ORIGINAL PAPER

NEW TAIL PROBABILITY TYPE INEQUALITIES AND COMPLETE CONVERGENCE FOR WOD RANDOM VARIABLES WITH APPLICATION TO LINEAR MODEL GENERATED BY WOD ERRORS

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Abstract. *In this paper, we prove an exponential type inequality for unbounded Widely Orthant Dependent (WOD) random variables. Then we will use this inequality for establishing the almost complete convergence for a sequence of widely dependent random variables (WOD). As an application, we combine Markov inequalities and the new exponential inequality to discuss almost complete convergence of errors generated by a sequence of dependent random variables (WOD) in linear models.*
Keywords: *autoregressive process; exponential inequalities; WOD sequence.*

1. INTRODUCTION

In many proofs of limit theorems, exponential inequality plays a key role. It gives a measure of complete convergence for partial sums in particular. We'll go through the definitions of widely upper orthant dependent and widely lower orthant dependent first.

Let $\{Z_n, n \geq 1\}$ be a random variables. We say that the $\{Z_n, n \geq 1\}$ are widely upper orthant dependent (WUOD, in short), if there exists a finite real sequence $\{g_U(n), n \geq 1\}$ satisfying for each $n \geq 1$,

$$\mathbb{P}(Z_1 > z_1, Z_2 > z_2, \dots, Z_n > z_n) \leq g_U(n) \prod_{i=1}^{n} \mathbb{P}(Z_i > z_i), \tag{1}$$

We say that the $\{Z_n, n \geq 1\}$ are widely lower orthant dependent (WLOD, in short), if there exists a finite real sequence $\{g_L(n), n \geq 1\}$ satisfying for each $n \geq 1$ and for all $z_i \in R$, $1 \leq i \leq n$,

$$\mathbb{P}(Z_1 \leq z_1, Z_2 \leq z_2, \dots, Z_n \leq z_n) \leq g_L(n) \prod_{i=1}^{n} \mathbb{P}(Z_i \leq z_i), \tag{2}$$

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We say that the $\{Z_n, n \geq 1\}$ are widely orthant dependent (WOD, in short), if they are both WUOD and WLOD, in this case $\{g_U(n). g_L(n), n \geq 1\}$ are called dominating coefficients.

An array $\{Z_{ni}, i \geq 1, n \geq 1\}$ of random variables is called row-wise WOD if for every $n \geq 1$, $\{Z_{ni}, i \geq 1\}$ is a sequence of WOD random variables. Now, we recall that the random variables $\{Z_n, n \geq 1\}$ are called negatively upper orthant dependent (NUOD) and negatively lower orthant dependent (NLOD), when $g_L(n) = g_U(n) = 1$ for any $n \geq 2$ in (1) and (2), respectively. In particulary, we say that the random variables $\{Z_n, n \geq 1\}$ are negatively orthant dependent (NOD), if they are both NUOD and NLOD [1]. If both (1.1) and (1.2) hold when $g_L(n) = g_U(n) = M$ for some constant $M > 1$ and for any $n \geq 2$, the random variables $\{Z_n, n \geq 1\}$ are called extended negatively upper ortant dependent (ENUOD) and extended negatively lower ortant dependent (ENLOD), respectively. We say that the random variables $\{Z_n, n \geq 1\}$ are extended negatively ortant dependent (ENOD), If they are both ENUOD and ENLOD [1].

Wang et al. [2] introduced the concept of Widely Orthant Dependent random variables (WOD) and various applications have since been discovered. The class of widely ortant dependent random variables (WOD) contains some common negatively dependent random variables, some positively dependent random variables and some others, it showed by some examples in [2]. Some basic renewal theorems for a random walk with widely dependent increments and gave some applications presented in [3]. The asymptotic of the finite-time ruin probability for a generalized renewal risk model with strong sub exponential claim independent sizes and widely lower orthant dependent inter-occurrence times, studied in [4]. [5] gave the asymptotically equivalent formula for the finite-time ruin probability under a dependent risk model with constant interest rate. [6] provided the asymptotic lower bounds of precise large deviations with nonnegative and dependent random variables. [7] considered uniform asymptotic for the finite-time ruin probabilities of two kinds of nonstandard bidimensional renewal risk models with constant interest forces and diffusion generated by Brownian motions. [8] established the Bernstein type inequality for widely ortant dependent random variables (WOD) and gave some applications, and so forth.

