ORIGINAL PAPER

EXISTENCE AND UNIQUENESS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH MULTIPOINT AND MULTI-TERM INTEGRAL BOUNDARY CONDITIONS

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Abstract. *We investigated the existence and uniqueness of solutions to a nonlinear fractional differential equation containing the Caputo operator supplemented with separated multipoint and multi-term integral boundary conditions. The existence results are obtained by utilizing the Banach contraction principle and the well-known fixed point theorem of Krasnoselskii for a sum of contraction and compact mappings in Banach spaces. Some examples are also presented to illustrate the main results.*

Keywords: *Caputo fractional derivative; solution; Banach contraction principle; compact.*

1. INTRODUCTION

In recent years, the interest in the study of fractional differential equations has been growing rapidly due their importance since they provided valuable tools for their applications in various sciences, such as gas dynamics, nuclear physics, electrodynamics of complex medium, polymer rheology. With this advantage, fractional order models are regarded as more realistic and practical. We refer a reader interested in the systematic development of the topic to the books [1-7].

Many scholars have studied the existence for nonlinear fractional differential equations with a variety boundary conditions by different approaches can easily be found in the literature on the topic. For some recent results, we can refer to [8-14] and references cited therein. However, most researchers tend to investigate either integral conditions or multi-point conditions, for example, in [15], the authors studied the existence of at least one, two or three positive solutions for the boundary value problem with three-point multi-term fractional integral boundary conditions

$$\begin{cases} D^q u(t) + f\big(t, u(t)\big) = 0, 1 < q \leq 2, 0 < t < 1, \\ u(0) = 0, u(1) = \sum_{i=1}^{m} \alpha_i (I^{p_i} u)(\eta), 0 < \eta < 1, \end{cases} \tag{1.1}$$

where D^q is the standard Riemann-Liouville fractional derivative of order q, I^{p_i} is the Riemann-Liouville fractional integral of order $p_i > 0$, $i = 0,1,2,\ldots,m$ and $\alpha_i \geq 0$, $i = 0,1,2,\ldots,m$ are real constants such that $\sum_{i=1}^{m} \frac{\alpha_i \eta^{p_i+q-1}}{\Gamma(p_i+q)} < 1$. The existence results are obtained by using the Krasnoselskii fixed point theorem and the Leggett-Williams fixed point theorem.

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In [16], Wang et al. considered the fractional differential equations that contain both the integral boundary condition and the multi-point boundary condition

$$\begin{cases} D^{\sigma}u(t) + f\big(t, u(t)\big) = 0, t \in [0, t], \\ u^{(i)}(0) = 0, i = 0,1,2,\dots, n-2, \\ u(1) = \displaystyle\sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} u(s)ds + \sum_{i=1}^{m-2} \gamma_i u(\eta_i) \end{cases} \tag{1.2}$$

where D^{σ} represents the standard Riemann-Liouville fractional derivative of order σ satisfying $n-1 < \sigma \le n$ with $n \ge 3$ and $n \in \mathbb{N}^+$. In addition, $0 < \eta_1 < \eta_2 <\ldots< \eta_{m-2} < 1$ and $\beta_i, \gamma_i > 0$ with $1 \le i \le m-2$, where m is an integer satisfying $m \ge 3$.

In [17], Li et al. investigated the existence of minimal and maximal positive solutions for the following boundary value problem of nonlinear fractional differential equation with integral boundary conditions

$$\begin{cases} \big({}^{C}D_{0+}^{q}u\big)(t) + f\big(t, u(t)\big) = 0, t \in [0, t], \\ u''(0) = 0, \\ \alpha u(0) - \beta u'(0) = \displaystyle\int_0^1 h_1(s)u(s)ds, \\ \gamma u(1) - \delta\big({}^{C}D_{0+}^{\sigma}u\big)(1) = \displaystyle\int_0^1 h_2(s)u(s)ds, \end{cases} \tag{1.3}$$

where ${}^{C}D_{0+}^{q}$ and ${}^{C}D_{0+}^{\sigma}$ denote the standard Caputo fractional derivatives of order q and order σ, respectively. Here $2 < q \le 3$, $0 < \sigma \le 1$ and α, β, γ and δ are nonnegative constants. The main tool used in this study is the monotone iterative method.

