ORIGINAL PAPER

AN EXISTENCE SOLUTION FOR A COUPLED SYSTEM WITH LAPLACIAN OPERATOR AND HILFER DERIVATIVES

MOHAMMED KAID[1], HOUARI FETTOUCH[1]

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Abstract. *In this paper, we study the existence of solutions for a coupled system of fractional differential equations with nonlocal integro multi point boundary conditions by using the Laplacian operator and the Hilfer derivatives. The presented results are obtained by the fixed point theorems of Krasnoselskii. An illustrative example is presented at the end to show the applicability of the obtained results. To the best of our knowledge, this is the first time where such problem is considered.*

Keywords: *Hilfer fractional derivative; Laplacian operator; fixed point theorem.*

1. INTRODUCTION

Fractional differential equations and systems can be found in a wide range of engineering and scientific fields. We recommend the reader to example [1-17] for some recent developments on this theory. Fractional integrals and derivatives are currently defined in a variety of ways, ranging from the most well-known Riemann-Liouville and Caputo fractional derivatives to less well-known approaches. R. Hilfer proposed the fractional Hilfer derivative of order α and type $\beta \in [0,1]$ in[18-20], which is a generalization of the Riemann-Liouville and Caputo derivatives. [18-20] and the literature cited above provide some features and applications of the Hilfer derivative. Several authors have looked at prime value difficulties with fractional Hilfer derivatives (see [21-23]). However, there are only a few papers on the subject in the literature.

However, in the literature there are few papers on the boundary value problems of the fractional Hilfer derivatives and many applications of Hilfer fractional differential equations can be found in many fields of mathematics, physics, etc. see, [24] and [25].

Beddani, H,. and Beddani, M in [26], proved the existence and uniqueness of solutions for the coupled system of $\varphi-$ Caputo fractional differential equations, of the form:

$$\begin{cases} {}^{c}\mathbf{D}_{a^+}^{\alpha_1;\varphi}\left({}^{c}\mathbf{D}_{a^+}^{\alpha_2;\varphi}+\mu_1\right)u(t)=f(t,u(t),v(t)), \quad t\in[a,b] \\ {}^{c}\mathbf{D}_{a^+}^{\alpha_3;\varphi}\left({}^{c}\mathbf{D}_{a^+}^{\alpha_4;\varphi}+\mu_2\right)v(t)=g(t,u(t),v(t)), \\ u(a)=u_a, \quad u(b)=A\sum_{i=1}^{n}u(\zeta_i), \\ v(a)=v_a, \quad v(b)=B\sum_{i=1}^{n}v(\zeta_i), \\ \mu_1,\mu_2>0, \quad 0\le a<\zeta_i<b<\infty, \text{ and } \varphi(b)-\varphi(a)=M>0. \end{cases}$$

[1]Abdelhamid Bni Badis University, Laboratory of Pure and Applied Mathematics, 27000 Mostaganem, Algeria.
E-mail: mohammed.kaid@univ-mosta.dz; houari.fettouch@univ-mosta.dz .

where ${}^{c}\mathbf{D}_{a^+}^{\alpha_i;\varphi}, i=\overline{1,4}$ are the $\varphi-$ Caputo fractional derivative of orders α_i, and $0<\alpha_i<1, i=\overline{1,4}$ $\mu_1,\mu_2,A,B\in \mathbb{R}_+^*, u_a, v_a \in R^m, m\in \mathbb{N}^*$, and $\varphi : J\to \mathbb{R}$ be an increasing function with $\varphi'(t)\neq 0$, for all $t\in J$, to be defined later， $g,f : J\times R^m\times R^m \to R^m$ is a given function .

In the present research, we study the existence and uniqueness of solutions for the following coupled system of $\varphi-$Hilfer-type fractional differential equations order of the form:

$$\begin{cases} {}^{H}\mathbf{D}_{a^+}^{\alpha_{11},\beta_{11};\varphi}\psi_{p_1}\left({}^{H}\mathbf{D}_{a^+}^{\alpha_{12},\beta_{12};\varphi}u_2\right)(t)=F_1(t,u_1(t),u_2(t)),\ \ t\in[a,b], \\ {}^{H}\mathbf{D}_{a^+}^{\alpha_{21},\beta_{21};\varphi}\psi_{p_2}\left({}^{H}\mathbf{D}_{a^+}^{\alpha_{22},\beta_{22};\varphi}u_2\right)(t)=F_2(t,u_1(t),u_2(t)),\ \ t\in[a,b], \\ u_k(a)=0, u_k(b)=\sum_{i=1}^{n}\left(\lambda_i u_1(\zeta_i)+\delta_i u_2(\eta_i)\right),\quad a<\zeta_i,\eta_i<b,\lambda_i,\delta_i\in\mathbb{R}_+^* \\ \psi_{p_k}\left({}^{H}\mathbf{D}_{a^+}^{\alpha_{k2},\beta_{k2};\varphi}u_k\right)(a)=\ \psi_{p_k}\left({}^{H}\mathbf{D}_{a^+}^{\alpha_{k2},\beta_{k2};\varphi}u_k\right)(b)=0, k=1,2 \end{cases} \tag{1.1}$$

Here, we take ${}^{H}\mathbf{D}_{a^+}^{\alpha_{k1},\beta_{k1};\varphi}$ and ${}^{H}\mathbf{D}_{a^+}^{\alpha_{k2},\beta_{k2};\varphi}$, $k=\overline{1,2}$ are the $\varphi-$Hilfer fractional derivative of orders α_{k1} and α_{k2} with $1<\alpha_{k1},\alpha_{k2}<2$ and β_{k1},β_{k2} two parameters $0\le\beta_{k1},\beta_{k2}\le 1,$ and $\psi_{p_k}(x)=|x|^{p_k-2}x$ denotes the p_k-Laplacian operator and satisfies $\frac{1}{p_k}+\frac{1}{q_k}=1,\left(\psi_{p_k}\right)^{-1}=\psi_{q_k},$ and $\varphi : (a,b]\to\mathbb{R}$ be an increasing function such that $\varphi'(t)\neq 0$, for all $t\in[a,b]$, and $F_k : [a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $(k=1,2)$ is given function. will be well defined later.