Our article is organized as follows, in the next section we give some lemmas necessary for the proof of the main results. We give the main results and their proofs in section number three. In the fourth section we apply the results obtained bellow, in AR(1) the autoregressive linear model, where the errors are a sequence of widely ortant dependent random variables.

2. LEMMAS

To prove the main results of the paper, we need the following important lemmas. The first one is a basic property for WOD random variables, which was obtained by [2].

Lemma 2.1. Let $\{Z_n, n \geq 1\}$ be WLOD (WUOD) with dominating coefficients $\{g_U(n), n \geq 1\}$, $\{g_L(n), n \geq 1\}$.

- If $f_n(.)$, $n \geq 1$ are nondecreasing, then $f_n(Z_n)$,$n \geq 1$ are still WLOD (WUOD) with dominating coefficients $\{g_U(n), n \geq 1\}$, $\{g_L(n), n \geq 1\}$.
- If $f_n(.)$, $n \geq 1$ are nonincreasing, then $f_n(Z_n)$,$n \geq 1$ are still WLOD (WUOD) with dominating coefficients $\{g_U(n), n \geq 1\}$, $\{g_L(n), n \geq 1\}$.

1) If $\{Z_n, n \geq 1\}$ are nonnegative and WUOD with dominating coefficient $\{g_U(n), n \geq 1\}$, then for each $n \geq 1$,

$$\mathbb{E} \prod_{i=1}^{n} Z_i \leq g_U(n) \prod_{i=1}^{n} \mathbb{E}(Z_i).$$

In particular, if $\{Z_n, n \geq 1\}$ are WUOD with dominating coefficients $(g_U(n), n \geq 1)$ then for each $n \geq 1$ and $s \geq 0$,

$$\mathbb{E}\exp\{ s\sum_{i=1}^{n} Z_i \} \leq g_U(n)\prod_{i=1}^{n}\mathbb{E}(sZ_i)$$

By lemma 2.1 we can get the following corollary immediately.

Corollary 2.1. [9] Let $\{Z_n, n \geq 1\}$ be sequence of WOD random variables.

1. If $f_n(.)$,$n \geq 1$ are all nondecreasing (or all nonincreasing), then $f_n(Z_n)$,$n \geq 1$ are still WOD.
2. For each $n \geq 1$ and $s \in \mathbb{R}$

$$\mathbb{E}\exp\{s\sum_{i=1}^{n} Z_i \} \leq g(n)\prod_{i=1}^{n}\mathbb{E}(sZ_i)$$

Lemma 2.2. [4] For all $z \in \mathbb{R}$

$$\exp\{z\} \leq 1 + z + |z|^{1+\beta}\exp\{2|z|\}.$$

where $0 < \beta \leq 1$.

Lemma 2.3. Let $Z_1, \dots, Z_n$ be Widely Orthant Dependent random variables $(Z \in R)$ with $\mathbb{E}(Z_i) = 0$ for $1 \leq i \leq n$, For all $0 < t \leq \delta/2$ where δ is a positive constant and t is a positive real. Set $S_n = \sum_{i=1}^{n} Z_i$ and $K_\beta = \sum_{i=1}^{n}[\mathbb{E}|Z_i|^{1+\beta}]^{\frac{1}{2}}[\mathbb{E}e^{2\delta|Z_i|}]^{\frac{1}{2}}$. Then for any $0 < t \leq \delta/2$ and for any $\xi > 0$,

$$\mathbb{P}(|S_n| > n\xi) \leq 2g(n)e^{-\beta K_\beta^{-\frac{1}{\beta}}\left(\frac{n\xi}{1+\beta}\right)^{1+\frac{1}{\beta}}} \tag{3}$$