Inspired and motivated by the aforementioned works, in this paper, we discuss the existence and uniqueness of positive solutions for the following nonlinear fractional differential equation

$${}^{C}D_{0+}^{\alpha}u(t) + f\big(t, u(t), u'(t)\big) = 0, 0 \le t \le 1, \tag{1.4}$$

supplemented with separated multipoint and multi-term integral boundary conditions

$$\begin{cases} u^{(i)}(0) = 0, i = 2,\dots, n-1, \\ u'(0) = \displaystyle\sum_{i=1}^{m-2} b_i u'(\xi_i), u(1) = \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} u(s)ds, \end{cases} \tag{1.5}$$

where ${}^{C}D_{0+}^{\alpha}$ is the Caputo fractional derivatives, $n-1 < \alpha \le n$, $n \ge 3$ is an integer, $0 = \eta_0 < \eta_1 < \eta_2\ldots< \eta_{m-2} < 1$, $0 = \xi_0 < \xi_1 < \xi_2\ldots< \xi_{m-2} < 1$, $a_i \ge 0$, $b_i \ge 0$, $(i =$ 1,...,m−2, 0≤i=1m−2bi<1 and 0≤i=1m−2$ai\eta i$−ηi−1<1, where m>2 is an integer, and $f: [0,1] \times \mathbb{R} \times \mathbb{R} \to\times \mathbb{R}$ is continuous.

This paper is organised as follows, in Section 2, we introduce some necessary basic knowledge and definitions for fractional calculus theory and we study the associated linear problem of the problem (1.4)-(1.5). In Section 3, we will prove the uniqueness of solution for the problem (1.4)-(1.5) by using Banach contraction mapping principle. Section 4 is devoted to the existence of at least one solution of the considered problem by applying Krasnoselskii fixed point theorem for a sum of contraction and compact mappings. In the last section, some examples are given to show the applicability of our existence results.

2. PRELIMINARIES

In this section, we present some basic definitions, notations and results of fractional calculus [4-6, 18] which are used throughout this paper and we study the associated linear problem of the boundary value problem (1.4)-(1.5).

Definition 2.1. *The Riemann-Liouville fractional integral of order* $\alpha > 0$ *for a function* $f: (0, +\infty) \to \mathbb{R}$ *is defined as*

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided the right side is pointwise defined on $(0, +\infty)$ where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. *For a function* $f: [0, +\infty) \to \mathbb{R}$, *the Caputo derivative of fractional order* α *is defined as*

$${}^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of the real number α, provided the right side is pointwise defined on $(0, +\infty)$.

Lemma 2.3. *Let* $\alpha > 0$ *and* $u \in AC^N[0,1]$. *Then the fractional differential equation*

$${}^{C}D^{\alpha} u(t) = 0,$$

has a unique solution

$$u(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{N-1} t^{N-1}, c_i \in \mathbb{R}, i = 1,2, \ldots, N,$$

where N is the smallest integer greater than or equal to α.

Remark 2.4. The following property (Dirichlet's formula) of the fractional calculus is well known ([19] p.57)

$$I^{\nu} I^{\mu} y(t) = I^{\nu+\mu} y(t), t \in [0,1], y \in L(0,1), \nu + \mu \geq 1,$$

which has the form

$$\int_0^t (t-s)^{\nu-1} \left(\int_0^s (s-\tau)^{\mu-1} y(\tau) d\tau \right) ds = \frac{\Gamma(\nu)\Gamma(\mu)}{\Gamma(\nu+\mu)} \int_0^t (t-s)^{\nu+\mu-1} y(s) ds.$$

Lemma 2.5 For $y \in C[0,1]$, the following boundary value problem

$$\begin{cases} {}^{C}D_{0+}^{\alpha} u(t) + y(t) = 0, t \in (0,1), \\ u^{(i)}(0) = 0, i = 2,3, \ldots, n-1, \\ u'^{(0)} = \sum_{i=1}^{m-2} b_i u_i'(\xi_i), u(1) = \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} u(s) ds, \end{cases} \tag{2.1}$$

has the unique solution

$$u(t)=c_0+c_1t-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds, \tag{2.2}$$

were

$$\begin{aligned}
c_0=&\frac{\int_0^1 (1-s)^{\alpha-1}y(s)ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)}\\
&-\frac{\sum_{i=1}^{m-2} a_i\left[\int_0^{\eta_i} (\eta_i-s)^{\alpha}-\int_0^{\eta_{i-1}} (\eta_{i-1}-s)^{\alpha}\right]y(s)ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+1)}\\
&+\frac{\left(1-\sum_{i=1}^{m-2} a_i\frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2} b_i\int_0^{\xi_i} (\xi_i-s)^{\alpha-2}y(s)ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha-1)},\\
c_1=&-\frac{\sum_{i=1}^{m-2} b_i\int_0^{\xi_i} (\xi_i-s)^{\alpha-2}y(s)ds}{(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha-1)}.
\end{aligned} \tag{2.3}$$

