varphi_{\mathrm{t}_3}(s,u)=\int \overline{\mathrm{w}}(s)ds+u\overline{D}_1(s).$$

The following equations can be given:

$$\overline{W}_2(s)=\overline{D}_1(s)=\frac{g(s)\mathrm{n}(s)+f(s)\overline{\mathrm{w}}(s)}{\sqrt{(g(s))^2+(f(s))^2}},$$
$$\mathrm{t}_1(s)=\mathrm{n}(s),\mathrm{t}_3(s)=\overline{\mathrm{w}}(s),$$

where $\overline{D}_1(s)$ is the *normalized Darboux vector* (i.e., spherical image of Darboux vector) along ς associated with alternative frame. Thus, the cylindrical and conical cases for the surfaces $\varphi_{\mathrm{t}_1}(s,u)=\int \mathrm{n}(s)ds+u\overline{D}_1(s)$ and $\varphi_{\mathrm{t}_3}(s,u)=\int \overline{\mathrm{w}}(s)ds+u\overline{D}_1(s)$ can be given by Theorem 5, and the conditions to being locally diffeomorphic to cuspidal edge $C\times\mathbb{R}$ or to swallowtail SW for these surfaces can be given by Theorem 6.

5.3. LEGENDRE-FRENET FRAME

Let $\gamma: I\subset\mathbb{R}\to\mathbb{S}^2$ and $\nu: I\to\mathbb{S}^2$ be smooth curves satisfying $<\gamma'(t),\nu(t)>=0$ for all $t\in I$, and let us take the set

$$\Delta=\{(x,y)\in\mathbb{S}^2\times\mathbb{S}^2:<x,y>=0\}.$$

Then, we call the pair $(\gamma,\nu): I\to\Delta$ a *Legendre curve*. Taken $\mu(t)=\gamma(t)\times\nu(t)$, then $\mu(t)\in\mathbb{S}^2$, $<\gamma(t),\mu(t)>=0$ and $<\nu(t),\mu(t)>=0$. In this case, we have a moving frame $\{\gamma(t),\nu(t),\mu(t)\}$ known as the *Legendre-Frenet frame* along the spine curve $\int\gamma$. This frame is also a RMF of the spine curve $\int\mu$, which can be denoted by RMF_μ, see [4].

Frenet-Serret type formulas of the spine curve $\int\gamma$ associated with Legendre-Frenet frame can be given in matrix form as

$$\begin{bmatrix}\gamma\\ \nu\\ \mu\end{bmatrix}'=\begin{bmatrix}0&0&m\\0&0&n\\-m&-n&0\end{bmatrix}\begin{bmatrix}\gamma\\ \nu\\ \mu\end{bmatrix}, \tag{43}$$

where $m=\langle\gamma',\mu\rangle$, $n=\langle\nu',\mu\rangle$. The pair (m,n) is called the *Legendre curvature* of Legendre curve (γ,ν), see [6, 17].

If we take the Legendre-Frenet frame $\{\gamma(t),\nu(t),\mu(t)\}$ as the moving frame $\{\mathrm{t}_1(s),\mathrm{t}_2(s),\mathrm{t}_3(s)\}$, then the developable surface $\psi_{\mathrm{t}_1}(s,u)=\int \mathrm{t}_1(s)ds+u\overline{W}_3(s)$, given by Eq. (38), can be represented using the Legendre-Frenet frame as

$$\psi_{\mathbf{t}_1}(s,u)=\int\gamma(s)ds+u\overline{D}_2(s)$$

while the developable surface $\psi_{\mathrm{t}_2}(s,u)=\int \mathrm{t}_2(s)ds+u\overline{W}_3(s)$, given by Eq. (39), can be represented using the Legendre-Frenet frame as

$$\psi_{\mathbf{t}_1}(s,u)=\int\nu(s)ds+u\overline{D}_2(s).$$

The following equations can be given:

$$\overline{W}_3(s)=\overline{D}_2(s)=\frac{n(s)\gamma(s)-m(s)\nu(s)}{\sqrt{(n(s))^2+(-m(s))^2}},$$
$$\mathrm{t}_1(s)=\gamma(s),\mathrm{t}_2(s)=\nu(s),$$

where $\overline{D}_2(s)$ is the *normalized Darboux vector* (i.e., spherical image of Darboux vector) along $\int\gamma$ associated with Legendre-Frenet frame. Thus, the cylindrical and conical cases for the surfaces $\psi_{\mathbf{t}_1}(s,u)=\int\gamma(s)ds+u\overline{D}_2(s)$ and $\psi_{\mathbf{t}_1}(s,u)=\int\nu(s)ds+u\overline{D}_2(s)$ can be given by Theorem 8, and the conditions to being locally diffeomorphic to cuspidal edge $C\times\mathbb{R}$ or to swallowtail SW for these surfaces can be given by Theorem 9.