Proof: For $0 < t \leq \delta/2$ and $\mathbb{E}(Z_i) = 0$, $i \geq 1$ and the fact that $1 + z \leq e^z$, therefore, by Markov's inequality and Lemma 2.1 it follows that

$$\mathbb{P}(S_n > n\xi) = \mathbb{P}(tS_n > tn\xi) = \mathbb{P}\left(e^{tS_n} > e^{tn\xi}\right)$$
$$\leq e^{-tn\xi}\mathbb{E}(e^{tS_n})$$
$$\leq g(n)e^{-tn\xi}\prod_{i=1}^{n}\mathbb{E}(e^{tZ_i})$$
$$\leq g(n)e^{-tn\xi}\prod_{i=1}^{n}\left[1 + t^{1+\beta}\mathbb{E}\left(|Z_i|^{1+\beta}e^{2t|Z_i|}\right)\right]$$
$$\leq g(n)e^{-tn\xi}\prod_{i=1}^{n}\left[1 + t^{1+\beta}\left(\left[\mathbb{E}|Z_i|^{2(1+\beta)}\right]^{\frac{1}{2}}\left[\mathbb{E}e^{4t|Z_i|}\right]^{\frac{1}{2}}\right)\right] \text{(By the Hölder inequality)}$$
$$\leq g(n)e^{-tn\xi}\prod_{i=1}^{n}\left[1 + t^{1+\beta}\left(\left[\mathbb{E}|Z_i|^{2(1+\beta)}\right]^{\frac{1}{2}}\left[\mathbb{E}e^{2\delta|Z_i|}\right]^{\frac{1}{2}}\right)\right]$$
$$\leq g(n)e^{-tn\xi}\prod_{i=1}^{n} e^{t^{1+\beta}\left([\mathbb{E}|Z_i|^{2(1+\beta)}]^{\frac{1}{2}}[\mathbb{E}e^{2\delta|Z_i|}]^{\frac{1}{2}}\right)} \text{(By } 1 + z \leq e^z\text{)}$$
$$\leq g(n)e^{-tn\xi + t^{1+\beta}K_\beta} \tag{4}$$

The right term is optimal for $t = \left(\frac{n\xi}{(1+\beta)K_\beta}\right)^{\frac{1}{\beta}}$.

We obtain

$$\mathbb{P}(S_n > n\xi) \le g(n)e^{-\beta K_\beta^{-\frac{1}{\beta}}\left(\frac{n\xi}{1+\beta}\right)^{1+\frac{1}{\beta}}}.$$

We know that

$$\mathbb{P}(|S_n| > n\xi) = \mathbb{P}(S_n > n\xi) + \mathbb{P}(-S_n < n\xi)$$

Since $\{-Z_n, n \ge 1\}$, is also WOD we obtain by (1.2) that

$$\mathbb{P}(S_n > n\xi) = \mathbb{P}(-S_n < n\xi) \le g(n)e^{-\beta K_\beta^{-\frac{1}{\beta}}\left(\frac{n\xi}{1+\beta}\right)^{1+\frac{1}{\beta}}}. \tag{5}$$

By (3) and (4) the result (5) follows

3. MAIN RESULTS AND PROOFS

Theorem 3.1 let $Z_1, \dots, Z_n$ be Widely Orthant Dependent random variables such that $\mathbb{E}(Z_i) = 0$ if there exists a positive constant δ such that $0 < t \le \delta/2$,then for any $0 < t \le \delta/2$ there exists a finite sequence g(n) such that

$$\mathbb{E}(e^{tS_n}) \le g(n)e^{t^{1+\beta}K_\beta} \tag{6}$$

Proof: From condition $\mathbb{E}(Z_i) = 0$ and using the lemma 2.2 and $1 + z \le e^z$, we have

$$\mathbb{E}(e^{tZ_i}) \le 1 + t^{1+\beta}\mathbb{E}\left(|Z_i|^{1+\beta}e^{2t|Z_i|}\right)$$

$$\le 1 + t^{1+\beta}\left(\left[\mathbb{E}|Z_i|^{2(1+\beta)}\right]^{\frac{1}{2}}\left[\mathbb{E}e^{4t|Z_i|}\right]^{\frac{1}{2}}\right) \text{ (by the Hölder inequality)}$$

$$\le e^{t^{1+\beta}\left(\left[\mathbb{E}|Z_i|^{2(1+\beta)}\right]^{\frac{1}{2}}\left[\mathbb{E}e^{2\delta|Z_i|}\right]^{\frac{1}{2}}\right)} \tag{7}$$

(By $1 + z \le e^z$)

For all $0 < t \le \delta/2$. by corollary 2.1 and (5)

$$\mathbb{E}\exp\{t\textstyle\sum_{i=1}^n Z_i\} \le g(n)\prod_{i=1}^n \mathbb{E}\exp(tZ_i)$$
$$\le g(n)e^{t^{1+\beta}K_\beta} \tag{8}$$