Proof: In view of Definition 2.1 and Lemma 2.3, it is clear that equation (2.1) is equivalent to the integral form,

$$u(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds+c_0+c_1t+\ldots+c_{n-1}t^{n-1},$$

where $c_0,c_1,\ldots,c_{n-1}\in\mathbb{R}$ are arbitrary constants.

Next, using the initial conditions: $u^{(i)}(0)=0$, $i=2,\ldots,n-1$, we get

$$c_2=c_3=\ldots=c_{n-1}=0,$$

that is,

$$u(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds+c_0+c_1t. \tag{2.4}$$

So, we get

$$u'(t)=-\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}y(s)ds+c_1. \tag{2.5}$$

By $u'(0)=\sum_{i=1}^{m-2} b_iu'(\xi_i)$, we obtain

$$c_1=-\frac{\sum_{i=1}^{m-2} b_i\int_0^{\xi_i} (\xi_i-s)^{\alpha-2}y(s)ds}{(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha-1)}. \tag{2.6}$$

Integrating the equation 2.4 from η_{i-1} to η_i for $0\le\eta_{i-1}\le\eta_i\le 1$, $i=1,\ldots,m-2$, and using Remark 2.4, we get

$$\int_{\eta_{i-1}}^{\eta_i} u(t)dt=-\frac{1}{\Gamma(\alpha)}\int_{\eta_{i-1}}^{\eta_i}\left(\int_0^s (s-\tau)^{\alpha-1}y(\tau)d\tau\right)ds+c_0\int_{\eta_{i-1}}^{\eta_i} ds+c_1\int_{\eta_{i-1}}^{\eta_i} sds$$

$$= -\frac{1}{\Gamma(\alpha)}\left[\int_0^{\eta_i}\left(\int_0^s (s-\tau)^{\alpha-1}y(\tau)d\tau\right)ds + \int_{\eta_{i-1}}^0 \left(\int_0^s (s-\tau)^{\alpha-1}y(\tau)d\tau\right)ds\right]$$
$$+c_0\int_{\eta_{i-1}}^{\eta_i} ds + c_1\int_{\eta_{i-1}}^{\eta_i} sds$$
$$= -\frac{1}{\Gamma(\alpha+1)}\int_0^{\eta_i}(\eta_i - s)^{\alpha}y(s)ds + \frac{1}{\Gamma(\alpha+1)}\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^{\alpha}y(s)ds$$
$$+c_0(\eta_i - \eta_{i-1}) + c_1\frac{\eta_i^2 - \eta_{i-1}^2}{2}.$$

Then, by the condition $u(1) = \sum_{i=1}^{m-2} a_i \int_{\eta_{i-1}}^{\eta_i} u(s)ds$, we get

$$-\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds + c_0 + c_1 = -\frac{1}{\Gamma(\alpha+1)}\sum_{i=1}^{m-2} a_i\int_0^{\eta_i}(\eta_i - s)^{\alpha}y(s)ds$$
$$+\frac{1}{\Gamma(\alpha+1)}\sum_{i=1}^{m-2} a_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^{\alpha}y(s)ds$$
$$+c_0\sum_{i=1}^{m-2} a_i(\eta_i - \eta_{i-1}) + c_1\sum_{i=1}^{m-2} a_i\frac{\eta_i^2-\eta_{i-1}^2}{2}.$$

which implies:

$$c_0 = \frac{\int_0^1 (1-s)^{\alpha-1}y(s)ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)} - \frac{\sum_{i=1}^{m-2} a_i\left[\int_0^{\eta_i}(\eta_i-s)^{\alpha} - \int_0^{\eta_{i-1}}(\eta_{i-1}-s)^{\alpha}\right]y(s)ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+1)}$$
$$+\frac{\left(1-\sum_{i=1}^{m-2} a_i\frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2} b_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-2}y(s)ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha-1)}.$$

