ORIGINAL PAPER

QUASI FOCAL CURVES OF ADJOINT CURVES OF TIMELIKE CURVES IN 3D MINKOWSKI SPACE

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Abstract. *In this work, we obtain new results by examining the adjoint curve and focal curve, which are examples of associated curves. In particular, we examine the focal curve of the adjoint curve of a timelike curve we have described with quasi-frame(Q-frame) elements in 3D Minkowski space M_1^3. We first characterize the focal curves of adjoint curves by considering the Q-frame in M_1^3. Next, we determine the focal curvatures of this curve. In the last section, we give some results by obtaining the focal curve in M_1^3 in terms of principal curve and adjoint curve.*

Keywords: *quasi frame, 3D Minkowski space, adjoint curve, focal curve, focal curvature.*

1. INTRODUCTION

The concept of the associated curve basically reveals the relationship between two curves, which is created with the help of a master curve. the differential geometry of curves and surfaces are reviewed in some aspects [1-14]. In addition, it has an important place in the literature in terms of its contribution to the investigation of the behavior of these two curves and the investigation of the surfaces that can be formed with these curves. Evolute-involute mates, Mannheim pairs, Bertrand mates, integral curves, and spherical images are the most common examples of associated curves. Integral curves, which will be the subject of our study, are very useful in terms of geometric expressions and solutions of related problems due to their contribution to the solutions of differential equations [15-19]. The adjoint curve, which is an important example of integral curves, is the associated curve defined as the integral of binormal vector of main curve with respect to an s parameter [20-22].

Another type of curve defined with the help of a principal curve is the focal curve. Let any unit velocity curve λ be defined, the focal curve λ_F of λ is formed by the center points of spheres oscillating along λ. Since the center of any of these spheres is in the plane normal to λ at the point of tangent, the focal curve is given by

$$\lambda_F(s) = (\lambda + \varphi_1\mathbf{N} + \varphi_2\mathbf{B})(s).$$

Here $\mathbf{N}$ and $\mathbf{B}$ vectors are normal and binormal elements of the Frenet-Serret(F-S) frame, s is arc lenght parameter and φ_1, φ_2 are the focal curvatures of λ [23, 24].

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In the light of these reminders, the idea of defining a new curve formed by a sphere making tangential oscillations to a curve that is an adjoint of the original curve is the main idea of our article. Therefore, in our study, we obtain new results by examining the focal curve under Q-frame of the adjoint curve of a timelike curve that we will choose in 3D Minkowski space. We first characterize the focal curves of adjoint curves by considering the Q-frame in $\mathbb{M}_1^3$. Next, we determine the focal curvatures of this curve. Finally, we present some results by obtaining the focal curve in $\mathbb{M}_1^3$ space in terms of principal curve and adjoint curve.

2. MATERIALS AND METHODS

In this part, we present Q-frame elements for a timelike curve in $\mathbb{M}_1^3$ and some fundamental informations.

Let $\lambda = \lambda(s)$ be a regular space curve. Then, F-S formulas are given as

$$\begin{bmatrix} \nabla_s \mathbf{T} \\ \nabla_s \mathbf{N} \\ \nabla_s \mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix},$$

and according to the arc length parameter $s,$ these formulas are computed as

$$\mathbf{T} = \lambda^{'}(s), \quad \mathbf{N} = \frac{\lambda^{''}(s)}{\left\| \lambda^{''}(s) \right\|}, \quad \mathbf{B} = \mathbf{T} \times \mathbf{N}.$$

Here, $\kappa \geq 0$ and τ are curvature and torsion, and $\mathbf{T} = \partial\lambda/\partial s, \mathbf{N}, \mathbf{B}$ vectors are tangent, normal and binormal elements for F-S frame [25].

Also, Q-frame components for curve λ is given as

$$\begin{aligned} \mathbf{T}_{Q_\lambda} &= \mathbf{T}, \\ \mathbf{N}_{Q_\lambda} &= \frac{\mathbf{T} \times \mu}{\left\| \mathbf{T} \times \mu \right\|}, \\ \mathbf{B}_{Q_\lambda} &= \mathbf{T}_{Q_\lambda} \times \mathbf{N}_{Q_\lambda}, \end{aligned}$$

where μ is projection vector. We take $\mu = (0,0,1)$, as the result will not matter for all cases. Then, for a timelike curve $\lambda = \lambda(s)$, the Q-frame formulas in $\mathbb{M}_1^3$ are given by

$$\begin{bmatrix} \nabla_s \mathbf{T}_{Q_\lambda} \\ \nabla_s \mathbf{N}_{Q_\lambda} \\ \nabla_s \mathbf{B}_{Q_\lambda} \end{bmatrix} = \begin{bmatrix} 0 & \lambda_1 & \lambda_2 \\ \lambda_1 & 0 & \lambda_3 \\ \lambda_2 & -\lambda_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}_{Q_\lambda} \\ \mathbf{N}_{Q_\lambda} \\ \mathbf{B}_{Q_\lambda} \end{bmatrix},$$

where $\lambda_1 = \kappa\cos(\psi)$, $\lambda_2 = -\kappa\sin(\psi)$, $\lambda_3 = \psi' + \tau$ (ψ is the angle between the $\mathbf{N}_{Q_\lambda}$ and $\mathbf{N}$) [26].