2. PRELIMINARY RESULT

In this section, we introduce some notations and definitions of $\varphi-$Hilfer Derivatives Calculus and present preliminary results needed in our proofs later, for details, see [3,22, 23]. Let $\varphi : [a,b]\to\mathbb{R}$ be an increasing function with $\varphi'(t)\neq 0$, for all $t\in J$, and let $C([a,b],R)$ be the Banach space.

For all $\upsilon>-1$ and $s,t\in[0,\infty),(t\ge s)$, we pose $\varphi_\upsilon(t,s)=(\varphi(t)-\varphi(s))^\upsilon$.

Definition 2.1. ([3,22]) Let (a,b), $(-\infty\le a<b\le\infty)$ be a finite or infinite interval of the half-axis $(0,\infty)$ and $\alpha>0$. In addition, let $\varphi(t)$ be a positive increasing function on $(a,b]$, which has a continuous derivative $\varphi'(t)$ on (a,b). The $\varphi-$Riemann-Liouville fractional integral of a function u with respect to another function φ on $[a,b]$ is defined by

$$\mathtt{I}_{a^+}^{\alpha;\varphi}u(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\varphi'(s)\varphi_{\alpha-1}(t,s)u(s)ds, \tag{2.1}$$

where $\Gamma(.)$ is the Gamma function.

Definition 2.2. ([3,22]) Let $n \in \mathrm{N}$ and let $\varphi, u \in C^n(J)$ be two functions such that e is increasing and $\varphi'(t) \neq 0$, for all $t \in (a,b]$. The left-sided $\phi-$Riemann Liouville fractional derivative of a function u of order α is defined by

$$\mathbf{D}_{a^+}^{\alpha;\varphi} u(t) = \left(\frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n \mathbf{I}_{a^+}^{n-\alpha;\varphi} u(t)$$
$$= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n \int_a^t \varphi'(s) \varphi_{n-\alpha-1}(t,s) u(s) ds,$$

where $n = [\alpha]+1, [\alpha]$ represents the integer part of the real number ☺

Definition 2.3. ([3,22]) Let $n-1 < \alpha < n$ with $n \in \mathrm{N}$, $[a,b]$ is the interval such that $-\infty \leq a < b \leq \infty$ and $\varphi, u \in C^n([a,b], \mathrm{R})$ two functions such that e is increasing and $\varphi'(t) \neq 0$, for all $t \in [a,b]$. The φ-Hilfer fractional derivative of a function u of order a and type $0 \leq \beta \leq 1$ is defined by

$${}^H\mathbf{D}_{a^+}^{\alpha,\beta;\varphi} u(t) = \mathbf{I}_{a^+}^{\beta(n-\alpha);\varphi} \left(\frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n \mathbf{I}_{a^+}^{(1-\beta)(n-\alpha);\varphi} u(t) = \mathbf{I}_{a^+}^{\gamma-\alpha;\varphi} \mathbf{D}_{a^+}^{\gamma;\varphi} u(t),$$

where $n = [\alpha]+1, \gamma - \alpha = \beta(n-\alpha)$.

Lemma 2.1. ([3,22]) Let $\alpha, \rho > 0$. Then, we have the following semigroup property given by

$$\mathbf{I}_{a^+}^{\alpha;\varphi} \mathbf{I}_{a^+}^{\rho;\varphi} u(t) = \mathbf{I}_{a^+}^{\alpha+\rho;\varphi} u(t), \quad t > a.$$

Next, we present the φ-fractional integral and derivatives of a power function.

Proposition 2.1. ([23]) Let $\alpha \geq 0, \sigma > 0$ and $t > a.$ Then, φ-fractional integral and derivative of a power function are given by

1) $\mathbf{I}_{a^+}^{\alpha,\varphi} \varphi_{\sigma-1}(t,a)(t) = \frac{\Gamma(\sigma)}{\Gamma(\alpha+\sigma)} \varphi_{\sigma+\alpha-1}(t,a).$
2) ${}^H\mathbf{D}_{a^+}^{\alpha,\beta;\varphi} \varphi_{\sigma-1}(t,a)(t) = \frac{\Gamma(\sigma)}{\Gamma(\sigma-\alpha)} \varphi_{\sigma-\alpha-1}(t,a), n-1 < \alpha < n, \sigma > n.$

Lemma 2.2. ([23]) If $u \in C^n([a,b], R), n-1 < \alpha < n, 0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta(n-\alpha)$. Then

$$\mathbf{I}_{a^+}^{\alpha,\varphi} ({}^H\mathbf{D}_{a^+}^{\alpha,\beta;\varphi} u)(t) = u(t) - \sum_{k=1}^{k=n} \frac{\varphi_{\gamma-k}(t,a)}{\Gamma(\gamma-k+1)} \nabla_\varphi^{[n-k]} \mathbf{I}_{a^+}^{(1-\beta)(n-\alpha);\varphi} u(a), \ t \in [a,b],$$

where $\nabla_\varphi^{[n]} u(t) := \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n u(t).$

Lemma 2.3. ([23]) Let $u \in C^n[a,b]$ and $0<q<1,$ we have

$$\left|\mathbf{I}_{a^+}^{q;\varphi}u(t_2)-\mathbf{I}_{a^+}^{q;\varphi}u(t_1)\right| \le \frac{2\|u\|}{\Gamma(q+1)}\varphi_q(t_2,t_1).$$

Lemma 2.4. ([24]) For the $p-$Laplacian operator ψ_p, the following conditions hold true:
(1) If $|\delta_1|,|\delta_2| \ge \rho>0,\ 1<p\le 2, \delta_1\delta_2>0,$ then

$$\left|\psi_p(\delta_1)-\psi_p(\delta_2)\right| \le (p-1)\rho^{p-2}|\delta_1-\delta_2|.$$