Theorem 3.2 Let $Z_1, \dots, Z_n$ be Widely Orthant Dependent random variables with $\mathbb{E}(Z_i) = 0$ and $\mathbb{E}(Z_i^2) < \infty$ such that $0 < t \le \delta/2$, where δ is positive constant and $K_\beta = \sum_{i=1}^n [\mathbb{E}|Z_i|^{1+\beta}]^{\frac{1}{2}} [\mathbb{E}e^{2\delta|Z_i|}]^{\frac{1}{2}}$ and for any $\xi > 0$ and $n \ge 1$,

$$\mathbb{P}\left(S_n/K_\beta > \xi\right) \le g(n)e^{-\beta K_\beta\left(\frac{\xi}{1+\beta}\right)^{1+\frac{1}{\beta}}} \tag{9}$$

Proof: For $0 < t \le \delta/2$, then from Markov's inequality it follows that

$$\mathbb{P}\left({S_n}/{K_\beta} > \xi\right) = \mathbb{P}\left(e^{tS_n} > e^{t\xi K_\beta}\right) \le e^{-t\xi K_\beta}\mathbb{E}\left(\prod_{i=1}^{n} e^{tZ_i}\right), \qquad (10)$$
$$\le g(n)e^{-t\xi K_\beta + t^{1+\beta}K_\beta}$$

By taking $t = \left(\frac{\xi}{1+\beta}\right)^{\frac{1}{\beta}}$ we obtain (9) from (10)

Theorem 3.3. Let $Z_1, \dots, Z_n$ be Widely Orthant Dependent random variables with $\mathbb{E}\left({Z_i}^2\right) < \infty$ and a real t, $0 < t \le \delta/2$, $0 < \beta \le 1$ where δ and β are positives constants and $D_\beta = \sum_{i=1}^{n}[\mathbb{E}|Z_i|^{2(1+\beta)}]^{\frac{1}{2}} [\mathbb{E}e^{2\delta|Z_i|}]$ then for any $\xi > 0$ we have

$$\mathbb{P}(|S_n - \mathbb{E}S_n| > n\xi) \le 2g(n)e^{-n^{(1+\frac{1}{\beta})}\beta D_\beta^{-\frac{1}{\beta}}\left[\frac{\xi}{2(1+\beta)}\right]^{1+\frac{1}{\beta}}} \qquad (11)$$

*Proof***:** By Markov's inequality and Corollary 2.1 and Lemma 2.2

$$\mathbb{P}(S_n - \mathbb{E}S_n > n\xi) \le e^{-tn\xi}\mathbb{E}\left(e^{t(S_n - \mathbb{E}S_n)}\right)$$
$$\le g(n)e^{-tn\xi}\prod_{i=1}^{n}\mathbb{E}\left(e^{t(Z_i - \mathbb{E}Z_i)}\right) \qquad (12)$$

On the other hand we have

$$\mathbb{E}\left(e^{t(Z_i - \mathbb{E}Z_i)}\right) \le \left[1 + t\mathbb{E}(Z_i - \mathbb{E}Z_i) + t^{1+\beta}\mathbb{E}\left(|Z_i - \mathbb{E}Z_i|^{1+\beta}e^{2t|Z_i - \mathbb{E}Z_i|}\right)\right]$$
$$\le \left[1 + t^{1+\beta}\left(\left[\mathbb{E}|Z_i - \mathbb{E}Z_i|^{2(1+\beta)}\right]^{\frac{1}{2}}\left[\mathbb{E}e^{4t|Z_i - \mathbb{E}Z_i|}\right]^{\frac{1}{2}}\right)\right] \text{(By the Hölder inequality)}$$
$$\le 1 + t^{1+\beta}\left([2^{2\beta+1}(\mathbb{E}|Z_i|^{2(1+\beta)} + \mathbb{E}|Z_i|^{2(1+\beta)})]^{\frac{1}{2}}\left[\mathbb{E}e^{4t|Z_i|}e^{4t\mathbb{E}|Z_i|}\right]^{\frac{1}{2}}\right) \text{(By the Cr-inequality)}$$
$$\le 1 + (2t)^{1+\beta}\left(\left[\mathbb{E}|Z_i|^{2(1+\beta)}\right]^{\frac{1}{2}}\left[\mathbb{E}e^{4t|Z_i|}\mathbb{E}e^{4t\mathbb{E}|Z_i|}\right]^{\frac{1}{2}}\right) \text{(By the Jensen inequality)}$$
$$\le 1 + (2t)^{1+\beta}\left(\mathbb{E}\left[|Z_i|^{2(1+\beta)}\right]^{\frac{1}{2}}\left[\mathbb{E}e^{4t|Z_i|}\right]\right)$$
$$\le 1 + (2t)^{1+\beta}\left(\left[\mathbb{E}|Z_i|^{2(1+\beta)}\right]^{\frac{1}{2}}\left[\mathbb{E}e^{2\delta|Z_i|}\right]\right) \le e^{(2t)^{1+\beta}\left(\mathbb{E}\left[|Z_i|^{2(1+\beta)}\right]^{\frac{1}{2}}\left[\mathbb{E}e^{2\delta|Z_i|}\right]\right)} \text{(By } 1 + z \le e^z\text{)}$$