Let $X = \{u: u, u' \in C[0,1]\}$ endowed with the norm defined $\|u\| = \sup_{t\in[0,1]}|u(t)| + \sup_{t\in[0,1]}|u'(t)|$ such that $\|u\| < \infty$. Then $(X, \|.\|)$ is a Banach space, and define the operator $T: X \longrightarrow X$ by

$$Tu(t) = \frac{\int_0^1 (1-s)^{\alpha-1}f\left(s,u(s),u'^{(s)}\right)ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)}$$
$$-\frac{\sum_{i=1}^{m-2} a_i\left[\int_0^{\eta_i}(\eta_i-s)^{\alpha} - \int_0^{\eta_{i-1}}(\eta_{i-1}-s)^{\alpha}\right]f\left(s,u(s),u'(s)\right)ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+1)}$$
$$+\frac{\left(1-\sum_{i=1}^{m-2} a_i\frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2} b_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-2}f\left(s,u(s),u'(s)\right)ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha-1)}$$
$$-\frac{\sum_{i=1}^{m-2} b_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-2}y(s)ds}{(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha-1)}t - \frac{\int_0^t (t-s)^{\alpha-1}f\left(s,u(s),u'(s)\right)ds}{\Gamma(\alpha)}.$$

Then u is a solution of the boundary value problem (1.4)-(1.5) if and only if it is a fixed point of the operator T. Throughout the paper, we let

$$M_1 = \frac{1}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+1)} + \frac{\sum_{i=1}^{m-2} a_i[\eta_i^{\alpha+1}-\eta_{i-1}^{\alpha+1}]}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+2)}$$
$$+\frac{\left(1-\sum_{i=1}^{m-2} a_i\frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2} b_i\xi_i^{\alpha-1}}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha)} + \frac{\sum_{i=1}^{m-2} b_i\xi_i^{\alpha-1}}{(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)}.$$

$$M_2 = \frac{\sum_{i=1}^{m-2} b_i\xi_i^{\alpha-1}}{(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}.$$

3. UNIQUENESS OF SOLUTION

In this section, we prove the uniqueness of solution to the problem (1.4)-(1.5) via Banach's contraction principle.

Theorem 3.1. *Assume*$(C1)$ there exists constant $K>0$ such that

$$|f(t,u,v)-f(t,\overline{u},\overline{v})| \le K(|u-v|+|\overline{u}-\overline{v}|), for\ any\ u,v,\overline{u},\overline{v}\in\mathbb{R}, t\in[0,1].$$

If

$$K(M_1+M_2)<1,$$

then the problem (1.4)-(1.5) has a unique solution on $[0,1]$.

Proof: Let us set $\sup_{t\in[0,1]}|f(t,0,0)| = N < \infty$ and choosing

$$r > \frac{N(M_1+M_2)}{1-K(M_1+M_2)}.$$

For $u\in X$, we observe that

$$\left|f\big(t,u(t),u'(t)\big)\right| \le \left|f\big(t,u(t),u'(t)\big)-f(t,0,0)\right| + |f(t,0,0)| \le L\|u\|+N.$$

Firstly, we show that $TB_r\subset B_r$, where $B_r=\{u\in X: \|u\|\}$. For $u\in B_r$, we have

$$|Tu(t)| \le \frac{\int_0^1 (1-s)^{\alpha-1}\left\|f\big(s,u(s),u'(s)\big)\right\|ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)}$$
$$+\frac{\sum_{i=1}^{m-2} a_i\left[\int_0^{\eta_i}(\eta_i-s)^{\alpha}-\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^{\alpha}\right]\left\|f\big(s,u(s),u'(s)\big)\right\|ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+1)}$$
$$+\frac{\left(1-\sum_{i=1}^{m-2} a_i\frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2} b_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-2}\left\|f\big(s,u(s),u'(s)\big)\right\|ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha-1)}$$
$$+\frac{\sum_{i=1}^{m-2} b_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-2}\left\|f\big(s,u(s),u'^{(s)}\big)\right\|ds}{(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha-1)}t$$

$$+\frac{\int_0^t (t-s)^{\alpha-1}\left\|f\big(s,u(s),u'(s)\big)\right\|ds}{\Gamma(\alpha)}$$