(2) If $p>2,\ |\delta_1|,|\delta_2| \le \rho_*>0,$ then

$$\left|\psi_p(\delta_1)-\psi_p(\delta_2)\right| \le (p-1)\rho_*^{p-2}|\delta_1-\delta_2|.$$

Lemma 2.5. [16] For nonnegative $a_i, i=1,...,k,$

$$\left(\sum_{i=1}^{k} a_i\right)^q \le k^{q-1}\left(\sum_{i=1}^{k} a_i^q\right), q\ge 1.$$

Lemma 2.6. [25] (Krasnoselskii fixed point theorem). Let Δ be a closed, bounded, convex and nonempty subset of a Banach space X. Let Γ,Π be operators such that:
(i) $\Gamma x+\Pi y\in\Delta$, $x,y\in\Delta.$
(ii) Γ is compact and continuous.
(iii) Π is a contraction mapping.
Then there exists $\xi\in\Delta$ such that $\xi=\Gamma\xi+\Pi\xi$.

In this subsection, we consider now the linear coupled system:

$$\begin{cases} {}^{H}\mathbf{D}_{a^+}^{\alpha_{k1},\beta_{k1};\varphi}\psi_{p_k}\left({}^{H}\mathbf{D}_{a^+}^{\alpha_{k2},\beta_{k2};\varphi}u_k\right)(t)= f_k(t),\ \ t\in[a,b], k=\overline{1,2} \\ u_k(a)=0, u_k(b)=\sum_{i=1}^{n}\left(\lambda_i u_1(\zeta_i)+\delta_i u_2(\eta_i)\right),\quad a<\zeta_i<b, \\ \psi_{p_k}\left({}^{H}\mathbf{D}_{a^+}^{\alpha_{k2},\beta_{k2};\varphi}u_k\right)(a)=\psi_{p_k}\left({}^{H}\mathbf{D}_{a^+}^{\alpha_{k2},\beta_{k2};\varphi}u_k\right)(b)=0, \end{cases} \tag{2.2}$$

where $f_k\ :\ (a,b]\to \mathrm{R}$ are continuous functions, and

$$\begin{cases} a\ge 0,\quad 1<\alpha_{k1},\alpha_{k2}<2,\quad 0\le\beta_{k1},\beta_{k2}\le 1, \\ \text{and} \\ \gamma_{k1}=\alpha_{k1}+\beta_{k1}(2-\alpha_{k1}),\quad \gamma_{k2}=\alpha_{k2}+\beta_{k2}(2-\alpha_{k2}). \end{cases} \tag{2.3}$$

Lemma 2.7. Let $f_k\in C([a,b])(k=\overline{1,2})$, the unique solution of the coupled system (2.2) is given by:

$$
\begin{aligned}
&u_k(t) \\
&= \frac{\varphi_{\gamma_{k2}-1}(t,a)}{\varphi_{\gamma_{k2}-1}(b,a)} \sum_{i=1}^{n} \left(\lambda_i u_1(\zeta_i) + \delta_i u_2(\eta_i) \right) + \frac{1}{\Gamma(\alpha_{k2})} \int_a^t \varphi'(s) \varphi_{\alpha_{k2}-1}(t,s) \\
&\times \left\{ \psi_{q_k} \left[\frac{1}{\Gamma(\alpha_{k1})} \int_a^s \varphi'(s) \varphi_{\alpha_{k1}-1}(s,x) f_k(z) dz - \frac{\varphi_{\gamma_{k1}-1}(s,a)}{\varphi_{\gamma_{k1}-1}(b,a)} \mathbf{I}_{a^+}^{\alpha_1;\varphi} f_k(b) \right] \right\} ds \\
&- \frac{\varphi_{\gamma_{k2}-1}(t,a)}{\Gamma(\alpha_{k2}) \varphi_{\gamma_{k2}-1}(b,a)} \int_a^b \varphi'(s) \varphi_{\alpha_{k2}-1}(b,y) \\
&\times \left\{ \psi_{q_k} \left[\frac{1}{\Gamma(\alpha_{k1})} \int_a^y \varphi'(s) \varphi_{\alpha_{k1}-1}(y,z) f_k(z) dz - \frac{\varphi_{\gamma_{k1}-1}(y,a)}{\varphi_{\gamma_{k1}-1}(b,a)} \mathbf{I}_{a^+}^{\alpha_1;\varphi} f_k(b) \right] \right\} dy.
\end{aligned}
$$

Proof: Assume u and v satisfies the system (2.2) and considering the first equation in the system (2.2). By applying the fractional integral operators $\mathbf{I}_{a^+}^{\alpha_1;\varphi}$, to (2.2) and using Lemma, we conclude that:

$$
\psi_{p_k} \left({}^H\mathbf{D}_{a^+}^{\alpha_{k2},\beta_{k2};\varphi} u_k \right)(t) = \mathbf{I}_{a^+}^{\alpha_{k1};\varphi} f_k(t) + \frac{c_1}{\Gamma(\gamma_{k1}-1)} \varphi_{\gamma_{k1}-2}(t,a) + \frac{c_2}{\Gamma(\gamma_{k1})} \varphi_{\gamma_{k1}-1}(t,a), \tag{2.4}
$$

for some real constantes c_1 and c_2. Now, using the first boundary condition (2.3)

$$
\psi_{p_k} \left({}^H\mathbf{D}_{a^+}^{\alpha_{k2},\beta_{k2};\varphi} u_k \right)(a) = 0,
$$

in (2.4), we get

$$
\frac{c_1}{\Gamma(\gamma_{k1}-1)} \varphi_{\gamma_{k1}-2}(a,a) + \frac{c_2}{\Gamma(\gamma_1)} \varphi_{\gamma_{k1}-1}(a,a) = 0.
$$

