ORIGINAL PAPER

ON COMPARISON OF SOLUTION OF ORDINARY DIFFERENTIAL EQUATION WITH HAAR WAVELET METHOD AND THE MODIFIED ISHIKAWA ITERATION METHOD

YASEMİN BAKIR[1], OYA MERT[2], ÖZLEM ORHAN[3]

Manuscript received: 03.02.2022; Accepted paper: 11.05.2022;
Published online: 30.06.2022.

Abstract. *In this study, we have used a newly modified Ishikawa iteration method and the Haar wavelet method to solve an ordinary linear differential equation with initial conditions. Using the modified Ishikawa iteration approach, we derive approximate solutions to the issue as well as the related iterative schemes. For this problem, the Ishikawa Iteration Method is applied for different lambda and gamma values and approximation solutions for these values are compared with the approximate solution of Haar wavelet collocation and its exact solution. Finally, the error tables are written and the graphs are shown.*

Keywords: *Ordinary differential equation; modified Ishikawa iteration; Haar wavelet collocation method.*

1. INTRODUCTION

Fixed point theory contains a considerable quantity of literature since it gives important tools for solving problems in different areas such as technology, engineering, economics, etc. However, if the presence of any mapping's fixed point is proven, finding the value of that fixed point becomes challenging, which is why iterative procedures are employed to calculate them. Someone can't list all the iterative processes that have been created. The Banach contraction theorem, which is a well-known theorem in analysis, uses a process called Picard iteration to approximate fixed points. Mann [1], Ishikawa [2], Agarwal [3], Noor [4], Abbas [5], Sahu [6], Khan [7], Gürsoy [8], Karakaya [9], and others are examples of well-known iterative processes.

So far, most academicians have studied a variety of numerical methods for solving different types of differential equations. L.E.J. Brouwer first proposed the fixed-point theory of normed linear spaces [10-16]. Following this, the use of normed direct spaces to approximate fixed-point theory [17,18], Banach spaces [6], and Hilbert spaces [19] is gaining popularity. In this paper, we take a differential equation and demonstrate how to use the Modified Ishikawa iteration approach to solve it. Later on, we investigate the Modified Ishikawa and Haar wavelet collocation methods compared to each other. Then, we present the tables and draw the graphs. As a result, we can identify whether iteration is more powerful or provides the most accurate approximation. Let's go through some of the key theorems and definitions.

[1] Doğuş University, Department of Computer Engineering, 34775 İstanbul, Turkey.
E-mail: ybakir@dogus.edu.tr.
[2] Tekirdağ Namık Kemal University, Department of Mathematics, 59030 Tekirdağ, Turkey.
E-mail: oyamert@nku.edu.tr.
[3] Bandırma Onyedi Eylül University, Department of Engineering Sciences, 10200 Balıkesir, Turkey.
E-mail: oorhan@bandirma.edu.tr.

2. PRELIMINARIES

Theorem 2.1. Let (X,d) be a complete metric space and $T: X \to X$ be a contraction with the Lipschitzian constant L. T then has a fixed point $u \in X$ that is unique.

Additionally, we have

$$\lim_{n\to\infty} T^n(x) = u$$

$$d(T^n(x), u) \le \frac{L^n}{1-L} d\big(x, T(x)\big)$$

for any $x \in X$ (see, [17]).

Corollary 2.2. Let (X,d) be a complete metric space and $B(x_0, r) = \{x \in X: d(x, x_0) < r\}$, where $x_0 \in X$ and $r > 0$. Assume $T: B(x_0, r) \to X$ is a contraction (that is, $d\big(T(x), T(y)\big) \le Ld(x,y)$ for all $x, y \in B(x_0, r)$ with $0 \le L < 1$) with $d(T(x_0), x_0) < (1-L)r$. Then T has a unique fixed point in $B(x_0, r)$ [17].

Definition 2.3. If the sequence $\{x_n\}_{n=0}^{\infty}$ has the condition $x_{n+1} = Tx_n$ for $n = 0,1,2, \ldots$, it is named as the Picard iteration [20].