$$\leq (K\|u\|+N)\Bigg[\frac{\int_0^1 (1-s)^{\alpha-1}ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)} + \frac{\sum_{i=1}^{m-2} a_i\left[\int_0^{\eta_i} (\eta_i-s)^{\alpha}-\int_0^{\eta_{i-1}} (\eta_{i-1}-s)^{\alpha}\right]ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+1)}$$

$$+\frac{\left(1-\sum_{i=1}^{m-2} a_i\frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2} b_i\int_0^{\xi_i} (\xi_i-s)^{\alpha-2}ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha-1)}$$

$$+\frac{\sum_{i=1}^{m-2} b_i\int_0^{\eta_i} (\xi_i^{\alpha-2}-s)^{\alpha-2}ds}{(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha-1)}t+\frac{\int_0^t (t-s)^{\alpha-1}ds}{\Gamma(\alpha)}\Bigg] \leq (Kr+N)M_1$$

and

$$|T'u(t)| \leq \frac{\sum_{i=1}^{m-2} b_i\int_0^{\xi_i} (\xi_i-s)^{\alpha-2}\left\|f\big(s,u(s),u'(s)\big)\right\|ds}{(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha-1)} + \frac{\int_0^t (t-s)^{\alpha-2}\left\|f\big(s,u(s),u'(s)\big)\right\|ds}{\Gamma(\alpha-1)}$$

$$\leq (K\|u\|+N)\left(\frac{\sum_{i=1}^{m-2} b_i\int_0^{\xi_i} (\xi_i-s)^{\alpha-2}ds}{(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha-1)}+\frac{\int_0^t (t-s)^{\alpha-2}ds}{\Gamma(\alpha-1)}\right)$$

$$\leq (K\|u\|+N)\left(\frac{\sum_{i=1}^{m-2} b_i\xi_i^{\alpha-1}}{(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)}\right)$$

$\leq (Kr+N)M_2$,
which implies that

$$\|Tu(t)\| \leq (Kr+N)(M_1+M_2) < r.$$

Now, for $u, v \in X$ and for each $t \in [0,1]$, we obtain

$$|(Tu)(t)-(Tv)(t)| \leq K\|u-v\|\Bigg[\frac{\int_0^1 (1-s)^{\alpha-1}ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)} + \frac{\sum_{i=1}^{m-2} a_i\left[\int_0^{\eta_i} (\eta_i-s)^{\alpha}-\int_0^{\eta_{i-1}} (\eta_{i-1}-s)^{\alpha}\right]ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+1)}$$

$$+\frac{\left(1-\sum_{i=1}^{m-2} a_i\frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2} b_i\int_0^{\xi_i} (\xi_i-s)^{\alpha-2}ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha-1)}$$

$$+\frac{\sum_{i=1}^{m-2} b_i\int_0^{\xi_i} (\xi_i-s)^{\alpha-2}ds}{(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha-1)}t+\frac{\int_0^t (t-s)^{\alpha-1}ds}{\Gamma(\alpha)}\Bigg]$$

$\leq K\|u-v\|M_1$,
also,

$$\|(T'u)(t)-(T'v)(t)\| \leq K\|u-v\|\left[\frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i-s)^{\alpha-2}ds}{(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha-1)}+\frac{\int_0^t (t-s)^{\alpha-2}ds}{\Gamma(\alpha-1)}\right]$$

$\leq K\|u-v\|M_2.$

Thus $\|Tu(t)-Tv(t)\| \leq K(M_1+M_2)\|u-v\|$. Since $K(M_1+M_2)<1$, thus T is a contraction. Hence, it follows by Banach's contraction principle that the boundary value problem (1.4)- (1.5) has a unique solution on $[0,1]$.

4. EXISTENCE OF SOLUTIONS

In this section, we establish the existence of at least one solution of the problem (1.4)-(1.5) by using Krasnoselskii's fixed point theorem [20].

Theorem 4.1 *(Krasnoselskii's fixed point theorem) Let M be a closed convex and nonempty subset of a Banach space E. Let A, B be the operators such that:*
$(i) Au+Bv \in M$ whenever $u, v \in M$,
$(ii) A$ is compact and continuous,
$(iii) B$ is a contraction mapping.
Then there exists $w \in M$ such that $w = Aw + Bw$.