We compensate for this result in the right-hand side of (10) we get

$$\mathbb{P}(S_n - \mathbb{E}S_n > n\xi) \le g(n)e^{-tn\xi}\prod_{i=1}^{n} e^{(2t)^{1+\beta}\left(\mathbb{E}\left[|Z_i|^{2(1+\beta)}\right]^{\frac{1}{2}}\left[\mathbb{E}e^{2\delta|Z_i|}\right]\right)}$$

$$\mathbb{P}(S_n - \mathbb{E}S_n > n\xi) \le g(n)e^{-tn\xi + (2t)^{1+\beta}D_\beta},$$

by taking $t = \left[\frac{n\xi}{2^{1+\beta}(1+\beta)D_\beta}\right]^{\frac{1}{\beta}}$ we obtain

$$\mathbb{P}(S_n - \mathbb{E}S_n > n\xi) \le g(n)e^{-n^{(1+\frac{1}{\beta})}\beta D_\beta^{-\frac{1}{\beta}}\left[\frac{\xi}{2(1+\beta)}\right]^{1+\frac{1}{\beta}}}$$

and

$$\mathbb{P}(|S_n - \mathbb{E}S_n| > n\xi) \le 2g(n)e^{-n^{(1+\frac{1}{\beta})}\beta D_\beta^{-\frac{1}{\beta}}\left[\frac{\xi}{2(1+\beta)}\right]^{1+\frac{1}{\beta}}}$$

Corollary 3.1. Let $\{Z_n, n \ge 1\}$ be a sequence of identically distributed WOD random variables. Assume that there exists a positive integer n_0. Such that $0 < t \le \delta/2$, for each $n \ge n_0$ where δ is positive constant and $0 < \beta \le 1$, Then for any $0 < t \le \delta/2$ and then for any $\xi > 0$ such that $\xi > 0$ we have

$$\mathbb{P}(|S_n - \mathbb{E}S_n| > n\xi) \le 2g(n)e^{-n^{(1+\frac{1}{\beta})}\beta D_\beta^{-\frac{1}{\beta}}\left[\frac{\xi}{2(1+\beta)}\right]^{1+\frac{1}{\beta}}} \tag{13}$$

Lemma 3.1. For any $z > 0$ and for any $q > 0$ such that $q < c < z$

$$e^{-z} \le \left(\frac{qe^{-1}}{z}\right)^q$$

Proof: For any $z > 0$ and for any $q > 0$ such that $q < c < z$

$$\ln(z) - \ln(q) = (z - q)\frac{1}{c}$$
$$\Leftrightarrow \ln\left(\frac{z}{q}\right) = \frac{z - q}{c} \le \frac{z - q}{q}$$
$$\Leftrightarrow \frac{z}{q} \le e^{\frac{z-q}{q}} = e^{\frac{z}{q}}e^{-1}$$
$$\Leftrightarrow \left(e^{-\frac{z}{q}}\right)^q \le \left(\frac{qe^{-1}}{z}\right)^q$$
$$\Leftrightarrow e^{-z} \le \left(\frac{qe^{-1}}{z}\right)^q$$

