ORIGINAL PAPER

QUESTING ELEVATED FAMILY OF PRODUCT ESTIMATORS OF POPULATION MEAN USING AUXILIARY VARIABLES

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Abstract. *The present manuscript deals with the enhanced estimation of population mean using the known negatively correlated auxiliary variable. We introduce a naïve class of product type estimators of population variance utilizing known auxiliary parameters. The expressions for the bias and the mean squared error (MSE) of the proposed family of estimators are retained up to the first order of approximation. The optimal values of the characterizing constants are obtained by the method of Maxima-Minima. The least value of MSE of the introduced class is obtained for these optimal values of the constants. The suggested family of estimators is compared with the competing product estimators of the population mean. The efficiency conditions are also obtained and these are verified through the real data set. The estimators with the lesser MSE are recommended for the practical utility in different areas of applications.*

Keywords: *primary variable; auxiliary variable; bias; MSE; percentage relative efficiency.*

1. INTRODUCTION

It is established in the theory of sampling that the proper use of auxiliary information enhances the efficiency of the estimator of the population mean. The auxiliary variable (X) may have either positive or negative correlation with the main variable (Y). The ratio estimators are used when positive correlation between Y and X while product estimators are used when there is negative correlation between Y and X. We are dealing with the product estimators in this paper. The population mean ($\bar{Y}$) of primary variable is estimated by the sample mean estimator ($\bar{y}$) when the population under consideration is large due to money and time limitations. The estimator $\bar{y}$ is unbiased for $\bar{Y}$ but has a reasonably large sampling variance. Thus we seek for an estimator having its sampling variance closer to $\bar{Y}$ than the $\bar{y}$. This aim is met through the use of negatively correlated X. Robson [1] utilized the negatively correlated X and suggested the conventional product estimator of $\bar{Y}$ and shown improvement over the estimator $\bar{y}$. Later on various authors utilized the negatively correlated auxiliary parameters and suggested different product type estimators of $\bar{Y}$ and shown improvements over $\bar{y}$ and other competing estimators. The some of the references for elevated estimation of

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$\bar{Y}$ using product type estimator may be made of Pandey and Dubey [2], Bahl and Tuteja [3], Gandge *et al.* [4], Singh and Tailor [5], Singh [6], Tailor *et al.* [7], Onyeka [8], Kumar and Chhaparwal [9], and Ijaz *et al.* [10].

In the recent years the authors have done some considerable work for enhanced estimation of $\bar{Y}$ using product type estimators. Kumar *et al.* [11] worked on the modified product type estimators of $\bar{Y}$ while Hasan *et al.* [12] suggested a novel modified product estimator of $\bar{Y}$ utilizing the known median of auxiliary variable. Kumar and Sharma [13] introduced a generalized class of product estimators for elevated estimation of $\bar{Y}$ and Hussain *et al.* [14] worked on the modified exponential product type estimators of $\bar{Y}$ using auxiliary parameters. Recently some of the authors used L-Moments for elevated estimation of population parameters under investigation. The latest references in this context includes, Shahzad *et al.* [15]. They used L-Moments for Calibrated Variance Estimators under stratified double sampling scheme. Shahzad *et al.* [16] utilized the L-moments and the auxiliary parameters for improved estimation of population variance for the characteristics under consideration.

In the present paper, we have introduced the Searls type estimators of $\bar{Y}$ using Ijaz *et al.* [10] estimators of $\bar{Y}$. We have obtained the bias and MSE of the introduced estimators up to the first degree of approximation. The whole paper has been presented in different sections. The review of competing estimators is presented in section 2. Section 3 shows the suggested estimators and the derivation of their biases and MSEs. A theoretical comparison between suggested estimators and the competing estimators are made in section 4. A numerical illustration has been carried out in section 5. Section 6 represents the results and discussions and conclusions in section 7.