One has

$$
\varphi_{\gamma_{k1}-1}(a,a) = 0 \text{ and } \quad \varphi_{\gamma_{k1}-2}(t,a) \to \infty, \quad t \to a,
$$

then $c_1 = 0.$ Using the second boundary condition

$$
\psi_{p_k} \left({}^H\mathbf{D}_{a^+}^{\alpha_{k2},\beta_{k2};\varphi} u_k \right)(b) = 0,
$$

in (2.2), we have

$$
\mathbf{I}_{a^+}^{\alpha_{k1};\varphi} f_k(b) + \frac{c_2}{\Gamma(\gamma_{k1})} \varphi_{\gamma_{k1}-1}(b,a) = 0,
$$

these imply that

$$
c_2 = -\frac{\Gamma(\gamma_{k1})}{\varphi_{\gamma_{k1}-1}(b,a)} \mathbf{I}_{a^+}^{\alpha_{k1};\varphi} f_k(b).
$$

Substituting the value of c_2 in (2.4), we obtain

$$\psi_{p_k}\left({}^{H}\mathbf{D}_{a^+}^{\alpha_{k2},\beta_{k2};\varphi}u_k\right)(t)=\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}f_k(t)-\frac{\varphi_{\gamma_{k1}-1}(t,a)}{\varphi_{\gamma_{k1}-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}f_k(b),$$

thus, we have

$$ {}^{H}\mathbf{D}_{a^+}^{\alpha_{k2},\beta_{k2};\varphi}u_k(t)=\psi_{q_k}\left[\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}f_k(t)-\frac{\varphi_{\gamma_{k1}-1}(t,a)}{\varphi_{\gamma_{k1}-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}f_k(b)\right], \tag{2.5}$$

by applying the fractional integral operators $\mathbf{I}_{a^+}^{\alpha_{k2};\varphi}$, to (2.5) and using Lemma 2, we get

$$u_k(t)=\mathbf{I}_{a^+}^{\alpha_{k2};\varphi}\left\{\psi_{q_k}\left[\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}f_k(t)-\frac{\varphi_{\gamma_{k1}-1}(t,a)}{\varphi_{\gamma_{k1}-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}f_k(b)\right]\right\} \tag{2.6}$$
$$+\frac{c_3}{\Gamma(\gamma_{k2}-1)}\varphi_{\gamma_{k2}-2}(t,a)+\frac{c_4}{\Gamma(\gamma_{k2})}\varphi_{\gamma_{k2}-1}(t,a),$$

for some real constantes c_3 and c_4, Using the second boundary condition $u(a)=0$ in (2.6), we obtain

$$\frac{c_3}{\Gamma(\gamma_{k2}-1)}\varphi_{\gamma_{k2}-2}(a,a)+\frac{c_4}{\Gamma(\gamma_{k2})}\varphi_{\gamma_2-1}(a,a)=0,$$

and we have

$$\varphi_{\gamma_{k2}-1}(a,a)=0,\quad \varphi_{\gamma_{k2}-2}(t,a)\to\infty,\ \ t\to a,$$

which implies that $c_3=0.$ Now, substituting the value of c_3 in (2.6), we obtain

$$u_k(t)=\mathbf{I}_{a^+}^{\alpha_{k2};\varphi}\left\{\psi_{q_k}\left(\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}f_k(t)-\frac{\varphi_{\gamma_{k1}-1}(t,a)}{\varphi_{\gamma_{k1}-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}f_k(b)\right)\right\} \tag{2.7}$$
$$+\frac{c_4}{\Gamma(\gamma_{k2})}\varphi_{\gamma_{k2}-1}(t,a).$$

In view of condition $u_k(b)=\sum_{i=1}^{n}\left(\lambda_i u_1(\zeta_i)+\delta_i u_2(\eta_i)\right)$ in (2.2), we get

$$\sum_{i=1}^{n}\left(\lambda_i u_1(\zeta_i)+\delta_i u_2(\eta_i)\right)$$
$$=\mathbf{I}_{a^+}^{\alpha_2;\varphi}\left\{\psi_{q_k}\left(\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}f_k(t)-\frac{\varphi_{\gamma_{k1}-1}(t,a)}{\varphi_{\gamma_{k1}-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}f_k(b)\right)\right\}_{t\to b}+\frac{c_4}{\Gamma(\gamma_{k2})}\varphi_{\gamma_{k2}-1}(b,a),$$

then

$$c_4=\frac{\Gamma(\gamma_{k2})}{\varphi_{\gamma_{k2}-1}(b,a)}\sum_{i=1}^{n}\left(\lambda_i u_1(\zeta_i)+\delta_i u_2(\eta_i)\right)$$
$$-\frac{\Gamma(\gamma_{k2})}{\varphi_{\gamma_2-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_{k2};\varphi}\left\{\psi_{q_k}\left[\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}f_k(t)-\frac{\varphi_{\gamma_{k1}-1}(t,a)}{\varphi_{\gamma_{k1}-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}f_k(b)\right]_{t=b}\right\},$$

substituting the value of c_4 in (2.7), we get

$$u_k(t)=\mathbf{I}_{a^+}^{\alpha_{k2};\varphi}\left\{\psi_{q_k}\left[\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}f_k(t)-\frac{\varphi_{\gamma_{k1}-1}(t,a)}{\varphi_{\gamma_{k1}-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}f_k(b)\right]\right\}$$
$$-\frac{\varphi_{\gamma_{k2}-1}(t,a)}{\varphi_{\gamma_{k2}-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_{k2};\varphi}\left\{\psi_{q_k}\left[\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}f_k(t)-\frac{\varphi_{\gamma_{k1}-1}(t,a)}{\varphi_{\gamma_{k1}-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}f_k(b)\right]_{t=b}\right\}$$
$$+\frac{\varphi_{\gamma_{k2}-1}(t,a)}{\varphi_{\gamma_{k2}-1}(b,a)}\sum_{i=1}^{n}\left(\lambda_i u_1(\zeta_i)+\delta_i u_2(\eta_i)\right).$$

The proof is finishe

3. MAIN RESULTS

In this section, we present our main results on the existence and the stability for the above problem. We begin by considering the space $\mathbf{C}=C([a,b],R)$ denotes the Banach space of all continuous mappings from $[a,b]$ to R endowed with the norm $\|u\|_C=sup_{\,t\in[a,b]}\|u(t)\|$. It is clear that the space $\mathbf{X}=\mathbf{C}\times\mathbf{C}$ endowed with the norm $\|(u_1,u_2)\|_{\mathbf{X}}=\|u_1\|_{\mathbf{C}}+\|u_2\|_{\mathbf{C}}$.