Theorem 3.4. let $\{Z_n, n \ge 1\}$ be a sequence WOD random variables with $\mathbb{E}(Z_i) = 0$, and $0 < t \le \delta/2$ where δ is positive constant and $0 < \beta \le 1$. Then for any $\Delta > 0$

$$\sum_{i=1}^{\infty} \mathbb{P}(|S_n| > n^\Delta \xi) < \infty \tag{14}$$

Proof: For any $\xi > 0$, from Lemma 2.3 we have

$$\sum_{i=1}^{\infty} \mathbb{P}(|S_n| > n^\Delta \xi) \le 2g(n)\sum_{i=1}^{\infty} e^{-\beta K_\beta^{-\frac{1}{\beta}}\left(\frac{n^\Delta \xi}{1+\beta}\right)^{1+\frac{1}{\beta}}}$$
$$= 2g(n)\sum_{i=1}^{\infty} e^{-n^{\Delta(1+\frac{1}{\beta})}\beta K_\beta^{-\frac{1}{\beta}}\left(\frac{\xi}{1+\beta}\right)^{1+\frac{1}{\beta}}} \tag{15}$$
$$\le 2g(n)\sum_{i=1}^{\infty} [e^{-\beta K_\beta^{-\frac{1}{\beta}}\left(\frac{\xi}{1+\beta}\right)^{1+\frac{1}{\beta}}}]^{n^{\Delta(1+\frac{1}{\beta})}}$$

$$\leq 2\, g(n) \sum_{i=1}^{\infty} [e^{-K}]^{n^{\Delta\left(1+\frac{1}{\beta}\right)}} < \infty,$$

where K is a positive number not depending on n.

By the lemma 3.1, then the right-hand side of (15) become

$$\sum_{i=1}^{\infty} \mathbb{P}(|S_n| > n^{\Delta}\xi) \leq 2g(n) \sum_{i=1}^{\infty} (qe^{-1})^q \frac{1}{K^q n^{\Delta q(1+\frac{1}{\beta})}}$$

$$\leq \frac{2g(n)(qe^{-1})^q}{K^q} \sum_{i=1}^{\infty} \frac{1}{n^{\Delta q\left(1+\frac{1}{\beta}\right)}} < \infty,$$

by suppose that $q > \frac{\beta}{\Delta(1+\beta)}$ Hence the proof is complete.

Theorem 3.5. Let $\{Z_n, n \geq 1\}$ be a sequence of identically distributed WOD random variables. Assume that there exists a positive integer n_0 such that $0 < t \leq \delta/2$,for each $n \geq n_0$, where δ is positive constant and $0 < \beta \leq 1$. Then for any $\Delta > 0$

$$\sum_{i=1}^{\infty} \mathbb{P}(|S_n - \mathbb{E}S_n| > n\xi) < \infty \tag{16}$$

Proof: We have from theorem 3.3**,** for any $\xi > 0$, there exists

$$\sum_{i=1}^{\infty} \mathbb{P}(|S_n - \mathbb{E}S_n| > n\xi) \leq 2g(n) \sum_{i=1}^{\infty} e^{-n^{(1+\frac{1}{\beta})}\beta D_\beta^{-\frac{1}{\beta}}\left[\frac{\xi}{2(1+\beta)}\right]^{1+\frac{1}{\beta}}}$$

$$\leq 2g(n) \sum_{i=1}^{\infty} \left(e^{-\beta D_\beta^{-\frac{1}{\beta}}\left[\frac{\xi}{2(1+\beta)}\right]^{1+\frac{1}{\beta}}} \right)^{n^{(1+\frac{1}{\beta})}} < \infty$$

after this result we get (14).

4. APPLICATION RESULTS IN THE FIRST-ORDER AUTOREGRESSIVE MODEL AR(1)

4.1 THE AR(1) MODEL

We consider an autoregressive model of first order AR(1) defined by

$$Z_n = \theta Z_{n-1} + \xi_i \,, i = 1,2, \ldots, \tag{17}$$

where $\{\xi_i, i \geq 1\}$ is a sequence of WOD random variables with $\xi_0 = Z_0 = 0, 0 \leq \mathbb{E}\xi_k^4 < \infty$, $k = 1,2, \ldots$ and θ is a parameter with $|\theta| < 1$ Hence (18) as follows:

$$Z_i = \sum_{j=1}^{\infty} \theta^j \xi_{i-j} \tag{18}$$

The coefficient θ is fitted least squares, giving the estimator

$$\hat{\theta} = \frac{\sum_{j=1}^{n} Z_j Z_{j-1}}{\sum_{j=1}^{n} Z_{j-1}^2} \tag{19}$$