3. FOCAL CURVES OF ADJOINT CURVES

Definition 1. 1 *Let λ be a regular curve arc-length parameterized, $\{\mathbf{T}_{Q_\lambda}, \mathbf{N}_{Q_\lambda}, \mathbf{B}_{Q_\lambda}\}$ be Q-frame of λ. Then, the adjoint curve of λ according to Q-frame is given as*

$$\beta(s) = \int_{s_0}^{s} \mathbf{B}_{Q_\lambda}(s)ds.$$

Theorem 2. 2 *Let λ be a timelike curve, $\{\mathbf{T}_{Q_\lambda}, \mathbf{N}_{Q_\lambda}, \mathbf{B}_{Q_\lambda}\}$ be Q-frame of λ and β be adjoint curve of λ according to Q-frame. Denote by $\{\mathbf{T}_{Q_\beta}, \mathbf{N}_{Q_\beta}, \mathbf{B}_{Q_\beta}\}$ be Q-frame of β. Then, Q-frame elements of β are given by*

$$\begin{aligned}
\mathbf{T}_{Q_\beta} &= \mathbf{B}_{Q_\lambda}, \\
\mathbf{N}_{Q_\beta} &= \frac{\lambda_2 \mathbf{T}_{Q_\lambda} - \lambda_3 \mathbf{N}_{Q_\lambda}}{\sqrt{\left|\lambda_3^2 - \lambda_2^2\right|}}, \\
\mathbf{B}_{Q_\beta} &= \frac{-\lambda_3 \mathbf{T}_{Q_\lambda} + \lambda_2 \mathbf{N}_{Q_\lambda}}{\sqrt{\left|\lambda_3^2 - \lambda_2^2\right|}}.
\end{aligned}$$

Proof: $\mathbf{T}_{Q_\beta} = \mathbf{B}_{Q_\lambda}$ is obtained by taking the derivative of both sides of the equation given in the last definition. Later on, by applying the equations given for the S-F formulas in the previous section for the quasi frame formulas here, the proof is easily obtained.

Definition 3. 3 *Let β be a regular curve and $\{\mathbf{T}_{Q_\lambda}, \mathbf{N}_{Q_\lambda}, \mathbf{B}_{Q_\lambda}\}$ be Q-frame of β. The focal curve of β is given by*

$$\beta_F = \beta + \varphi_1 \mathbf{N}_{Q_\beta} + \varphi_2 \mathbf{B}_{Q_\beta}, \tag{3.1}$$

where the coefficients φ_1, φ_2 are the focal curvatures of β [24].

Theorem 4. 4 *Let β be adjoint curve of timelike curve λ, β_F its focal curve on $\mathbb{M}_1^3$ and β_1, β_2, β_3, λ_1, λ_2, λ_3 be curvatures of β and λ according to Q-frame. Then,*

$$\beta_F = \beta + [\frac{\lambda_3 + \lambda_3\beta_1 e^{-\int \frac{\beta_1\beta_3}{\beta_2} ds} (-\int (\frac{\beta_3}{\beta_2} e^{\int \frac{\beta_1\beta_3}{\beta_2} ds}) ds + C)}{\beta_2\sqrt{|\lambda_3^2 - \lambda_2^2|}} \tag{3.2}$$
$$+ \frac{\lambda_2}{\sqrt{|\lambda_3^2 - \lambda_2^2|}} e^{-\int \frac{\beta_1\beta_3}{\beta_2} ds} (-\int (\frac{\beta_3}{\beta_2} e^{\int \frac{\beta_1\beta_3}{\beta_2} ds}) ds + C)]\mathbf{T}_{Q_\lambda}$$
$$+ [\frac{\lambda_3}{\sqrt{|\lambda_3^2 - \lambda_2^2|}} e^{-\int \frac{\beta_1\beta_3}{\beta_2} ds} (-\int (\frac{\beta_3}{\beta_2} e^{\int \frac{\beta_1\beta_3}{\beta_2} ds}) ds + C)$$
$$- \frac{\lambda_2 + \lambda_2\beta_1 e^{-\int \frac{\beta_1\beta_3}{\beta_2} ds} (-\int (\frac{\beta_3}{\beta_2} e^{\int \frac{\beta_1\beta_3}{\beta_2} ds}) ds + C)}{\beta_2\sqrt{|\lambda_3^2 - \lambda_2^2|}}]\mathbf{N}_{Q_\lambda},$$

where C is a constant of integration.

