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DIFFERENCE-CUM-EXPONENTIAL EFFICIENT ESTIMATOR OF POPULATION VARIANCE

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Abstract. *In the current investigation, we have suggested a Difference Cum-Exponential Type Efficient Estimator of Population variance of the study variable using information on the auxiliary variable. Up to the first order of approximation, the proposed estimator's bias and mean square error (MSE) expressions are derived and suggested optimum estimator is also found, with its optimal qualities are investigated. The suggested estimator is proven to be more competent than sample variance, classic ratio estimators based on Isaki , Singh et al. and Kadilar and Cingi estimators in [1-3]. Numerical study is also carried out by using real data sets.*

Keywords*: ratio estimator; exponential estimator; bias; mean square error; efficiency.*

1. INTRODUCTION

In general, supplementary data is used is survey sampling to improve sampling strategies and attain higher accuracy in estimations of particular population parameters like the mean, variance of the research variable. This evidence can be employed during the design stage (leading to stratified, systematic, or probability proportional to size sample designs, for example) as well as the estimating stage. When auxiliary data is employed at the estimation stage, it is widely known that the ratio, product and regression estimation approaches are frequently used in various circumstances. This is an example of in sampling theory, estimating population variance is a hot topic, and numerous efforts have been made to increase the precision of the estimates. A considerable deal has been written about survey sampling in the literature. A variety of strategies have been utilised, including different estimators and the use of auxiliary data. In the current manuscript, we have suggesteda Difference Cum-Exponential Type Efficient Estimator of Population variance. Here the keypurpose of this paper is to propose a new estimator and to increase the efficiency of the estimator for the population variance.

2. REVIEW OF LITERATURE

Let the finite population U consists of N units $U_1, U_2, \ldots U_N$ and the simple random sampling without replacement (SRSWOR) technique is used to select a sample of n units

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from this population. Assume that Y and X represent the study and explanatory variables, respectively, and these variables are closely connected.

It is well identified that the variance of the sample variance estimator, $t_0 = s_y^2$

$$V(t_0) = \gamma S_y^4 \left[(\lambda_{40} - 1)\right] \tag{2.1}$$

Based on Robson in [4] stated estimator for the population mean, product type-estimator for the population variance as

$$t_1 = s_y^2 \left(\frac{s_x^2}{S_x^2} \right) \tag{2.2}$$

where, $s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(y_i - \overline{y}\right)^2$, $S_x^2 = \frac{1}{n-1}\sum_{i=1}^{N}\left(X_i - \overline{X}\right)^2$, $s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2$,
$\overline{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$, $\overline{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i$, $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} n_i$, and $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$

The MSE of the estimator, Eq. (2.2) presents as

$$MSE(t_1) = \gamma S_y^4 \left[(\lambda_{40} - 1) + (\lambda_{04} - 1) + 2(\lambda_{22} - 1)\right] \tag{2.3}$$

where,

$$\lambda_{rs} = \frac{\mu_{rs}}{\mu_{20}^{r/2}\,\mu_{02}^{s/2}},\ \mu_{rs} = \frac{1}{N-1}\sum\left(Y_i - \overline{Y}\right)^r,\ f = \frac{n}{N} \text{ and } \gamma = \frac{(1-f)}{n}$$

Based on Bahl and Tuteja and Singh *et al.* in [5-2] suggested the exponential-ratio estimator for the population variance as

$$t_2 = s_y^2 \exp\left[\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right] \tag{2.4}$$

The MSE of the estimator, Eq. (2.4) presents as

$$MSE(t_2) = \gamma S_y^4 \left[(\lambda_{40} - 1) + \frac{(\lambda_{04} - 1)}{4} - (\lambda_{22} - 1)\right] \tag{2.5}$$

Based on Bahl and Tuteja in [5], the exponential-product type estimator for the population variance can be define is stated as

$$t_3 = s_y^2 \exp\left[\frac{s_x^2 - S_x^2}{s_x^2 + S_x^2}\right] \tag{2.6}$$

The MSE of the estimator, Eq. (2.6) presents as

$$MSE(t_3) = \gamma S_y^4 \left[(\lambda_{40} - 1) + \frac{(\lambda_{04} - 1)}{4} + (\lambda_{22} - 1) \right] \tag{2.7}$$