2. MATERIALS AND METHODS

2.1. REVIEW OF EXISTING ESTIMATORS

Under this section, we have presented different competing product type estimators of $\bar{Y}$ using known auxiliary parameters. The estimators along with their MSE are presented in the section.

Robson [1] utilized the negatively correlated auxiliary parameter and suggested the following usual product estimator of $\bar{Y}$ as,

$$t_{pr} = \bar{y}\left(\frac{\bar{x}}{\bar{X}}\right)$$

The MSE of t_{pr} is obtained as,

$$MSE(t_{pr}) = \lambda \bar{Y}^2 (C_y^2 + C_x^2 + 2C_{yx}) \tag{1}$$

where $\lambda = \left(\frac{1}{n} - \frac{1}{N}\right)$ with n as sample size and N as population size, $\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} y_i$ is the population mean of y, $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ is the sample mean of y, $C_y = \frac{\sigma_y}{\bar{Y}}$ is the coefficient of variation of y, $\sigma_y = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(y_i - \bar{Y})^2}$ is the standard deviation y, $\bar{X} = \frac{1}{N}\sum_{i=1}^{N} x_i$ is the population mean of x, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the sample mean of x, $C_x = \frac{\sigma_x}{\bar{X}}$ is the coefficient of variation of x, $\sigma_x = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{X})^2}$ is the standard deviation of x, $C_{yx} = \rho C_y C_x$, ρ is the correlation coefficient between y and x.

Pandy and Dubey [2] used the known C_x and suggested the following product type estimator as,

$$t_1 = \bar{y}\left(\frac{\bar{x} + C_x}{\bar{X} + C_x}\right)$$

The MSE of the estimator t_1 is,

$$MSE(t_1) = \lambda \bar{Y}^2 (C_y^2 + R_1^2 C_x^2 + 2R_1 C_{yx}), \tag{2}$$

where, $R_1 = \frac{\bar{X}}{\bar{X} + C_x}$.

Singh and Tailor [5] suggested the estimator given below by utilizing the known correlation coefficient between Y and X as,

$$t_2 = \bar{y}\left(\frac{\bar{x} + \rho}{\bar{X} + \rho}\right)$$

The MSE of t_2, is given by,

$$MSE(t_2) = \lambda \bar{Y}^2 (C_y^2 + R_2^2 C_x^2 + 2R_2 C_{yx}), \tag{3}$$

where $R_2 = \frac{\bar{X}}{\bar{X} + \rho}$.

Singh [6] introduced the following estimators by utilizing the known standard deviation of X as,

$$t_3 = \bar{y}\left(\frac{\bar{x} + S_x}{\bar{X} + S_x}\right)$$

The MSE of the estimator t_3 is,

$$MSE(t_3) = \lambda \bar{Y}^2 (C_y^2 + R_3^2 C_x^2 + 2R_3 C_{yx}), \tag{4}$$

where, $R_3 = \frac{\bar{X}}{\bar{X} + S_x}$.

Gandge *et al.* [4] suggested the following three Searls [17] type product estimators of $\bar{Y}$ using product estimators by Robson [1], Pandey and Dubey [2], and Singh and Tailor [5] estimators respectively as,

$$t_4 = L_4 \bar{y}\left(\frac{\bar{x}}{\bar{X}}\right),\ t_5 = L_5 \bar{y}\left(\frac{\bar{x} + C_x}{\bar{X} + C_x}\right),\ t_6 = L_6\, \bar{y}\left(\frac{\bar{x} + \rho}{\bar{X} + \rho}\right)$$

where L_i, $i=4,5,6$ are the Searls characterizing constants to be determined such that the MSEs of the above estimators are least.
The MSEs of above estimators respectively are,

$$MSE_{\min}(t_i)=\lambda\bar{Y}^2\left[1-\frac{A_i^2}{B_i}\right],\ i=4,5,6 \tag{5}$$

where $L_4=\dfrac{A_4}{B_4}=\dfrac{1+\lambda C_{yx}}{1+\lambda C_y^2+\lambda C_x^2+4\lambda C_{yx}}$, $L_5=\dfrac{A_5}{B_5}=\dfrac{1+\lambda R_2 C_{yx}}{1+\lambda C_y^2+\lambda R_2^2 C_x^2+4\lambda R_2 C_{yx}}$,
$L_6=\dfrac{A_6}{B_6}=\dfrac{1+\lambda R_3 C_{yx}}{1+\lambda C_y^2+\lambda R_3^2 C_x^2+4\lambda R_3 C_{yx}}$.