Example 10. Let $\gamma: I=[0,2\pi)\subset\mathbb{R}\to\mathbb{S}^2$ and $\nu: I\to\mathbb{S}^2$ be smooth curves given by

$$\gamma(t)=\tfrac{1}{4}(3\cos(t)-\cos(3t),3\sin(t)-\sin(3t),2\sqrt{3}\cos(t)),$$

$$\nu(s)=\tfrac{1}{4}(3\sin(t)+\sin(3t),-3\cos(t)-\cos(3t),-2\sqrt{3}\sin(t)).$$

Then, we obtain $<\gamma(t),\nu(t)>=0$ and $<\gamma'(t),\nu(t)>=0$. This means that the pair $(\gamma,\nu): I\to\Delta\subset\mathbb{S}^2\times\mathbb{S}^2$ is a *Legendre curve*. Taken $\mu(t)=\gamma(t)\times\nu(t)$, we get $\mu(t)=(\sqrt{3}\cos(2t),\sqrt{3}\sin(2t),-1)/2$ for all $t\in I$. The *Legendre curvature* of Legendre curve (γ,ν) is obtained as

$$(m(t),n(t))=\sqrt{3}(\sin(t),\cos(t)).$$

Spherical image of Darboux vector along the spine curve $\int\gamma$ associated with Legendre-Frenet frame $\{\gamma(t),\nu(t),\mu(t)\}$ will be obtained as

$$\overline{D}_2(t)=\frac{n(t)\gamma(t)-m(t)\nu(t)}{\sqrt{(n(t))^2+(-m(t))^2}}=\frac{1}{2}\big(\cos(2t),\sin(2t),\sqrt{3}\big).$$

If we take the Legendre-Frenet frame as the moving frame $\{\mathrm{t}_1(s),\mathrm{t}_2(s),\mathrm{t}_3(s)\}$ along the unit spine curve $\int\gamma$, then the developable surface $\psi_{\mathrm{t}_1}(s,u)=\int\mathrm{t}_1(s)ds+u\overline{W}_3(s)$, given by Eq. (38), can be represented using the Legendre-Frenet frame as

$$\psi_{\mathbf{t}_1}(s,u)=\int\gamma(s)ds+u\overline{D}_2(s)$$
$$=\frac{1}{4}\left(3\sin(s)-\frac{1}{3}\sin(3s),-3\cos(s)+\frac{1}{3}\cos(3s),2\sqrt{3}\sin(s)\right)$$
$$+\frac{u}{2}\big(\cos(2s),\sin(2s),\sqrt{3}\big),$$

see Fig. 3, while the developable surface $\psi_{\mathbf{t}_2}(s,u)=\int\mathbf{t}_2(s)ds+u\overline{W}_3(s)$, given by Eq. (39), can be represented using the Legendre-Frenet frame as

$$\psi_{\mathbf{t}_2}(s,u)=\int\nu(s)ds+u\overline{D}_2(s)$$

$$= \frac{1}{4}\left(-3\cos(s) - \frac{1}{3}\cos(3s), -3\sin(s) - \frac{1}{3}\sin(3s), 2\sqrt{3}\cos(s)\right) + \frac{u}{2}\left(\cos(2s), \sin(2s), \sqrt{3}\right),$$

for all $s \in I$, see Fig. 4. The following equations can be given:

$$\overline{W}_3(s) = \overline{D}_2(s),\ \ \mathrm{t}_1(s) = \gamma(s),\ \ \mathrm{t}_2(s) = \nu(s), n(s) = \varkappa_1(s), m(s) = \varkappa_2(s). \tag{44}$$