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Now, we need to make the following assumptions:

$(\mathbf{A}_1)$: There exist positive real constants $N_k>0$, $k=1,2$, such that, for all $t\in[a,b]$, and $u_k,v_k\in\mathrm{R}(k=1,2)$, we have

$$|F_k(t,u_1,u_2)-F_k(t,v_1,v_2)|\le N_k\ \left(|u_1-v_1|+|u_2-v_2|\right)\ .$$

$(\mathbf{A}_2)$: There exist positive real constants $\Upsilon_{1k},\Upsilon_{2k}>0$, $k=1,2$, such that, for all $t\in[a,b]$, and $u_k\in\mathrm{R}(k=1,2)$, we have

$$|F_k(t,u_1,u_2)|\le\Upsilon_{1k}\|u_1\|+\Upsilon_{2k}\|u_2\|\ .$$

Now, we define the following quantities:

$$K=\varphi(b)-\varphi(a)$$
$$\Delta_k\geq\frac{2K^{\alpha_{k1}}N_k}{\Gamma(1+\alpha_{k1})},k=1,2.$$
$$\mathbf{B}_{k1}=2\frac{6^{q_k-2}K^{\alpha_{k2}}}{\Gamma(1+\alpha_{k2})}\left(\frac{K^{\alpha_{k1}}\left(\Upsilon_{1k}+\Upsilon_{2k}\right)}{\Gamma(1+\alpha_{k1})}\right)^{q_k-1},k=1,2$$
$$\mathbf{B}_{k2}=4\frac{6^{q_k-2}}{\Gamma(1+\alpha_{k2})}\left(\frac{K^{\alpha_{k1}}\left(\Upsilon_{1k}+\Upsilon_{2k}\right)}{\Gamma(1+\alpha_{k1})}\right)^{q_k-1},k=1,2$$

and

$$\mathbf{B}_{k3}=\frac{\left(q_k-1\right)\Delta_k^{q_k-2}K^{\alpha_{k2}}}{\Gamma(1+\alpha_{k2})}\left[\frac{2K^{\alpha_{k1}}N_k}{\Gamma(1+\alpha_{k1})}\right]+\sum_{i=1}^{n}\left(\lambda_i+\delta_i\right),k=1,2$$

Based on the above hypotheses, we present to the reader the following result. Now, consider the following operator $\mathbf{T}:\mathbf{X}\to\mathbf{X}$ by:

$$\begin{aligned}
&\mathbf{T}_k(u_1,u_2)(t)\\
&=\frac{1}{\Gamma(\alpha_{k2})}\int_a^t\varphi'(s)\varphi_{\alpha_{k2}-1}(t,s)\\
&\times\left\{\psi_{q_k}\left[\frac{1}{\Gamma(\alpha_{k1})}\int_a^s\varphi'(s)\varphi_{\alpha_{k1}-1}(s,z)F_{u_k}(z)dz-\frac{\varphi_{\gamma_{k1}-1}(t,a)}{\varphi_{\gamma_{k1}-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}F_{u_k}(b)\right]\right\}ds\\
&+\frac{\varphi_{\gamma_{k2}-1}(t,a)}{\varphi_{\gamma_{k2}-1}(b,a)}\sum_{i=1}^{n}\left(\lambda_iu_1\left(\zeta_i\right)+\delta_iu_2\left(\eta_i\right)\right)-\frac{\varphi_{\gamma_{k2}-1}(t,a)}{\Gamma(\alpha_{k2})\varphi_{\gamma_{k2}-1}(b,a)}\int_a^b\varphi'(s)\varphi_{\alpha_{k2}-1}(b,y)\\
&\times\left\{\psi_{q_k}\left[\frac{1}{\Gamma(\alpha_{k1})}\int_a^s\varphi'(s)\varphi_{\alpha_{k1}-1}(s,z)F_{u_k}(z)dz-\frac{\varphi_{\gamma_{k1}-1}(t,a)}{\varphi_{\gamma_{k1}-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}F_{u_k}(b)\right]\right\}ds.
\end{aligned}$$

where

$$F_{u_k}(t)=F_k(t,u_1,u_2).$$

Here, we divide the operator $\mathbf{T}(u_1,u_2)(t)$ as follows

$$\begin{cases}\mathbf{T}(u_1,u_2)(t)=(\mathbf{T}_1(u_1,u_2)(t),\mathbf{T}_2(u_1,u_2)(t)),\\ \text{and}\\ \mathbf{T}_k(u_1,u_2)(t)=\mathbf{P}_{1k}(u_1,u_2)(t)+\mathbf{P}_{2k}(u_1,u_2)(t),k=\overline{1,2}.\end{cases}\tag{3.1}$$

where

$$\begin{aligned}&\mathbf{P}_{1k}(u_1,u_2)(t)\\&=\frac{1}{\Gamma(\alpha_{k2})}\int_a^t \varphi'(s)\varphi_{\alpha_{k2}-1}(t,s)\\&\times\left\{\psi_{q_k}\left[\frac{1}{\Gamma(\alpha_{k1})}\int_a^s \varphi'(s)\varphi_{\alpha_{k1}-1}(s,z)F_{u_k}(z)dz-\frac{\varphi_{\gamma_{k1}-1}(t,a)}{\varphi_{\gamma_{k1}-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}F_{u_k}(b)\right]\right\}ds\end{aligned}$$