Theorem 4.2 *Assume that $(C1)$ and the following hypothesis holds*$(C2)$ There exists $\mu \in L^1([0,1],\mathbb{R}^+)$ such that

$$\|f(t,u(t),v(t))\| \leq \mu(t), for\ all\ (t,u,v) \in [0,1]\times X\times X.$$

If

$$K\left(M_1+M_2-\frac{\alpha+1}{\Gamma(\alpha+1)}\right)<1.$$

Then, the boundary value problem (1.4)-(1.5) has at least one solution on $[0,1]$.

Proof: Let us fix

$$r \geq \|\mu\|_{L^1}(M_1+M_2)$$

and consider $B_r = \{u \in X: \|u\| \leq r\}$. We define the operators A and B on B_r as

$$(Au)(t) = -\frac{\int_0^t (t-s)^{\alpha-1} f(s,u(s))ds}{\Gamma(\alpha)},$$

$$(Bu)(t) = \frac{\int_0^1 (1-s)^{\alpha-1} f(s,u(s),u'(s))ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha)}$$
$$-\frac{\sum_{i=1}^{m-2} a_i[\int_0^{\eta_i} (\eta_i-s)^{\alpha}-\int_0^{\eta_{i-1}} (\eta_{i-1}-s)^{\alpha}] f(s,u(s),u'(s))ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)\Gamma(\alpha+1)}$$
$$+\frac{\left(1-\sum_{i=1}^{m-2} a_i \frac{\eta_i^2-\eta_{i-1}^2}{2}\right)\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i-s)^{\alpha-2} f(s,u(s))ds}{\left(1-\sum_{i=1}^{m-2} a_i(\eta_i-\eta_{i-1})\right)(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha-1)}$$

$$-\frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-2} f\big(s, u(s), u'(s)\big) ds}{(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha-1)} t.$$

For $u, v \in B_r$, we have

$$|Au(t) + Bu(t)| \le \frac{\int_0^t (t-s)^{\alpha-1} \big|f\big(s, u(s)u'(s)\big)\big| ds}{\Gamma(\alpha)}$$
$$+\frac{\sum_{i=1}^{m-2} a_i \big[\int_0^{\eta_i} (\eta_i - s)^{\alpha} - \int_0^{\eta_{i-1}} (\eta_{i-1} - s)^{\alpha}\big] \big|f\big(s, u(s)u'(s)\big)\big| ds}{\Big(1-\sum_{i=1}^{m-2} a_i(\eta_i - \eta_{i-1})\Big)\Gamma(\alpha+1)}$$
$$+\frac{\Big(1-\sum_{i=1}^{m-2} a_i \frac{\eta_i^2 - \eta_{i-1}^2}{2}\Big) \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-2} \big|f\big(s, u(s)u'(s)\big)\big| ds}{\Big(1-\sum_{i=1}^{m-2} a_i(\eta_i - \eta_{i-1})\Big)(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha-1)}$$
$$+\frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-2} \big|f\big(s, u(s)u'(s)\big)\big| ds}{(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha-1)} t \le \|\mu\|_{L^1} M_1,$$

and

$$|A'u(t) + B'v(t)| \le \frac{\int_0^t (t-s)^{\alpha-2} \big|f\big(s, u(s)u'^{(s)}\big)\big| ds}{\Gamma(\alpha-1)}$$
$$+\frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-2} \big|f\big(s, u(s)u'(s)\big)\big| ds}{(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha-1)} \le \|\mu\|_{L^1} M_2.$$

Consequently, $\|Au + Bv\| \le \|\mu\|_{L^1}(M_1 + M_2) \le r$. Thus, $Au + Bv \in B_r$.. Now, for $u, v \in X$, we have

$$|Bu(t) - Bv(t)| \le K\|u-v\| \Bigg[\frac{1}{\Big(1-\sum_{i=1}^{m-2} a_i(\eta_i - \eta_{i-1})\Big)\Gamma(\alpha)}$$
$$+\frac{\sum_{i=1}^{m-2} a_i(\eta_i^{\alpha+1} - \eta_{i-1}^{\alpha+1})}{\Big(1-\sum_{i=1}^{m-2} a_i(\eta_i - \eta_{i-1})\Big)\Gamma(\alpha+2)}$$

$$+\frac{\Big(1-\sum_{i=1}^{m-2} a_i \frac{\eta_i^2 - \eta_{i-1}^2}{2}\Big) \sum_{i=1}^{m-2} b_i \xi_i^{\alpha-1}}{\Big(1-\sum_{i=1}^{m-2} a_i(\eta_i - \eta_{i-1})\Big)(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha)} + \frac{\sum_{i=1}^{m-2} b_i \xi_i^{\alpha-1}}{(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha-1)}\Bigg].$$