Definition 2.4. Let $x_0 \in X$ be arbitrary. If the sequence $\{x_n\}_{n=0}^{\infty}$ has the condition

$$x_{n+1} = (1-\alpha_n)x_n + \{\alpha_n\}Ty_n$$

$$y_n = (1-\beta_n)x_n + \{\beta_n\}Tx_n$$

for $n = 0,1,2, \ldots$, it is named as Ishikawa iteration [2] where (α_n) and (β_n) are positive number sequences that provide the following criteria:

(i) $0 \le \alpha_n \le \beta_n < 1$, $n \in Z^+$
(ii) $\lim_{n\to\infty} \beta_n = 0$
(iii) $\sum_{n\ge 0} \alpha_n \beta_n = \infty$

Definition 2.5. If $y_0 \in X$, $\lambda, \gamma \in [0,1]$ and T is given contraction mapping in terms of Picard iteration and the $\{y_n\}_{n=0}^{\infty}$ sequence satisfies the following criteria:

$$\left.\begin{aligned} y_{n+1} &= \lambda y_{n-1} + (1-\lambda)Ty_{n-1} \\ y_n &= (1-\gamma)y_{n-2} + \gamma T y_{n-2} \end{aligned}\right\} \text{ for } n = 2,4, \ldots$$

$$y_{n+1} = y_0 + \int_{x_0}^{x} F\big(t, y_n(t)\big)dt \quad \text{for } n = 0$$

$$Ty_{n-1} = y_n \quad , \quad 0 < \lambda, \gamma < 1$$

It is named as Modified Ishikawa iteration where $T = \int_{x_0}^{x} F\big(t, y_n(t)\big)dt$ [18]. Consider the following example to demonstrate the performance of the modified Ishikawa iteration approach in solving linear differential equations, as well as to explain the method given in this study's accuracy and efficiency. We utilize this iteration method and the Haar wavelet collocation method in this example. Following that, a comparison of the two approaches is given as figures and tables.

3. NUMERICAL APPLICATION

Example 3.1. Consider $y'(t) + ty(t) = 2t^3$ with the initial condition $y(0) = 1$. First, the exact solution of the equation is found as $y(t) = 2t^2 - 4 + 5e^{\frac{-t^2}{2}}$.

Now, for special values of λ and γ, by applying the Modified Ishikawa Iteration Method to the problem, then

$$\left.\begin{aligned}
y_1 &= 1 + 1/2(t^4 - t^2)\\
y_2 &= 1 + 1/4(t^4 - t^2)\\
y_3 &= 1 + 3/8(t^4 - t^2)\\
y_4 &= 1 + 5/16(t^4 - t^2)\\
y_5 &= 1 + 11/32(t^4 - t^2)\\
y_6 &= 1 + 21/64(t^4 - t^2)\\
y_7 &= 1 + 43/128(t^4 - t^2)\\
y_8 &= 1 + 85/256(t^4 - t^2)\\
y_9 &= 1 + 171/512(t^4 - t^2)\\
y_{10} &= 1 + 341/1024(t^4 - t^2)
\end{aligned}\right\} \quad for\ \lambda = 0.5\ and\ \gamma = 0.5$$

$$\left.\begin{aligned}
y_1 &= 1 + 1/2(t^4 - t^2)\\
y_2 &= 1 + 1/8(t^4 - t^2)\\
y_3 &= 1 + 5/16(t^4 - t^2)\\
y_4 &= 1 + 11/64(t^4 - t^2)\\
y_5 &= 1 + 31/128(t^4 - t^2)\\
y_6 &= 1 + 97/512(t^4 - t^2)\\
y_7 &= 1 + 221/1024(t^4 - t^2)\\
y_8 &= 1 + 803/4096(t^4 - t^2)\\
y_9 &= 1 + 1687/8192(t^4 - t^2)\\
y_{10} &= 1 + 6505/32768(t^4 - t^2)
\end{aligned}\right\} \quad for\ \lambda = 0.5\ and\ \gamma = 0.25$$

$$\left.\begin{aligned}
y_1 &= 1 + 1/2(t^4 - t^2)\\
y_2 &= 1 + 1/4(t^4 - t^2)\\
y_3 &= 1 + 5/8(t^4 - t^2)\\
y_4 &= 1 + 7/16(t^4 - t^2)\\
y_5 &= 1 + 37/64(t^4 - t^2)\\
y_6 &= 1 + 65/128(t^4 - t^2)\\
y_7 &= 1 + 269/512(t^4 - t^2)\\
y_8 &= 1 + 529/1024(t^4 - t^2)\\
y_9 &= 1 + 2125/4096(t^4 - t^2)\\
y_{10} &= 1 + 4241/8192(t^4 - t^2)
\end{aligned}\right\} \quad \text{for } \lambda = 0.25 \text{ and } \gamma = 0.5$$