Ijaz *et al.* [10] suggested the following three product type estimators of $\bar{Y}$ utilizing known standard deviation, median and quartile deviation of X as,

$$t_7=\bar{y}\left(\frac{\bar{x}-S_x}{\bar{X}-S_x}\right),\ t_8=\bar{y}\left(\frac{\bar{x}-S_x M_d}{\bar{X}-S_x M_d}\right),\ t_9=\bar{y}\left(\frac{\bar{x}-S_x Q_d}{\bar{X}-S_x Q_d}\right)$$

where S_x is the standard deviation, M_d is the median and Q_d is the quartile deviation of X respectively.

The MSEs of these estimators respectively are,

$$MSE(t_i)=\lambda\bar{Y}^2(C_y^2+R_i^2C_x^2+2R_iC_{yx}), \tag{6}$$

where $R_7=\dfrac{\bar{X}}{\bar{X}-S_x}$, $R_8=\dfrac{\bar{X}}{\bar{X}-S_x M_d}$, $R_9=\dfrac{\bar{X}}{\bar{X}-S_x Q_d}$.

2.2. PROPOSED FAMILY OF ESTIMATORS

Motivated by Ijaz *et al.* [10], we introduce the following generalized ratio type estimators using Ijaz *et al.* [10] estimators as,

$$t_p=\bar{y}\left(\frac{\bar{x}-S_x G(x)}{\bar{X}-S_x G(x)}\right)^{\delta}$$

where δ is the characterizing constants to be determined such that the MSE of t_p is a minimum and $G(x)$ is specific auxiliary parameter, which may take values as 1, M_d and Q_d as discussed in Ijaz *et al.* [10]. To get the bias and MSE of t_p, we apply the following standard approximations, given as, $\bar{y}=\bar{Y}(1+e_0)$, $\bar{x}=\bar{X}(1+e_1)$ with $E(e_0)=E(e_1)=0$ and $E(e_0^2)=\lambda C_y^2$, $E(e_1^2)=\lambda C_x^2$, $E(e_0e_1)=\lambda C_{yx}$.

Expressing t_p in terms of e_0 and e_1, we have,

$$t_p=\bar{Y}(1+e_0)(1+\mathrm{R}e_1)^{\delta},$$

where, R are the constants of Ijaz *et al.* [10].

Expanding $(1+\mathrm{Re}_1)^{\delta}$ and using the condition $|e_1|<1$ and retaining the terms till the first degree approximation, we get,

$$t_p = \bar{Y}(1+e_0)[1+\delta \mathrm{Re}_1 + \frac{\delta(\delta-1)}{2}R^2e_1^2]$$

$$t_p = \bar{Y}[1+e_0+\delta \mathrm{Re}_1+\delta \mathrm{Re}_0\, e_1 + \frac{\delta(\delta-1)}{2}R^2e_1^2]$$

Subtracting $\bar{Y}$ on both sides of above equation, we get

$$t_p - \bar{Y} = \bar{Y}[e_0+\delta \mathrm{Re}_1+\delta \mathrm{Re}_0\, e_1 + \frac{\delta(\delta-1)}{2}R^2e_1^2] \tag{7}$$

Taking expectation on both sides of (7) after putting values of different expectations, we have the bias of t_p as,

$$B(t_p) = \lambda\bar{Y}[\delta RC_{yx} + \frac{\delta(\delta-1)}{2}R^2C_x^2] \tag{8}$$