Proof: Let β be a regular curve and β_F is focal curve of β in $\mathbb{M}_1^3$. Then, applying the derivative to the equation in the above definition of focal curve, we obtain

$$\beta_F' = (1 + \beta_1\varphi_1 + \beta_2\varphi_2)\mathbf{T}_{Q_\beta} + (\varphi_1' - \beta_3\varphi_2)\mathbf{N}_{Q_\beta} + (\varphi_2' + \beta_3\varphi_1)\mathbf{B}_{Q_\beta}$$

In this equation, by vanishing the tangent and normal components, we get

$$1 + \beta_1\phi_1 + \beta_2\phi_2 = 0,$$

$$\phi_1' - \beta_3\phi_2 = 0.$$

With the aid of these equations, we obtain

$$\varphi_2 = -\frac{1 + \varphi_1\beta_1}{\beta_2},$$

$$\varphi_1' + \frac{\beta_1\beta_3}{\beta_2}\varphi_1 = -\frac{\beta_3}{\beta_2}.$$

By solution this differantial equation, focal curvatures is found as

$$\varphi_1 = e^{-\int \frac{\beta_1\beta_3}{\beta_2} ds} (-\int e^{\int \frac{\beta_1\beta_3}{\beta_2} ds} \frac{\beta_3}{\beta_2} ds + C),$$

$$\varphi_2 = -\frac{1}{\beta_2} - \frac{\beta_1}{\beta_2} e^{-\int \frac{\beta_1\beta_3}{\beta_2} ds} \left(-\int e^{\int \frac{\beta_1\beta_3}{\beta_2} ds} \frac{\beta_3}{\beta_2} ds + C\right).$$

By means of obtained equations and using Theroem 2, we express (3.2). This completes the proof of the theorem. As a consequence of this theorem, we express the following result:

Corollary 5. 5 *Let* β *be adjoint curve of* λ *and* β_F *its focal curve on* $\mathbb{M}_1^3$*. Then, the focal curvatures of* β *are*

$$\varphi_1 = e^{-\int \frac{\beta_1\beta_3}{\beta_2} ds} \left(-\int e^{\int \frac{\beta_1\beta_3}{\beta_2} ds} \frac{\beta_3}{\beta_2} ds + C\right),$$

$$\varphi_2 = -\frac{1}{\beta_2} - \frac{\beta_1}{\beta_2} e^{-\int \frac{\beta_1\beta_3}{\beta_2} ds} \left(-\int e^{\int \frac{\beta_1\beta_3}{\beta_2} ds} \frac{\beta_3}{\beta_2} ds + C\right).$$

From Theorem 4 and Corollary 5, we obtain the following result:

Corollary 6 6 *Let* $\beta : I \to \mathbb{M}_1^3$ *be a unit speed curve and* β_F *its focal curve on* $\mathbb{M}_1^3$*. If the curvatures of* β *are constant, then the focal curve of* β *is*

$$\beta_F = \beta - \left[\frac{\lambda_2}{\beta_1\sqrt{\left|\lambda_3^2 - \lambda_2^2\right|}} + \frac{\lambda_2\beta_2 C + \lambda_3\beta_1 C}{\beta_2\sqrt{\left|\lambda_3^2 - \lambda_2^2\right|} e^{\frac{\beta_1\beta_3}{\beta_2}s}}\right]\mathbf{T}_{Q_\lambda}$$
$$+\left[\frac{\lambda_3}{\beta_1\sqrt{\left|\lambda_3^2 - \lambda_2^2\right|}} + \frac{\lambda_3\beta_2 C + \lambda_2\beta_1 C}{\beta_2\sqrt{\left|\lambda_3^2 - \lambda_2^2\right|} e^{\frac{\beta_1\beta_3}{\beta_2}s}}\right]\mathbf{N}_{Q_\lambda}.$$

CONCLUSIONS

In the differential geometry, curves theory is a ultimate important applications. Special curves and their interpretations have been studied for a long time and are still being studied. The operation of this special curves is seen in nature, computer aided design, mechanic tools and computer graphics etc. [27-40].

In our paper, we obtain new results by examining the adjoint curve and focal curve, which are examples of associated curves. In particular, we examine the focal curve of the adjoint curve of a timelike curve we have described with quasi-frame(Q-frame) elements in 3D Minkowski space $\mathbb{M}_1^3$.

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