and

$$\begin{aligned}&\mathbf{P}_{2k}(u_1,u_2)(t)\\&=\frac{\varphi_{\gamma_{k2}-1}(t,a)}{\varphi_{\gamma_{k2}-1}(b,a)}\sum_{i=1}^{n}\left(\lambda_i u_1(\zeta_i)+\delta_i u_2(\eta_i)\right)\\&-\frac{\varphi_{\gamma_{k2}-1}(t,a)}{\Gamma(\alpha_{k2})\varphi_{\gamma_{k2}-1}(b,a)}\int_a^b \varphi'(s)\varphi_{\alpha_{k2}-1}(b,y)\\&\times\left\{\psi_{q_k}\left[\frac{1}{\Gamma(\alpha_{k1})}\int_a^s \varphi'(s)\varphi_{\alpha_{k1}-1}(s,z)F_{u_k}(z)dz-\frac{\varphi_{\gamma_{k1}-1}(t,a)}{\varphi_{\gamma_{k1}-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}F_{u_k}(b)\right]\right\}ds.\end{aligned}$$

Our first result concerning the existence of solutions of the problem (3.1) for which we have used the fixed point theorem of Krasnoselskii's is as follows

Theorem 1. Let $F_{u_k}:[a,b]\times\mathrm{R}^2\to\mathrm{R}$ be continuous functions which satisfy conditions A_1-A_2 .If

$$\left(\sum_{i=1}^{n}\left(\lambda_i+\delta_i\right)\right)<\frac{1}{6},\ \text{and}\ \left(\mathbf{B}_{13}+\mathbf{B}_{23}\right)<1.$$

Then, problem (1.1) admits at least one solution on $[a,b]$

Proof: The proof will be given in several steps. Let $U_r=\{(u_1,u_2)\in\mathbf{X},\|(u_1,u_2)\|\le r\},$ so that

$$\max\left\{\left(6\mathbf{B}_{11}\right)^{2-q_1},\left(6\mathbf{B}_{21}\right)^{2-q_2}\right\}\le r$$

First step: We prove that

$$\|\mathbf{T}(u_1,u_2)(t)\|_{\mathbf{X}}\le r.$$

From (3.1), we obtain

$$\|\mathbf{T}(u_1,u_2)(t)\|_{\mathbf{X}}=\|\mathbf{T}_1(u_1,u_2)(t)\|_{\mathbf{C}}+\|\mathbf{T}_2(u_1,u_2)(t)\|_{\mathbf{C}},$$

and

$$\|\mathbf{T}_k(u_1,u_2)(t)\|_{\mathbf{C}}\le\sup_{t\in[a,b]}|\mathbf{P}_{1k}(u_1,u_2)(t)|+\sup_{t\in[a,b]}|\mathbf{P}_{2k}(u_1,u_2)(t)|,k=\overline{1,2}.$$

Let $(u_1,u_2),(v_1,v_2)\in U_r$. by $\psi_{q_k}(z)=z|z|^{q_k-2}$, and Lemma 5, we have

$$\begin{aligned}
&\left|\mathbf{P}_{1k}(u_1,u_2)(t)\right| \\
&\leq \frac{1}{\Gamma(\alpha_{k2})}\left|\int_a^t \varphi'(s)\varphi_{\alpha_{k2}-1}(t,s)\right. \\
&\left.\times\left\{\psi_{q_k}\left[\frac{1}{\Gamma(\alpha_{k1})}\int_a^s \varphi'(s)\varphi_{\alpha_{k1}-1}(s,z)F_{u_k}(z)dz-\frac{\varphi_{\gamma_{k1}-1}(t,a)}{\varphi_{\gamma_{k1}-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}F_{u_k}(b)\right]\right\}ds\right| \\
&\leq \frac{K^{\alpha_{k2}}}{\Gamma(1+\alpha_{k2})}\left|\frac{1}{\Gamma(\alpha_{k1})}\int_a^s \varphi'(s)\varphi_{\alpha_{k1}-1}(s,z)F_{u_k}(z)dz-\frac{\varphi_{\gamma_{k1}-1}(t,a)}{\varphi_{\gamma_{k1}-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}F_{u_k}(b)\right|^{q_k-1} \\
&\leq \frac{3^{q_k-2}K^{\alpha_{k2}}}{\Gamma(1+\alpha_{k2})}\left(\left|\frac{1}{\Gamma(\alpha_{k1})}\int_a^s \varphi'(s)\varphi_{\alpha_{k2}-1}(s,z)F_{u_k}(z)dz\right|^{q_k-1}+\left|\mathbf{I}_{a^+}^{\alpha_1;\varphi}F_{u_k}(b)\right|^{q_k-1}\right) \\
&\qquad\leq \frac{3^{q_k-2}K^{\alpha_{k2}}}{\Gamma(1+\alpha_{k2})}\left(2\left|\frac{K^{\alpha_{k1}}\left(\Upsilon_{1k}+\Upsilon_{2k}\right)\delta}{\Gamma(1+\alpha_{k1})}\right|^{q_k-1}\right) \\
&\qquad\leq \frac{3^{q_k-2}K^{\alpha_{k2}}}{\Gamma(1+\alpha_{k2})}\left(2\left(\frac{K^{\alpha_{k1}}\left(\Upsilon_{1k}+\Upsilon_{2k}\right)}{\Gamma(1+\alpha_{k1})}\right)^{q_k-1}\right)\delta^{q_k-1} \qquad (3.3)\\
&\qquad\leq \mathbf{B}_{k1}r^{q_k-1},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\mathbf{P}_{2k}(u_1,u_2)(t)\right| \\
&\leq \left|\frac{\varphi_{\gamma_{k2}-1}(t,a)}{\varphi_{\gamma_{k2}-1}(b,a)}\sum_{i=1}^{n}\left(\lambda_i u_1(\zeta_i)+\delta_i u_2(\eta_i)\right)\right| \\
&+\left|\frac{\varphi_{\gamma_{k2}-1}(t,a)}{\Gamma(\alpha_{k2})\varphi_{\gamma_{k2}-1}(b,a)}\int_a^b \varphi'(s)\varphi_{\alpha_{k2}-1}(b,y)\right. \\
&\left.\times\left\{\psi_{q_k}\left[\frac{1}{\Gamma(\alpha_{k1})}\int_a^s \varphi'(s)\varphi_{\alpha_{k1}-1}(s,z)F_{u_k}(z)dz-\frac{\varphi_{\gamma_{k1}-1}(t,a)}{\varphi_{\gamma_{k1}-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}F_{u_k}(b)\right]\right\}ds\right| \\
&\leq \left(\sum_{i=1}^{n}\left(\lambda_i+\delta_i\right)\right)r+\mathbf{B}_{k1}r^{q_k-1}, \qquad (3.4)
\end{aligned}$$