$$\left.\begin{aligned}
y_1 &= 1 + 1/2(t^4 - t^2)\\
y_2 &= 1 + 1/8(t^4 - t^2)\\
y_3 &= 1 + 13/32(t^4 - t^2)\\
y_4 &= 1 + 25/128(t^4 - t^2)\\
y_5 &= 1 + 181/512(t^4 - t^2)\\
y_6 &= 1 + 481/2048(t^4 - t^2)\\
y_7 &= 1 + 2653/8192(t^4 - t^2)\\
y_8 &= 1 + 8425/32768(t^4 - t^2)\\
y_9 &= 1 + 40261/131072(t^4 - t^2)\\
y_{10} &= 1 + 141361/524288(t^4 - t^2)
\end{aligned}\right\} \text{ for } \lambda = 0.75 \text{ and } \gamma = 0.25$$

$$\left.\begin{aligned}
y_1 &= 1 + 1/2(t^4 - t^2)\\
y_2 &= 1 + 3/8(t^4 - t^2)\\
y_3 &= 1 + 13/32(t^4 - t^2)\\
y_4 &= 1 + 51/128(t^4 - t^2)\\
y_5 &= 1 + 205/512(t^4 - t^2)\\
y_6 &= 1 + 819/2048(t^4 - t^2)\\
y_7 &= 1 + 5734/8192(t^4 - t^2)\\
y_8 &= 1 + 20478/32768(t^4 - t^2)\\
y_9 &= 1 + 84370/131072(t^4 - t^2)\\
y_{10} &= 1 + 335022/524288(t^4 - t^2)
\end{aligned}\right\} \text{ for } \lambda = 0.25 \text{ and } \gamma = 0.75$$

are calculated. Let us now present a table showing the absolute error about Example 3.1 for various values of λ and γ as follows:

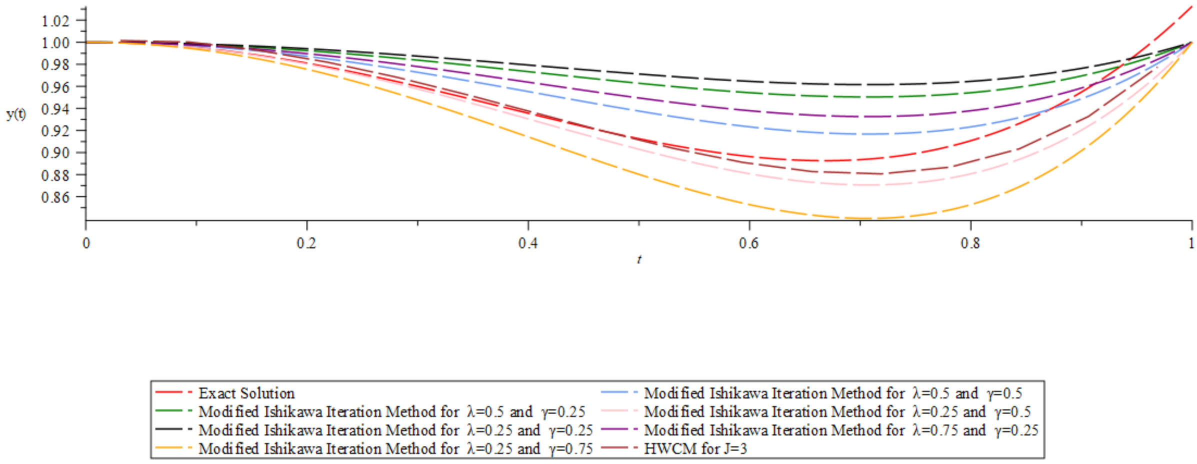

Figure 1. Comparison of Haar wavelet method and modified Ishikawa iteration method by using various values of λ and γ.