Kadilar and Cingi in [3], the ratio estimator for the population variance can be stated as

$$t_4 = s_y^2 \left[\frac{(S_x^2)^2}{(s_x^2)^2} \right] \tag{2.8}$$

The MSE of the estimator, Eq. (2.8) presents as

$$MSE(t_4) = \gamma S_y^4 [(\lambda_{40} - 1) + 4(\lambda_{04} - 1) - 4(\lambda_{22} - 1)] \tag{2.9}$$

Kadilar and Cingi in [3], the product-type estimator for the population variance, can be stated as

$$t_5 = s_y^2 \left[\frac{(s_x^2)^2}{(S_x^2)^2} \right] \tag{2.10}$$

The MSE of the estimator, Eq. (2.10) presents as

$$MSE(t_5) = \gamma S_y^4 [(\lambda_{40} - 1) + 4(\lambda_{04} - 1) + 4(\lambda_{22} - 1)] \tag{2.11}$$

3. PROPOSED ESTIMATOR

We suggest the estimator of the population variance as a result and motivated Tiwari *et al.* in [6],

$$t_{RK} = \eta s_y^2 + \delta (S_x^2 - s_x^2) \left[1 - \exp\left(\frac{S_x - s_x}{S_x + s_x} \right) \right]^{-1} \tag{3.1}$$

where *k*, *t* are suitably chosen constants.

For study the large sample assets of the suggested class of estimators t_6, we express $s_y^2 = S_y^2(1 + \varepsilon_0)$ and $s_x^2 = S_x^2(1 + \varepsilon_0)$ such that $E(\varepsilon_1) = 0$ for i=0,1 and $E(\varepsilon_0^2) = \gamma(\lambda_{40} - 1)$, $E(\varepsilon_1^2) = \gamma(\lambda_{04} - 1), E(\varepsilon_0 \varepsilon_1) = \gamma(\lambda_{22} - 1)$.

Express (3.1) in terms of ε_i's

$$t_{RK} = \eta S_y^2 (1 + \varepsilon_0) + \delta S_x^2 \varepsilon_1 \left[1 - \exp\left(\frac{-\varepsilon_1}{2 + \varepsilon_1} \right) \right]^{-1}$$

$$t_{RK} = \eta S_y^2(1+\varepsilon_0) + \delta S_x^2 \varepsilon_1 \left[1 - \exp\left\{\frac{-\varepsilon_1}{2}\left(1+\frac{\varepsilon_1}{2}\right)^{-1}\right\}\right]^{-1}$$

Expression in terms of ε’s and ignoring terms having ε’s degree greater than two to get first order approximation, we have

$$t_{RK} = \eta S_y^2(1+\varepsilon_0) + 2\delta S_x^2\left[1 - \left(\frac{3\varepsilon_1}{4} + \frac{\varepsilon_1^2}{2}\right)\right]^{-1}$$

On simplifying, we get

$$t_{RK} - S_y^2 = \left[S_y^2(\eta - 1) - 2\delta S_x^2\right] + \eta S_y^2 \varepsilon_0 - \delta S_x^2 \left(\frac{3\varepsilon_1}{2}\right)\frac{\delta S_x^2}{8}\varepsilon_1^2 \tag{3.2}$$

To obtain the bias of t_{rk} up to first order of Approximation, take expectation on both sides of Eq. (3.2), we have

$$Bias(t_{RK}) = \left[S_y^2(\eta - 1) - 2\delta S_x^2\right] - \frac{\delta S_x^2}{8}\gamma(\lambda_{04} - 1) \tag{3.3}$$

By squaring Eq. (3.3) on both sides and terminating the terms having ε’s degree more than two, we have

$$\begin{aligned}\left(t_{RK} - S_y^2\right)^2 &= \left[S_y^2(\eta-1) - 2\delta S_x^2\right]^2 + \eta^2 S_y^4 \varepsilon_1^2 - \eta\delta S_{xy}^2 3\varepsilon_0\varepsilon_1 \\ &\quad + \left[\frac{11}{4}t^2 - \frac{1}{4}\delta(\eta-1)R\right] S_x^4 \varepsilon_2^2\end{aligned} \tag{3.4}$$