Striction curves of the surfaces $\psi_{\mathbf{t}_1}$ and $\psi_{\mathbf{t}_2}$ are obtained, respectively, as

$$\psi^*_{\mathbf{t}_1}(s) = (\frac{1}{2}\sin(s) + \frac{1}{6}\sin(3s), -\frac{1}{2}\cos(s) - \frac{1}{6}\cos(3s), \sqrt{3}\sin(s)),$$

see Fig. 3, and

$$\psi^*_{\mathbf{t}_2}(s) = (-\frac{1}{2}\cos(s) + \frac{1}{6}\cos(3s), -\frac{1}{2}\sin(s) + \frac{1}{6}\sin(3s), \sqrt{3}\cos(s)),$$

see Fig. 4. We can write the Frenet-Serret formulas given by Eq. (37) similar to the Frent-Serret type formulas given by Eq. (43) as

$$\begin{bmatrix} \overline{W}_3 \\ \mathrm{t}_3 \\ u_3 \end{bmatrix}' = \begin{bmatrix} 0 & 0 & -\tau_3 \\ 0 & 0 & \kappa_3 \\ \tau_3 & -\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} \overline{W}_3 \\ \mathrm{t}_3 \\ u_3 \end{bmatrix}.$$

This means that the continous functions $\varkappa_1$, $\varkappa_2$ and $\varkappa_3$ used in Eqs. (40), (41) and (42) will be obatined as

$$\varkappa_1 = 0,\ \ \varkappa_2 = -\tau_3,\ \ \varkappa_3 = \kappa_3.$$

Therefore, the invariants characterizing the singularities of the developable surfaces $\psi_{\mathbf{t}_1}$ and $\psi_{\mathbf{t}_2}$, which are given by Eqs. (40), (41) and (42), can be given as

$$\zeta_{\mathbf{t}_1}(s) = \frac{\kappa_3(s)}{\sqrt{(-\tau_3(s))^2 + (\kappa_3(s))^2}} - \frac{\tau_3(s)\tau_3'(s) + \kappa_3(s)\kappa_3'(s)}{((-\tau_3(s))^2 + (\kappa_3(s))^2)^{3/2}},$$

$$\zeta_{\mathbf{t}_2}(s) = \frac{\tau_3(s)}{\sqrt{(-\tau_3(s))^2 + (\kappa_3(s))^2}} - \left(\frac{\kappa_3(s)}{\tau_3(s)\sqrt{(-\tau_3(s))^2 + (\kappa_3(s))^2}}\right)',$$

$$\tau_3(s) = \frac{-\tau_3(s)\kappa_3'(s) + \tau_3'(s)\kappa_3(s)}{(-\tau_3(s))^2 + (\kappa_3(s))^2},$$

where $\tau_3(s) \neq 0$, for all $s \in I$. From Eq. (44), we know that $\varkappa_2(s) = m(s) = \sqrt{3}\sin(s)$ and $\varkappa_3(s) = n(s) = \sqrt{3}\cos(s)$ which yield

$$\zeta_{\mathbf{t}_1}(s) = \cos(s),\ \ \zeta_{\mathbf{t}_2}(s) = -2\sin(s),\ \tau_3(s) = -1.$$

Since $\tau_3(s) = -1 \neq 0$ for all $s \in I$, according to the second case of Condition 1 of Theorem 8, surfaces $\psi_{\mathbf{t}_1}$ and $\psi_{\mathbf{t}_2}$ are not cylindrical. And since $\zeta_{\mathbf{t}_1}(s) = \cos(s)$ and $\zeta_{\mathbf{t}_2}(s) = -2\sin(s)$ are not equal to zero for all $s \in I$, according to the second case of Condition 2 of Theorem 8, surfaces $\psi_{\mathbf{t}_1}$ and $\psi_{\mathbf{t}_2}$ are not conical. We can calculate that

$$u_0 = -\frac{\varkappa_2(s_0)}{\tau_3(s_0)\sqrt{(\varkappa_2(s_0))^2 + (\varkappa_3(s_0))^2}} = \frac{1}{\sqrt{3}}$$
$$(\text{resp.,}\quad u_0 = -\frac{\varkappa_3(s_0)}{\tau_3(s_0)\sqrt{(\varkappa_2(s_0))^2 + (\varkappa_3(s_0))^2}} = \cos(s_0)),$$

and this means that the singular points of the surface $\psi_{\mathbf{t}_1}$ (resp., $\psi_{\mathbf{t}_2}$) are $(s, 1/\sqrt{3})$ (resp., $(s, \cos(s))$), for all $s \in I$. According to the first condition of Theorem 9, the non-singular points of the surface $\psi_{\mathbf{t}_1}$ (resp., $\psi_{\mathbf{t}_2}$) are