Table 1. Error table for modified Ishikawa iteration technique with various values of λ and γ

Modified Ishikawa Iteration Method						
t_s	$\lambda = 0.5$, $\gamma = 0.5$	$\lambda = 0.5$, $\gamma = 0.25$	$\lambda = 0.25$, $\gamma = 0.5$	$\lambda = 0.25$, $\gamma = 0.25$	$\lambda = 0.75$, $\gamma = 0.25$	$\lambda = 0.25$, $\gamma = 0.75$
0.03125	1.628001×10^{-4}	3.028532×10^{-4}	2.48908×10^{-5}	3.353653×10^{-4}	1.717318×10^{-4}	1.357331×10^{-4}
0.09375	1.4452195×10^{-3}	2.6958416×10^{-3}	2.307840×10^{-4}	2.9861649×10^{-3}	1.524979×10^{-3}	1.2205599×10^{-3}
0.15625	3.9044275×10^{-3}	7.3236156×10^{-3}	6.777496×10^{-4}	8.117355×10^{-3}	4.122488×10^{-3}	3.3837886×10^{-3}

Modified Ishikawa Iteration Method						
t_s	$\lambda = 0.5$, $\gamma = 0.5$	$\lambda = 0.5$, $\gamma = 0.25$	$\lambda = 0.25$, $\gamma = 0.5$	$\lambda = 0.25$, $\gamma = 0.25$	$\lambda = 0.75$, $\gamma = 0.25$	$\lambda = 0.25$, $\gamma = 0.75$
$\mathbf{0.21875}$	7.3335868×10^{-3}	$1.38741949 \times 10^{-2}$	1.4317183×10^{-3}	1.5392551×10^{-2}	7.750717×10^{-3}	6.6081361×10^{-3}
$\mathbf{0.28125}$	1.1433382×10^{-2}	$2.18905518 \times 10^{-2}$	2.5806494×10^{-3}	2.4318109×10^{-2}	1.2100293×10^{-2}	1.0856741×10^{-2}
$\mathbf{0.34375}$	1.5824818×10^{-2}	$3.07834061 \times 10^{-2}$	4.2217236×10^{-3}	3.4255935×10^{-2}	1.677881×10^{-2}	1.6060361×10^{-2}
$\mathbf{0.40625}$	2.0065742×10^{-2}	$3.98477459 \times 10^{-2}$	6.4448352×10^{-3}	4.4439997×10^{-2}	2.132735×10^{-2}	2.2100857×10^{-2}
$\mathbf{0.46875}$	2.3670646×10^{-2}	$4.82826381 \times 10^{-2}$	9.3127720×10^{-3}	5.3996136×10^{-2}	2.5240288×10^{-2}	2.8791379×10^{-2}
$\mathbf{0.53125}$	2.6133239×10^{-2}	$5.52137908 \times 10^{-2}$	1.2838658×10^{-2}	6.1964633×10^{-2}	2.7987865×10^{-2}	3.5853806×10^{-2}
$\mathbf{0.59375}$	2.6951186×10^{-2}	$5.97183009 \times 10^{-2}$	1.6961205×10^{-2}	6.7324952×10^{-2}	2.9040925×10^{-2}	4.2894000×10^{-2}
$\mathbf{0.65625}$	2.565240×10^{-2}	$6.08509465 \times 10^{-2}$	2.1518418×10^{-2}	6.9022035×10^{-2}	2.7897214×10^{-2}	4.9375508×10^{-2}
$\mathbf{0.71875}$	2.1822279×10^{-2}	$5.76713856 \times 10^{-2}$	2.6220401×10^{-2}	6.5993499×10^{-2}	2.4108574×10^{-2}	5.4592370×10^{-2}
$\mathbf{0.78125}$	1.5131047×10^{-2}	$4.92715871 \times 10^{-2}$	3.0621919×10^{-2}	5.7197069×10^{-2}	1.7308378×10^{-2}	5.7641682×10^{-2}
$\mathbf{0.84375}$	5.3608974×10^{-3}	$3.48028771 \times 10^{-3}$	3.4095361×10^{-2}	4.1637622×10^{-2}	7.2385743×10^{-3}	5.7396553×10^{-3}
$\mathbf{0.90625}$	7.5680171×10^{-3}	$1.35019836 \times 10^{-3}$	3.5804685×10^{-2}	1.839323×10^{-2}	6.2242672×10^{-3}	5.2480062×10^{-3}
$\mathbf{0.96875}$	2.3573059×10^{-2}	$1.52844555 \times 10^{-2}$	3.4680917×10^{-2}	1.3360315×10^{-2}	2.3044449×10^{-2}	4.1240746×10^{-2}