To get the MSE of t_{rk} to the first order of approximation, take expectation on both sides of Eq. (3.4), we have

$$\begin{aligned}MSE(t_{RK}) &= \left[S_y^2(\eta-1) - 2\delta S_x^2\right]^2 + \eta^2 S_y^4 \gamma(\lambda_{40}-1) - \eta\delta S_{xy}^2 3\gamma(\lambda_{22}-1) \\ &\quad + \left[\frac{11}{4}\delta^2 - \frac{1}{4}\delta(\eta-1)R\right] S_x^4 \gamma(\lambda_{04}-1)\end{aligned} \tag{3.5}$$

$$MSE(t_{RK}) = S_y^4\left[1 - 2\eta + \eta^2 \phi_1 + \delta\phi_2 \delta^2 \phi_3 - \eta\delta\phi_4\right] \tag{3.6}$$

where,

$$\phi_1 = \gamma(\lambda_{40} - 1), \phi_2 = \frac{4}{R} + \frac{\gamma(\lambda_{04}-1)}{4R}, \phi_3 = \frac{4}{R^2} + \frac{11}{4R^2}\gamma(\lambda_{04}-1)$$

$$\phi_4 = \frac{4}{R} + \frac{\gamma(\lambda_{04}-1)}{4R} + \frac{3}{R}\gamma(\lambda_{22}-1), \ R = \frac{S_y^2}{S_x^2}$$

To minimize $MSE(t_{RK})$, we have to differentiate $MSE(t_{RK})$ in Eq. (3.6) partially with respect to k and t, and equating to zero, we get

$$2\varphi_1\eta - \varphi_4\delta = 2 \tag{3.7}$$

and

$$-\phi_4\eta + 2\phi_3\delta = -\phi_2 \tag{3.8}$$

On solving above simultaneous Eqs.(3.7) and (3.8) for *k* and *t*, the optimal values of *k* and *t* are obtained as

$$\eta_0 = \frac{4\phi_3 - \phi_2\phi_4}{4\phi_1\phi_3 - \phi_4^2}, \delta_0 = \frac{2\phi_4 - 2\phi_1\phi_2}{4\phi_1\phi_3 - \phi_4^2} \tag{3.9}$$

The minimum *MSE* (t_{RK}) can be obtained by using the optimum values of *k* and *t* from Eq. (3.9) in Eq. (3.6), as

$$MSE\min(t_{RK}) = S_y^4\left[1 - \frac{4\phi_3 + \phi_1\phi_2^2 - 2\phi_2\phi_4}{4\phi_1\phi_3 - \phi_4^2}\right] \tag{3.10}$$

4. EFFICIENCY COMPARISON

In this section, the performance of the suggested estimator has been demonstrated over the existing estimators in the literature as follows:

From (2.1) and (3.10)

I. $[Var(t\_0) - 〖MSE〗 \_min\ (t\_RK)] > 0$

$$\omega_0 - \omega_6 > 0 \tag{3.1}$$

From (2.3) and (3.10)

II. $[MSE(t_1) - MSE_{min}(t_{RK})] > 0$

$$\omega_1 - \omega_6 > 0 \tag{3.2}$$

From (2.5) and (3.10)

III. $[MSE(t_2) - MSE_{min}(t_{RK})] > 0$

$$\omega_2 - \omega_6 > 0 \tag{3.3}$$

From (2.7) and (3.10)

IV. $[MSE(t_3) - MSE_{min}(t_{RK})] > 0$

$$\omega_3 - \omega_6 > 0 \tag{3.4}$$

From (2.9) and (3.10)

V. $[MSE(t_4) - MSE_{min}(t_{RK})] > 0$

$$\omega_4 - \omega_6 > 0 \quad (3.5)$$

From (2.11) and (3.10)