From (7), up to first degree of approximation, we have

$$t_p - \bar{Y} \cong \bar{Y}[e_0+\delta \mathrm{Re}_1] \tag{9}$$

Squaring on both sides of (9), we get the MSE of t_p as,

$$MSE(t_p) = \lambda\bar{Y}^2[e_0^2+\delta^2R^2e_1^2+2\delta \mathrm{Re}_0\, e_1]$$

Putting the values of different expectations, we have,

$$MSE(t_p) = \lambda\bar{Y}^2[C_y^2+\delta^2R^2C_x^2+2\delta RC_{yx}] \tag{10}$$

The optimum value of the characterizing constant δ which minimizes $MSE(t_p)$ is obtained using the method of Maxima-Minima, under which, we differentiate the $MSE(t_p)$ with respect to δ and equate it to zero, the following normal equation is obtained as,

$$\frac{\partial MSE(t_p)}{\partial\delta} = 0$$

Solving it, we get,

$$\delta = \frac{-C_{yx}}{RC_x^2} = \delta_{opt} \tag{11}$$

and for this optimum value of δ, the least value of $MSE(t_p)$ from (10) is given by,

$$MSE_{\min}(t_p) = \lambda\bar{Y}^2\left(C_y^2 - \frac{C_{yx}^2}{C_x^2}\right) \tag{12}$$

This will be the least MSE for all mentioned and other such type of product type estimators for all type of auxiliary parameters.

3. RESULTS AND DISCUSSION

3.1. EFFICIENCY COMPARISON

In this section, different estimators of $\bar{Y}$ under comparison are theoretically compared with the introduced estimators and the efficiency conditions for the suggested estimator to be better than the estimators in competition are obtained.

The proposed estimators t_p is has lesser MSE than the conventional product estimator of Robson [1] under the condition if,

$$MSE(t_{pr}) - MSE_{\min}(t_p) > 0,$$

or,

$$[C_y^2 + C_x^2 + 2C_{yx}] - \left(C_y^2 - \frac{C_{yx}^2}{C_x^2} \right) > 0 \tag{13}$$

The proposed estimators t_p have lesser MSE than the Pandey and Dubey [2] estimator if,

$$MSE(t_1) - MSE_{\min}(t_p) > 0,$$

or,

$$[C_y^2 + R_1^2 C_x^2 + 2R_1 C_{yx}] - \left(C_y^2 - \frac{C_{yx}^2}{C_x^2} \right) > 0 \tag{14}$$

The introduced estimators t_p has improvement over the Singh and Tailor [5] estimators for the condition if,

$$MSE(t_2) - MSE_{\min}(t_p) > 0,$$

or,

$$[C_y^2 + R_2^2 C_x^2 + 2R_2 C_{yx}] - \left(C_y^2 - \frac{C_{yx}^2}{C_x^2} \right) > 0 \tag{15}$$

The suggested estimators t_p performs better than the Singh [6] estimator if,

$$MSE(t_3) - MSE_{\min}(t_p) > 0,$$

or,

$$[C_y^2 + R_3^2 C_x^2 + 2R_3 C_{yx}] - \left(C_y^2 - \frac{C_{yx}^2}{C_x^2} \right) > 0 \tag{16}$$

The introduced estimators t_p is more efficient than the Gandge *et al.* [4] estimators for the condition if,

$$MSE(t_i) - MSE_{\min}(t_p) > 0, \; i = 4,5,6,$$

or,

$$[C_y^2 + R_i^2 C_x^2 + 2R_i C_{yx}] - \left(C_y^2 - \frac{C_{yx}^2}{C_x^2} \right) > 0 \tag{17}$$