From (18) and (20) we obtain that

$$\hat{\theta} - \theta = \frac{\sum_{j=1}^{n} \xi_j Z_{j-1}}{\sum_{j=1}^{n} Z_{j-1}^2} \tag{20}$$

Theorem 4.1 Let the conditions of theorem 3.3 be satisfied then for any $\frac{(\mathbb{E}Z_1)}{\rho^2} < \xi^2 < \frac{\sum_{j=1}^{n} \mathbb{E}\, Z_{j-1}^2}{2n^2}$ positive, we have

$$\mathbb{P}\left(\sqrt{n}\left|\hat{\theta} - \theta\right| > \rho\right) \leq g(n)\left[e^{-n^{\left(1+\frac{1}{\beta}\right)}D} + e^{-n^{2\left(1+\frac{1}{\beta}\right)}C}\right] \tag{21}$$

where $C = \beta C_\beta^{-\frac{1}{\beta}}\left(\frac{\xi^2}{1+\beta}\right)^{1+\frac{1}{\beta}}$ and $D = \beta D_\beta\left[\frac{\rho^2\xi^2 - \mathbb{E}Z_1}{2(1+\beta)D_\beta}\right]^{1+\frac{1}{\beta}}$,

Proof: From (21) it follows that

$$\mathbb{P}\left(\sqrt{n}\left|\hat{\theta} - \theta\right| > \rho\right) \leq \mathbb{P}\left(\left|\frac{\frac{1}{\sqrt{n}}\sum_{j=1}^{n} \xi_j Z_{j-1}}{\frac{1}{n}\sum_{j=1}^{n} Z_{j-1}^2}\right| > \rho\right) \tag{22}$$

Therefore, by virtue if the probability properties and Hölder's inequalities, we have that for any $\xi > 0$

$$\mathbb{P}\left(\sqrt{n}\left|\hat{\theta} - \theta\right| > \rho\right) \leq \mathbb{P}\left(\frac{1}{n}\sum_{j=1}^{n} Z_j \geq \xi^2\rho^2\right) + \mathbb{P}\left(\frac{1}{n^2}\sum_{j=1}^{n} Z_{j-1}^2 \leq \xi^2\right) \tag{23}$$

$$= \mathbb{P}\left(\sum_{j=1}^{n} Z_j \geq (\xi^2\rho^2)n\right) + \mathbb{P}\left(\sum_{j=1}^{n} Z_{j-1}^2 \leq \xi^2 n^2\right) \tag{24}$$

$$= L_{n1} + L_{n2} \tag{25}$$

Now, we start by estimate L_{n1} then estimate L_{n2}. We have that

$$\begin{aligned} L_{n1} &= \mathbb{P}\left(\sum_{j=1}^{n} Z_j \geq (\xi^2\rho^2)n\right) \\ &= \mathbb{P}\left(\sum_{j=1}^{n} (Z_j - \mathbb{E}Z_j + \mathbb{E}Z_j) \geq (\xi^2\rho^2)n\right) \qquad (26) \\ &= \mathbb{P}\left(\sum_{j=1}^{n} (Z_j - \mathbb{E}Z_j) \geq (\xi^2\rho^2 - \mathbb{E}Z_1)n\right) \end{aligned}$$

By using the theorem 3.3, the right hand side of (27) become

$$L_{n1} = \mathbb{P}\left(\sum_{j=1}^{n} Z_j \geq (\xi^2\rho^2)n\right) \leq 2g(n)\left[e^{-n^{\left(1+\frac{1}{\beta}\right)}\beta D_\beta^{-\frac{1}{\beta}}\left(\frac{\rho^2\xi^2-\mathbb{E}Z_1}{2(1+\beta)}\right)^{1+\frac{1}{\beta}}}\right] \quad (27)$$