by (3 .2) , (3.3) and (3.4) we get

$$\left\|\mathbf{T}(u_1,u_2)(t)\right\|_{\mathbf{X}} \leq 2\left(\sum_{i=1}^{n}\left(\lambda_i+\delta_i\right)\right)r+2\mathbf{B}_{11}r^{q_1}+2\mathbf{B}_{21}r^{q_2}\leq r.$$

Second Step: $\mathbf{P}_2$ is a contraction. $\left(\mathbf{P}_2=\mathbf{P}_{21}+\mathbf{P}_{22}\right)$

Let $(u_1,u_2),(v_1,v_2)\in U_r$, we have the following estimate

$$\begin{aligned}&\left|\mathbf{P}_{2k}(u_1,u_2)(t)-\mathbf{P}_{2k}(v_1,v_2)(t)\right|\\&\leq\sum_{i=1}^{n}\left(\lambda_i\left|u_1(\zeta_i)-v_1(\zeta_i)\right|+\delta_i\left|u_2(\eta_i)-v_2(\eta_i)\right|\right)\\&+\frac{K^{\alpha_{k2}}}{\Gamma(1+\alpha_{k2})}\\&\times\left|\psi_{q_k}\left[\frac{1}{\Gamma(\alpha_{k1})}\int_a^s\varphi'(s)\varphi_{\alpha_{k1}-1}(s,z)F_{u_k}(z)dz-\frac{\varphi_{\gamma_{k1}-1}(t,a)}{\varphi_{\gamma_{k1}-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}F_{u_k}(b)\right]\right.\\&\left.-\psi_{q_k}\left[\frac{1}{\Gamma(\alpha_{k1})}\int_a^s\varphi'(s)\varphi_{\alpha_{k1}-1}(s,z)F_{v_k}(z)dz-\frac{\varphi_{\gamma_{k1}-1}(t,a)}{\varphi_{\gamma_{k1}-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}F_{v_k}(b)\right]\right|,\end{aligned}$$

by Lemma 4 , we get

$$\begin{aligned}&\left|\mathbf{P}_{2k}(u_1,u_2)(t)-\mathbf{P}_{2k}(v_1,v_2)(t)\right|\\&\leq\sum_{i=1}^{n}\left(\lambda_i\left|u_1(\zeta_i)-v_1(\zeta_i)\right|+\delta_i\left|u_2(\eta_i)-v_2(\eta_i)\right|\right)+\frac{(q_k-1)\Delta^{q_k-2}K^{\alpha_{k2}}}{\Gamma(1+\alpha_{k2})}\\&\times\left[\frac{2K^{\alpha_{k1}}}{\Gamma(1+\alpha_{k1})}\left|F_{u_k}(t)-F_{v_k}(t)\right|\right]\end{aligned}$$

$$\begin{aligned}&\leq\left(\sum_{i=1}^{n}\lambda_i\right)\|u_1-v_1\|+\left(\sum_{i=1}^{n}\delta_i\right)\|u_2-v_2\|\\&+\frac{(q_k-1)\Delta^{q_k-2}K^{\alpha_{k2}}}{\Gamma(1+\alpha_{k2})}\left[\frac{2K^{\alpha_{k1}}N_k}{\Gamma(1+\alpha_{k1})}\right]\left(\|u_1-v_1\|+\|u_2-v_2\|\right)\end{aligned}$$

$$\leq\left(\frac{(q_k-1)\Delta_k^{q_k-2}K^{\alpha_{k2}}}{\Gamma(1+\alpha_{k2})}\left[\frac{2K^{\alpha_{k1}}N_k}{\Gamma(1+\alpha_{k1})}\right]+\sum_{i=1}^{n}\lambda_i+\delta_i\right)\times\left(\|u_1-v_1\|+\|u_2-v_2\|\right)$$

$$\leq\mathbf{B}_{k3}\left(\|u_1-v_1\|+\|u_2-v_2\|\right).$$

So,

$$\left|(\mathbf{P}_2(u_1,u_2)(t)-(\mathbf{P}_2(v_1,v_2)(t)\right|\leq\left(\mathbf{B}_{13}+\mathbf{B}_{23}\right)\left(\|u_1-v_1\|+\|u_2-v_2\|\right)\leq\|u_1-v_1\|+\|u_2-v_2\|.$$

Since $\left(\mathbf{B}_{13}+\mathbf{B}_{23}\right)<1$, the operator $\mathbf{P}_2$ is a contraction.

Third Step: P_1 is compact and continuous.