The proposed estimators t_p performs better than the Ijaz *et al.* [10] estimators if,

$$MSE(t_i) - MSE_{\min}(t_p) > 0,\ i = 7,8,9,$$

or,

$$[C_y^2 + R_i^2 C_x^2 + 2R_i C_{yx}] - \left(C_y^2 - \frac{C_{yx}^2}{C_x^2} \right) > 0 \tag{18}$$

3.1. NUMERICAL COMPARISON

To verify the efficiency conditions of the introduced estimators over the competing estimator and further to see which estimator is best among the considered estimators for enhanced estimation of $\bar{Y}$, we have taken into consideration, a natural population given in Madala [18]. The parameters of the considered population are shown in Table 1. The MSE of different estimators in competition and the suggested estimators and the PRE of these estimators with respect to traditional product estimator of Robson [1] are shown in Table 2.

Table 1. Parameters of the natural population given in Maddala [18]

$N = 16$	$n = 4$	$\bar{Y} = 7.6375$
$\bar{X} = 7.6375$	$\rho = -0.6823$	$S_y = 1.7398$
$S_x = 7.4375$	$C_y = 0.2278$	$C_x = 0.0986$
$M_d = 3.9667$	$Q_d = 4.9584$	$\lambda = 0.1875$

Table 2. MSE and PRE of competing and suggested estimators of $\bar{Y}$

S. No.	Estimator	MSE	PRE
1.	t_{pr}	0.338664236	100.00
2.	t_1	0.338824365	99.95
3.	t_2	0.337552597	100.33
4.	t_3	0.350518851	96.62
5.	t_4	0.338058556	100.18
6.	t_5	0.338216576	100.13
7.	t_6	0.336962917	100.51
8.	t_7	0.326529579	103.72
9.	t_8	0.31876063	106.24
10.	t_9	0.303807719	111.47
11.	t_p	0.303341315	111.64

The MSE and PRE of competing and suggested estimators with respect to Robson (1957) [1] product estimator are also shown in the form of graphs in Figs. 1-2.

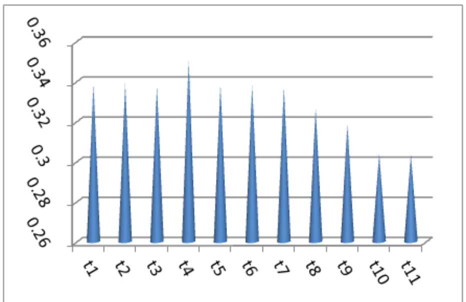

Figure 1. MSE of different estimator

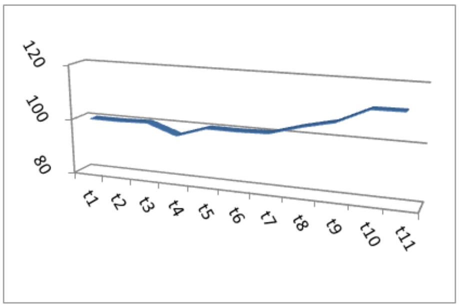

Figure 2. PRE of different estimator with respect to t_{pr}.

4. CONCLUSIONS

In the present paper, we introduced a generalized product type of estimator for the enhanced estimation of $\bar{Y}$ utilizing some known conventional and non-conventional auxiliary parameters. We studied the bias and the MSE of the introduced estimator till the first degree approximation. The expression for the MSE of the introduced estimator has been derived in which the characterizing scalar is involved. The optimum value of the characterizing scalar is obtained and the least value of the MSE for this optimum value of characterizing constant is also obtained. The suggested estimator is compared theoretically with the estimators in competition and the efficiency conditions are obtained. These efficiency conditions are verified through the real data set from Maddala [18]. It may be observed from Table 2 that the suggested estimator is having least MSE and highest PRE among the family of all estimators of $\bar{Y}$ in competition and further it is worth notable that the suggested estimator has the least MSE among the family of all product type estimators utilizing all type of conventional and non-conventional auxiliary parameters. Thus it is recommended to use the suggested family of estimators for the enhanced estimation of $\bar{Y}$ in different areas of applications.

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