VI. $[MSE(t_5) - MSE_{min}(t_{RK})] > 0$

$$\omega_5 - \omega_6 > 0 \quad (3.6)$$

$$\omega_0 = \sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2$$

$$\omega_1 = \left[(\lambda_{40}-1)+(\lambda_{04}-1)+2(\lambda_{22}-1)\right]$$

$$\omega_2 = \left[(\lambda_{40}-1)+\frac{(\lambda_{04}-1)}{4}-(\lambda_{22}-1)\right]$$

$$\omega_3 = \left[(\lambda_{40}-1)+\frac{(\lambda_{04}-1)}{4}+(\lambda_{22}-1)\right]$$

$$\omega_4 = \left[(\lambda_{40}-1)+4(\lambda_{04}-1)-4(\lambda_{22}-1)\right]$$

$$\omega_5 = \left[(\lambda_{40}-1)+4(\lambda_{04}-1)+4(\lambda_{22}-1)\right]$$

$$\omega_6 = \left[1-\frac{4\phi_3+\phi_1\phi_2^2-2\phi_2\phi_4}{4\phi_1\phi_3-\phi_4^2}\right]$$

It is observed that $\hat{\hat{Y}}_{RK(min)}$ is always more proficient than the traditional estimators t_0, t_1, t_2, t_3, t_4 and t_5, because the conditions from (3.1) to (3.6) are always satisfied.

5. EMPIRICAL STUDY

A data set is considered to exhibits theenactment of the suggested estimator; we use some real-life population.Description of the populations is given below:

Population I: in [7]
Y: Figure of inhabitants in 1930
X: Figure of inhabitants in 1920

Table 5.1. Statistical Description

N = 196	n = 49	λ_{40} = 8.5362	λ_{04} = 7.3617	λ_{22}=7.8780	S_y= 123.12	S_x= 104.57

Population II: in [8]
Y: Output for 80 factories.
X: Fixed Capital.

Table 5.2. Statistical Description

N = 80	n = 20	λ40 = 2.2667	λ04= 2.8664	λ22=2.2209	S_y= 18.3569	S_x= 11.4563

Population III: in [9]
Y: leaf area for newly developed strain.
X: weight of leaves

Table 5.3. Statistical Description

N = 39	n = 14	λ_{40} = 2.4032	λ_{04}= 2.993	λ_{22}=2.4882	S_y= 25.2442	S_x= 13.1413

Population IV: in [10]
An artificial population containing the following description below.

Table 5.4. Statistical Description

N = 200	n = 60	λ_{40} = 2.56	λ_{04}= 3.74	λ_{22}=2.47	S_y= 14.109	S_x= 6.967

Table 5.5. Competences of estimators and suggested estimator with respect to S_y^2

Estimators	**PRE**			
	POPULATION I	POPULATION II	POPULATION III	POPULATION IV
$t_0 = S_y^2$	-	-	-	-
t_1	27.252	22.723	22.019	21.547
t_2	335.147	247.209	339.557	201.29
t_3	47.088	42.878	41.397	41.992
t_4	137.748	32.912	41.001	23.494
t_5	12.458	9.3031	9.155	8.4783
$\boldsymbol{t_{RK}}$	**789.457**	**139.831**	**155.938**	**173.007**

Table 5.5, revealed the percent relative proficiencies (PRE) of estimators for population I. Here wesee that the suggested exponential ratio type estimator t_{RK} proved to be the best estimator in the sense of having highest percent relative efficiency than usual unbiased estimators$t_0 = S_y^2$, t_1, t_2, t_3,t_4 and t_5 for the population I.

Also the relative proficiencies of estimators for population II-IV. It is seen that the proposed exponential ratio type estimator t_{RK} proved to be the best estimator in the sense of having highest percent relative efficiency than usual unbiased estimators$t_0 = S_y^2$, t_1, t_3,t_4 and t_5 for the population II-IV.

6. CONCLUSION

Finally, we conclude that when auxiliary data is available, there is sufficient flexibility in selecting the suggested estimator of population variance of variable under to be study. Thus suggested exponential ratio type estimator t_{RK} is thought to be the best estimator in terms of percent relative efficiency when compared to existing estimators.

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