$$(s, \mathbb{R} - \{1/\sqrt{3}\})$$
$$(\text{resp.,}\quad (s, \mathbb{R} - \{\cos(s)\})),$$

for all $s \in I$. According to the second condition of Theorem 9, surface $\psi_{\mathbf{t}_1}$ (resp., $\psi_{\mathbf{t}_2}$) is locally diffeomorphic to cuspidal edge $C \times \mathbb{R}$ at points

$$(\mathbb{R} - \{\pm\frac{\pi}{2} + 2\pi k\}, \frac{1}{\sqrt{3}})$$
$$(\text{resp.,}\quad (s_0, \cos(s_0)), \qquad \text{where} \quad s_0 = \mathbb{R} - (\{2\pi k\} \cup \{\pi + 2\pi k\})),$$

for integer k. According to the third condition of Theorem 9, surface $\psi_{\mathbf{t}_1}$ (resp., $\psi_{\mathbf{t}_2}$) is locally diffeomorphic to cuspidal edge SW at points

$$(\pm\frac{\pi}{2} + 2\pi k, \frac{1}{\sqrt{3}})$$
$$(\text{resp.,}\quad (s_0, \cos(s_0)), \quad \text{where} \quad s_0 = \{\frac{\pi}{2} + 2\pi k\} \cup \{\frac{3\pi}{2} + 2\pi k\}),$$

for integer k.

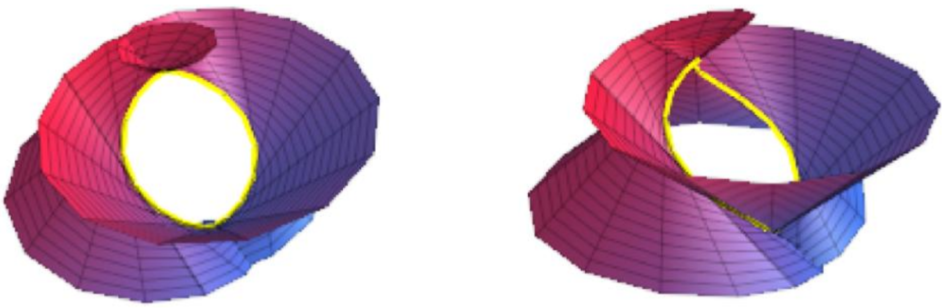

Figure 3. Developable surface $\psi_{\mathbf{t}_1}$ from two different viewing positions, where the yellow curves represent the striction curves(i.e., locus of singular points) of $\psi_{\mathbf{t}_1}$.

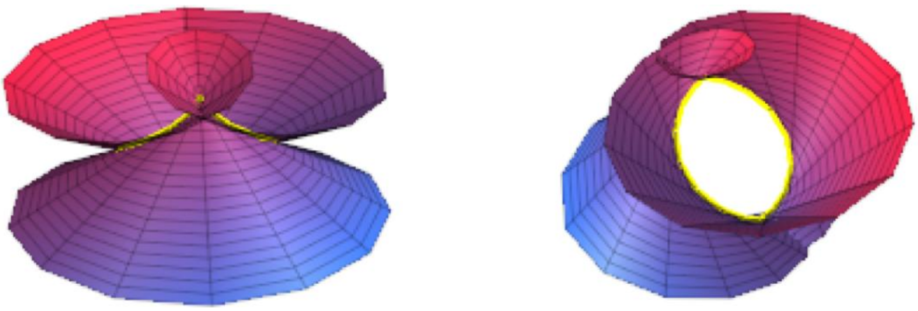

Figure 4. Developable surface $\psi_{\mathbf{t}_2}$ from two different viewing positions, where the yellow curves represent the striction curves(i.e., locus of singular points) of $\psi_{\mathbf{t}_2}$.

Conflict of interest *On behalf of all authors, the corresponding author states that there is no conflict of interest.*

6. CONCLUSION

In this study, six different developable surfaces are introduced by taken the integral curves of the coordinate axes of the moving frame as their base curves and the spherical images of the Darboux vector fields of RMFs as their director curves. We have characterized the singularities of these surfaces. We have also given the conditions to be contour generators of the base curves of these surfaces.

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