Also,

$$|B'u(t) - B'v(t)| \le K\|u-v\| \frac{\sum_{i=1}^{m-2} b_i \xi_i^{\alpha-1}}{(1-\sum_{i=1}^{m-2} b_i)\Gamma(\alpha)}$$
$$\le K\|u-v\| \left(M_2 - \frac{1}{\Gamma(\alpha)}\right).$$

Thus $\|Bu - Bv\| \le K\|u-v\| \left(M_1 + M_2 - \frac{\alpha+1}{\Gamma(\alpha+1)}\right)$. Since

$$K\left(M_1 + M_2 - \frac{\alpha+1}{\Gamma(\alpha+1)}\right) < 1,$$

we conclude that B is a contraction mapping.

The continuity of A arise from the continuity of f. For $u \in X$

$$|Au(t)| \leq \frac{\|\mu\|_{L^1}}{\Gamma(\alpha+1)},$$

also

$$|A'u(t)| \leq \frac{\|\mu\|_{L^1}}{\Gamma(\alpha)}$$

Thus $\|Au\| \leq \frac{\|\mu\|_{L^1}(\alpha+1)}{\Gamma(\alpha+1)}$. Then A is uniformly bounded on B_r.Now we prove the compactness of the operator A. Let

$$\rho = \sup\{|f(t,u(t),u'(t))| : t \in [0,1], u \in B_r\}.$$

For $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$, we have

$$\begin{aligned}
&|Au(t_1) - Au(t_2)| \\
&= \left| -\frac{\int_0^{t_1} (t_1-s)^{\alpha-1} f(s,u(s)u'^{(s)})ds}{\Gamma(\alpha)} + \frac{\int_0^{t_2} (t_2-s)^{\alpha-1} f(s,u(s)u'(s))ds}{\Gamma(\alpha)} \right| \\
&\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [-(t_1-s)^{\alpha-1} + (t_2-s)^{\alpha-1}] f(s,u(s)u'(s))ds \right. \\
&\left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f(s,u(s)u'(s))ds \right| \\
&\leq \frac{\rho}{\Gamma(\alpha+1)} |t_2^\alpha - t_1^\alpha|.
\end{aligned}$$

Obviously, the right-hand side of the above inequality tends to zero as $t_2 \to t_1$.Similarly, we have

$$\begin{aligned}
&|A'u(t_1) - A'u(t_2)| \\
&= \left| -\frac{\int_0^{t_1} (t_1-s)^{\alpha-2} f(s,u(s)u'^{(s)})ds}{\Gamma(\alpha-1)} + \frac{\int_0^{t_2} (t_2-s)^{\alpha-2} f(s,u(s)u'(s))ds}{\Gamma(\alpha-1)} \right| \\
&\leq \left| \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} [-(t_1-s)^{\alpha-2} + (t_2-s)^{\alpha-2}] f(s,u(s)u'(s))ds \right. \\
&\left. + \int_{t_1}^{t_2} (t_2-s)^{\alpha-2} f(s,u(s)u'(s))ds \right| \leq \frac{\rho}{\Gamma(\alpha)} |t_2^{\alpha-1} - t_1^{\alpha-1}|.
\end{aligned}$$

Again, it is seen that the right-hand side of the above inequality tends to zero as $t_2 \to t_1$. Thus $\|(Au)(t_1) - (Au)(t_2)\| \longrightarrow 0$, as $t_2 \to t_1$. This show that A is relatively compact on B_r. Hence, by the Arzela-Ascoli theorem, we have A is compact on B_r. As a consequence of Theorem 4.1, we conclude that the boundary value problem (1.4)-(1.5) has at least one solution on [0,1].