Then we shall bind the right hand side of L_{n2}. By Markov's inequality, for any $0 < t \leq \delta/2$ it follows that

$$\begin{aligned} L_{n2} &= \mathbb{P}\left(\sum_{j=1}^{n} Z_{j-1}^2 \leq \xi^2 n^2\right) \\ &= \mathbb{P}\left(n^2\xi^2 - \sum_{j=1}^{n} Z_{j-1}^2 \geq 0\right) \\ &\leq \mathbb{E}\left(\mathbb{I}_{\left\{n^2\xi^2-\sum_{j=1}^{n} Z_{j-1}^2 \geq 0\right\}}\right) \\ &\leq \mathbb{E}\left(\exp\{t(n^2\xi^2 - \sum_{j=1}^{n} Z_{j-1}^2)\}\right) \\ &\leq e^{tn^2\xi^2}\mathbb{E}\left(e^{-t\sum_{j=1}^{n} Z_{j-1}^2}\right) \\ &\leq e^{tn^2\xi^2}\mathbb{E}\left(\prod_{j=1}^{n} e^{-tZ_{j-1}^2}\right) \end{aligned}$$

By using corollary 2.1 and lemma 2.2 the right hand side of the expression L_{n2} become

$$L_{n2} \leq g(n)\, e^{tn^2\xi^2}\prod_{j=1}^{n}\mathbb{E}\left(e^{-tZ_{j-1}^2}\right) \quad (28)$$

$$\leq g(n)\, e^{tn^2\xi^2}\prod_{j=1}^{n}\mathbb{E}\left(1 - tZ_{j-1}^2 + t^{1+\beta}\left|Z_{j-1}\right|^{2(1+\beta)} e^{2tZ_{j-1}^2}\right) \quad (29)$$

$$\leq g(n)\, e^{tn^2\xi^2}\prod_{j=1}^{n}\left(1 - t\mathbb{E}Z_{j-1}^2 + t^{1+\beta}\left[\mathbb{E}\left|Z_{j-1}\right|^{4(1+\beta)}\right]^{\frac{1}{2}}\left[\mathbb{E}e^{4tZ_{j-1}^2}\right]^{\frac{1}{2}}\right) \quad (30)$$

$$\leq g(n)\, e^{tn^2\xi^2}\prod_{j=1}^{n}\left(1 - t\mathbb{E}Z_{j-1}^2 + t^{1+\beta}\left[\mathbb{E}\left|Z_{j-1}\right|^{4(1+\beta)}\right]^{\frac{1}{2}}\left[\mathbb{E}e^{2\delta Z_{j-1}^2}\right]^{\frac{1}{2}}\right) \quad (31)$$

$$\leq g(n)\, e^{tn^2\xi^2}\prod_{j=1}^{n} e^{-t\mathbb{E}Z_{j-1}^2 + t^{1+\beta}\left[\mathbb{E}\left|Z_{j-1}\right|^{4(1+\beta)}\right]^{\frac{1}{2}}\left[\mathbb{E}e^{2\delta Z_{j-1}^2}\right]^{\frac{1}{2}}} \quad (32)$$

$$\leq g(n)\, e^{tn^2\xi^2 - t\sum_{j=1}^{n}\mathbb{E}Z_{j-1}^2 + t^{1+\beta}C_\beta}. \quad (33)$$

By taking $t = \left[\frac{\sum_{j=1}^{n} \mathbb{E}Z_{j-1}^2 - n^2\xi^2}{(1+\beta)C_\beta}\right]^{\frac{1}{\beta}}$ in (34), then for any $\rho > 0$

$$\mathbb{P}\left(\sqrt{n}\left|\hat{\theta} - \theta\right| > \rho\right) \leq g(n)\left[e^{-n^{\left(1+\frac{1}{\beta}\right)}D} + e^{-n^{2\left(1+\frac{1}{\beta}\right)}C}\right] \quad (34)$$

Corollary 4.1 The sequence $\hat{\theta}$ defined in (20) completely converges to the parameter θ of the first-order autoregressive process, and then we have

$$\sum_{n}^{\infty} \mathbb{P}\left(\sqrt{n}\left|\hat{\theta} - \theta\right| > \rho\right) < +\infty \quad (35)$$

Proof: By using theorem 3.6 and $\mathbb{E}Z_j^2 < \infty$, $C < \infty$ and $D < \infty$ we get the result of (35) immediately.

5. CONCLUSION

In this article we established the asymptotic distribution of the WOD random error in first-order autoregressive processes using the exponential inequalities and some inequalities for probability of orthant dependent variables. Then we study the complete convergence for widely orthant dependent random variables and its applications in autoregressive AR(1) models.

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