Since F_{u_k} are a continuous functions, this implies that the operator $\mathbf{P}_2$ is continuous on U_r and by (3.4) we have

$$\left|\mathbf{P}_2(u_1,u_2)(t)\right| \le 2\left(\sum_{i=1}^{n}\left(\lambda_i+\delta_i\right)\right)r+\mathbf{B}_{11}r^{q_1}+\mathbf{B}_{21}r^{q_2}. \tag{3.5}$$

Moreover, $\mathbf{P}_1(u_1,u_2)=\mathbf{P}_{11}(u_1,u_2)+\mathbf{P}_{12}(u_1,u_2)$ is uniformly bounded by (3.5).
Next, we show equicontinuity and $t_1,t_2\in[0,1]$, such that $t_1<t_2$ we have

$$\left|\mathbf{P}_{1k}(u_1,u_2)(t_2)-\mathbf{P}_{1k}(u_1,u_2)(t_1)\right|$$

$$=\frac{1}{\Gamma(\alpha_{k2})}\left|\int_a^{t_2}\varphi'(s)\varphi_{\alpha_{k2}-1}(t_2,s)\right.$$

$$\times\left\{\psi_{q_k}\left[\frac{1}{\Gamma(\alpha_{k1})}\int_a^s\varphi'(s)\varphi_{\alpha_{k1}-1}(s,z)F_{u_k}(z)dz-\frac{\varphi_{\gamma_{k1}-1}(t,a)}{\varphi_{\gamma_{k1}-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}F_{u_k}(b)\right]\right\}ds$$

$$-\int_a^{t_1}\varphi'(s)\varphi_{\alpha_{k2}-1}(t_1,s)$$

$$\left.\times\left\{\psi_{q_k}\left[\frac{1}{\Gamma(\alpha_{k1})}\int_a^s\varphi'(s)\varphi_{\alpha_{k1}-1}(s,z)F_{u_k}(z)dz-\frac{\varphi_{\gamma_{k1}-1}(t,a)}{\varphi_{\gamma_{k1}-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_{k1};\varphi}F_{u_k}(b)\right]\right\}ds\right|$$

$$\le\frac{2\times 3^{q_k-2}}{\Gamma(1+\alpha_{k2})}\left(2\left(\frac{K^{\alpha_{k1}}\left(\Upsilon_{1k}+\Upsilon_{2k}\right)}{\Gamma(1+\alpha_{k1})}\right)^{q_k-1}\right)r^{q_k}\left(\varphi(t_2)-\varphi(t_2)\right)^{\alpha_{k2}}$$

$$\le r^{q_k-1}\mathbf{B}_{k2}\left(\varphi(t_2)-\varphi(t_2)\right)^{\alpha_{k2}}.$$

So,

$$\left|\mathbf{P}_1(u_1,u_2)(t_2)-\mathbf{P}_1(u_1,u_2)(t_1)\right|\le r^{q_1}\mathbf{B}_{12}\left(\varphi(t_2)-\varphi(t_2)\right)^{\alpha_{12}}+r^{q_2}\mathbf{B}_{22}\left(\varphi(t_2)-\varphi(t_2)\right)^{\alpha_{22}}.$$

Consequently,

$$\left|\mathbf{P}_1(u_1,u_2)(t_2)-\mathbf{P}_1(u_1,u_2)(t_1)\right|\to 0,\ \text{ as } t_1\to t_2.$$

This shows that $\mathbf{P}_1U_r$ is equicontinuous. Hence, by Arzelià-Ascoli theorem $\mathbf{P}_1$ is completely continuous on U_r. As a consequence of Krasnoselskii's fixed point theorem, we conclude that has a fixed point which is a solution of (1.1). The proof of Theorem 1 is thus completely achieved.

4. EXAMPLE

Consider the following problem for all $t \in [0,1]$:

$$\begin{cases} D_{0^+}^{\frac{1}{2};t^2} \varphi_p\left(D_{0^+}^{\frac{1}{4};t^2} u(t)\right) = \dfrac{\sqrt{u(t)}}{u^2(t)+3} + \dfrac{Sin\, v(t)}{16+t^2}, \\ D_{0^+}^{\frac{1}{4};t^2} \varphi_p\left(D_{0^+}^{\frac{1}{2};t^2} v(t)\right) = \dfrac{u(t)+2}{u(t)+3} + \dfrac{Sin\, v(t)}{\sqrt{v^2(t)+3}}, \\ u(0) = v(0) = 0, \\ u(1) = \sum_{i=1}^{n} \dfrac{1}{4(i!)}\big(u(\delta_i)+v(\delta_i)\big), \varphi_p\left(D_{0^+}^{\frac{1}{4};t^2} u(t)\right) = 0, \\ v(1) = \sum_{i=1}^{n} \dfrac{1}{8(i!)}\big(u(\delta_i)-v(\delta_i)\big), = \varphi_p\left(D_{0^+}^{\frac{1}{2};t^2} v(t)\right) = 0, \end{cases}$$

Clearly,

$$\begin{cases} F_1\big(t,u(t),v(t)\big) = \dfrac{\sqrt{u(t)}}{u^2(t)+3} + \dfrac{Sin\, v(t)}{16+t^2}, \\ F_1\big(t,u(t),v(t)\big) \leq \dfrac{1}{4} sup_{t\in[0,1]}|u(t)| + \dfrac{1}{16} sup_{t\in[0,1]}|v(t)|. \end{cases}$$

and

$$\begin{cases} F_2\big(t,u(t),v(t)\big) = \dfrac{u(t)+2}{u(t)+3} + \dfrac{Sin\, v(t)}{\sqrt{v^2(t)+3}}, \\ F_2\big(t,u(t),v(t)\big) \leq \dfrac{3}{4} sup_{t\in[0,1]}|u(t)| + \dfrac{1}{2} sup_{t\in[0,1]}|v(t)|. \end{cases}$$

Thus, the assumptions ($A_1 - A_2$) are satisfied and Theorem - implies that the problem has a unique solution on [0,1].

5. CONCLUSION

In this article, we have demonstrated the existence of solutions of a coupled system of fractional differential equations with nonlocal integro multi point boundary conditions by using the *p*-Laplacian operator and the *φ*-Hilfer derivatives, using the fixed point theorem of Krasnoselskii's, we have completed our work by example is presented at the end to show the applicability of the obtained results.

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