5. EXAMPLES

Example 5.1 *Consider the following fractional differential equation*

$$ {}^{C}D_{0+}^{\frac{5}{2}}u(t)+\frac{\cos(\pi t)e^{-\pi t}}{\sqrt{2}\pi+\cosh(t)}\left(1-\frac{0{,}9}{1+|u(t)|+|v(t)|}\right), t\in[0{,}1], \tag{5.1}$$

subject to the following boundary conditions

$$\begin{cases} u''(0)=0 \\ u'(0)=0{,}2u'(0{,}1)+0{,}1u'(0{,}2)+0{,}25u'(0{,}6) \\ u(1)=2\int_0^{0,3}u(s)ds+0{,}4\int_{0,3}^{0,5}u(s)ds+0{,}6\int_{0,5}^{0,7}u(s)ds \end{cases}. \tag{5.2}$$

Here $f(t,u,u')=\frac{\cos(\pi t)e^{-\pi t}}{\sqrt{2}\pi+\cosh(t)}\left(1-\frac{0{,}9}{1+|u|+|u'|}\right)$ for $t\in[0{,}1]$, $u,u'\in\mathbb{R}$, $\xi_1=0{,}1$, $\xi_2=0{,}2$, $\xi_3=0{,}6$, $b_1=0{,}2$, $b_2=0{,}4$, $b_3=0{,}25$, $\eta_1=0{,}3$, $\eta_2=0{,}5$, $\eta_3=0{,}7$, $a_1=2$, $a_2=0{,}4$ and $a_3=0{,}6$, then, by calculations we obtain that $M_1=3{,}07$ and $M_2=0{,}972008$.

For every $u_i,v_i\in\mathbb{R}$, $i=1{,}2$, we have

$$\begin{aligned}|f(t,u_1,v_1)-f(t,u_2,v_2)| &= \frac{\cos(\pi t)e^{-\pi t}}{\sqrt{2}\pi+\cosh(t)}\left|\frac{0{,}9}{1+|u_2|+|v_2|}-\frac{0{,}9}{1+|u_1|+|v_1|}\right| \\ &\le \frac{0{,}9\cos(\pi t)e^{-\pi t}}{\sqrt{2}\pi+\cosh(t)}\left(\frac{|u_1-u_2|+|v_1-v_2|}{(1+|u_1|+|v_1|)(1+|u_2|+|v_2|)}\right) \\ &\le \frac{0{,}9}{\sqrt{2}\pi}(|u_1-u_2|+|v_1-v_2|).\end{aligned}$$

Then the condition $(C1)$ is satisfied with $K=\frac{0{,}9}{\sqrt{2}\pi}$.Since

$$K(M_1+M_2)\simeq 0{,}818794<1,$$

it follows from Theorem 3.1 that problem (5.1)-(5.2) has a unique solution on $[0{,}1]$.

Example 5.2. *Consider the following fractional differential equation*

$$ {}^{C}D_{0+}^{\frac{5}{2}}u(t)+\frac{t^2}{2}\left(1-\frac{0{,}9}{1+|u(t)|+|v(t)|}\right), \tag{5.3}$$

subject to the boundary conditions in (5.2). Here $f(t,u,u')=\frac{t^2}{5}\left(1-\frac{0{,}9}{1+|u(t)|+|v(t)|}\right)$ for $t\in[0{,}1]$, $u,u'\in\mathbb{R}$.

For every $u_i,v_i\in\mathbb{R}$, $i=1{,}2$, we have

$$|f(t,u_1,v_1)-f(t,u_2,v_2)|\le K(|u_1-u_2|+|v_1-v_2|),$$

where $K=\frac{1}{5}$, then condition $(C1)$is satisfied, and

$$\left|f\big(t,u(t),v(t)\big)\right|=\left|\frac{t^2}{5}\left(1-\frac{0{,}9}{1+|u(t)|+|v(t)|}\right)\right|\le\frac{t}{5},$$

by taking $\mu(t)=\frac{t}{5}$, condition $(C2)$ is satisfied. Since $K\left(M_1+M_2-\frac{3{,}5}{\Gamma(3{,}5)}\right)\simeq 0{,}597771<1$, it follows from Theorem 4.2, that problem (5.3)-(5.2) has at least one solution $[0{,}1]$.

CONCLUSION

In this paper, some results on the existence and uniqueness of solutions for a nonlinear higher order fractional differential equation involving the left Caputo fractional derivative with both multi-point and multi-strip boundary conditions are obtained. Under sufficient conditions, we have applied the Banach contraction principle to obtain the uniqueness of the solution and by using the fixed point theorem of Krasnoselskii for a sum of contraction and compact mappings in Banach spaces, we prove the existence of at least one solution of our suggested problem. Two examples are given to show the applicability of our results.

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