Table 2. Error table for Haar wavelet collocation method for $K = 3$

t_s	Exact Solution of $z(t)$	Solution of $z(t_i)$ with HWCM for $K = 3$	Absolute Error for HWCM for $K = 3$ $\lvert z(t_s) - \mathbf{z}(\mathbf{t})\rvert$
0.03125	1.00150855711906	0.999512315	$1.99624211906246 \times 10^{-3}$
0.09375	1.00013892515156	0.995653678	$4.48524715156240 \times 10^{-3}$
0.15625	0.992616342144688	0.988163987	$4.45235514468756 \times 10^{-3}$
0.21875	0.981790231305937	0.977493981	$4.29625030593750 \times 10^{-3}$
0.28125	0.968209909041000	0.964308829	$3.90108004099998 \times 10^{-3}$
0.34375	0.952607797224875	0.949475329	$3.13246822487501 \times 10^{-3}$
0.40625	0.935899423201719	0.934045392	$1.85403120171868 \times 10^{-3}$
0.46875	0.919183419784094	0.919236230	$5.28102159061961 \times 10^{-5}$
0.53125	0.903741525251875	0.906407794	$2.66626874812503 \times 10^{-3}$
0.59375	0.891038583355000	0.897038025	$5.99944164499999 \times 10^{-3}$
0.65625	0.882722543313750	0.892696563	$9.97401968625000 \times 10^{-3}$
0.71875	0.880624459815625	0.895017549	$1.43930891843750 \times 10^{-2}$
0.78125	0.886758493017813	0.905672191	$1.89136979821874 \times 10^{-2}$
0.84375	0.903321908545312	0.926341721	$2.30198124546875 \times 10^{-2}$
0.90625	0.932695077492500	0.958691349	$2.59962715074999 \times 10^{-2}$
0.96875	0.977441476420000	1.004345753	$2.69042765799999 \times 10^{-2}$

4. CONCLUSION

It is demonstrated that the modified Ishikawa iteration approach is effective in solving the problem. However, comparing it to the Haar wavelet collocation method developed for solving linear differential equations reveals that the Haar wavelet approximation has a lower error rate.

REFERENCES

[1] Mann, W.R., *Proc. Am. Math. Soc.,* **4**, 506, 1953.
[2] Ishikawa, S., *Proc. Am. Math. Soc*., **44**(1), 147, 1974.
[3] Agarwal, R.P., O'Regan, D., Sahu, D.R., *J. Nonlinear Convex Anal.,* **8**(1), 61, 2007.
[4] Noor, M.A., *J. Math. Anal. Appl*., **251**(1), 217, 2000.
[5] Abbas, M., Nazir., T., *Mat. Vesn.*, **66**(2), 223, 2014.
[6] Sahu, D.R., Petrusel, A., *NonlinearAnalysis: Theory, Methods &Applications*, **74**(17), 6012, 2011.
[7] Khan, S.H., *Fixed Point Theory Appl.,* **2013**, 69, 2013.
[8] Gursoy, F., Karakaya, V., *arXiv*:1403.2546v2 [math.FA] 28 Apr 2014.
[9] Karakaya, V., Atalan, Y., Dogan, K., Bouzara, N.E.H., *Istanbul Commerce University, Journal of Science*, **15**(30), 35, 2016.
[10] Brouwer, L.E.J., *Proc. Kon. Nederl. Akad. Wetensch*, **11**, 788, 1909.
[11] Brouwer, L.E.J., *Proc. Kon. Nederl. Akad. Wetensch*, **12**, 286, 1910.
[12] Brouwer, L.E.J., *Proc. Kon. Nederl. Akad. Wetensch*, **13**, 767, 1911.
[13] Brouwer, L.E.J., *Proc. Kon. Nederl. Akad. Wetensch*, **14**, 300, 1912.
[14] Brouwer, L.E.J., *Proc. Kon. Nederl. Akad. Wetensch*, **15**, 352, 1913.
[15] Brouwer, L.E.J., *Proc. Kon. Nederl. Akad. Wetensch*, **22**, 811, 1920.
[16] Brouwer, L.E.J., *Proc. Kon. Nederl. Akad. Wetensch*, **23**, 232, 1921.
[17] Agarwal, R.P., Meehan, M., O'Regan, D., *Fixed Point Theory and Applications*, Vol. **141**, Cambridge University Press, 2004.
[18] Bildik, N., Bakır, Y., Mutlu, A., *Fixed Point Theory Appl*, **2013**, 52, 2013.
[19] Shang, X., Qin, M., Su, Y., *J. Math. Inequal.,* **1**(2), 195, 2007.
[20] Soltuz, S.M., *Math. Commun*., **10**(1), 81, 2005.