ORIGINAL PAPER

SOME IMPROVED ESTIMATORS FOR THE MEAN ESTIMATION UNDER STRATIFIED SAMPLING BY USING TRANSFORMATIONS

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Abstract. *In this research, we proposed some new estimators of finite population mean by using the transformations of coefficient of variation,* C_x*, the coefficient of kurtosis,* $\beta_{2(x)}$ *and some real numbers in stratified random sampling without replacement scheme. The bias and mean square error (MSE) of the proposed estimator are obtained by the first order approximation. It is found that the proposed estimators are more efficient than the traditional mean, ratio, Bahl and tuteja [1], regression, Koyuncu and Kadilar [2], Singh and Solanki [3-4] estimators. We have applied a real data set using stratified random sampling technique for measuring the efficiency of the estimators considered here.*

Keywords: *Kurtosis; stratified random sampling; mean square estimator; efficiency.*

1. INTRODUCTION

The estimation of population mean can be computed by several different sampling designs. In modern surveys, the stratified sampling has achieved more attention to improve the precision in estimation. Stratified sampling can be used by combined and separate ratio-type estimators. Our current research is based on the modification of the combination of combined ratio and product estimators, for estimating the population mean under stratified random sampling scheme. We used auxiliary information to improve precision of estimates and get more efficient results. Some known parameters of auxiliary variable 'x' such as coefficient of variation c_x, coefficient of kurtosis $\beta_{2(x)}$ are used for the estimation of population mean. Different existing ratio and product type estimators in stratified sampling are considered for comparison with the proposed estimators. The numerical results of the real-life data set are used to support the theoretical findings.

The use of auxiliary information improves precision of estimates and efficiency of estimators in estimation process. Some researchers used different transformations for some known parameter of auxiliary variable 'x'. For traditional mean estimators see, Bahl and Tuteja [1], traditional regression, Koyuncu and Kadilar [2], Singh and Solanki [3-4] estimators are given below.

Suppose a finite population $\zeta=\left[u_1^*,u_2^*,u_3^*,...,u_N^*\right]$ of size N. Let Y be taken as study variable and X be taken as auxiliary variable having values y_i and x_i in unit u_i^* (i=1, 2,… N). The population divides into L groups and n_h random sample drawn without replacement from N_h population in stratum h, (h=1, 2, 3, … L). Where $\overline{X},\overline{Y}$ are the population means of x and y

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respectively, assuming population of size N, is divided into L strata containing N_h units, where (h=1,2,…,L), such as $\sum_{h=1}^{L} N_h = N$ and $\sum_{h=1}^{L} n_h = n$, where $\bar{x}_{st} = \sum_{h=1}^{L} W_h \bar{x}_h$ and $\bar{y}_{st} = \sum_{h=1}^{L} W_h \bar{y}_h$.

$$\lambda_0 = \frac{\bar{y}_{st} - \bar{Y}}{\bar{Y}}, \quad E(\lambda_0) = 0, \quad E(\lambda_0^2) = \sum_{h=1}^{L} W_h^2 f_h' \frac{S_{yh}^2}{\bar{Y}^2} = \eta_0$$

$$\lambda_1 = \frac{\bar{x}_{st} - \bar{X}}{\bar{X}}, \quad E(\lambda_1) = 0, \quad E(\lambda_1^2) = \sum_{h=1}^{L} W_h^2 f_h' \frac{S_{xh}^2}{\bar{X}^2} = \eta_1$$

$$E(\lambda_0 \lambda_1) = \sum_{h=1}^{L} W_h^2 f_h' \frac{S_{yxh}}{\bar{X}\bar{Y}} = \eta_{01}$$

$$f_h = \frac{n_h}{N_h}, \quad f_h' = \left(\frac{1 - f_h}{n_h} \right)$$

$$\Omega_1 = \sum_{h=1}^{L} W_h S_{xh}, \ \Omega_2 = \sum_{h=1}^{L} W_h C_{xh}, \ \Omega_3 = \sum_{h=1}^{L} W_h \beta_{1h}(x)$$

$$\Omega_4 = \sum_{h=1}^{L} W_h \beta_{2h}(x), \ \Omega_5 = \sum_{h=1}^{L} W_h \rho_h, \ \Omega_6 = \sum_{h=1}^{L} W_h \Delta_h(x)$$

1.1. MEAN ESTIMATORS UNDER STRATIFIED RANDOM SAMPLING

$$\bar{y}_{st} = \sum_{h=1}^{L} W_h \bar{y}_h$$

where $\bar{y}_h = \frac{1}{n_h} \sum_{i=1}^{n_h} y_{hi}$

Variance of unbiased sample mean is

$$\operatorname{var}(\bar{y}_{st}) = \bar{Y}^2 \eta_1 \tag{1}$$

Ratio estimator under stratified random sampling

$$G_{R(st)} = \bar{y}_{st} \left(\frac{\bar{X}}{\bar{x}_{st}} \right)$$

The expressions of bias and MSE of Ratio estimators are

$$\begin{aligned} bias(G_{R(st)}) &= \bar{Y}(\eta_0 - \eta_{01}) \\ MSE(G_{R(st)}) &= \bar{Y}^2(\eta_0 + \eta_1 - 2\eta_{01}) \end{aligned} \tag{2}$$

Bahl and tuteja [1] combined ratio exponential estimator under stratified random sampling

$$G_{Bt(st)} = \bar{y}_{st} \exp\left(\frac{\bar{X} - \bar{x}_{st}}{\bar{X} + \bar{x}_{st}} \right)$$

The expressions of bias and MSE of Bahl and tuteja [1] exponential estimator are

$$\begin{aligned} bias\left(G_{Bt(st)}\right) &= \bar{Y}\left(\frac{3}{8}\eta_0 - \frac{1}{2}\eta_{01}\right) \\ MSE\left(G_{Bt(st)}\right) &= \bar{Y}^2\left(\eta_0 + \frac{1}{4}\eta_1 - \eta_{01}\right) \end{aligned} \tag{3}$$

Traditional regression estimator under stratified random sampling

$$G_{\text{Reg}(st)} = \bar{y}_{st} + b_{st}\left(\bar{X} - \bar{x}_{st}\right)$$

The expression of MSE of regression estimator is

$$MSE\left(G_{\text{Re}g(st)}\right) = \bar{Y}^2 \eta_1 \left(1 - \rho_{st}^{\ 2}\right) \tag{4}$$

where $\rho_{st} = \dfrac{\eta_{01}}{\sqrt{\eta_0}\sqrt{\eta_1}}$.

Family of ratio estimators of Koyuncu & Kadilar [2] in stratified random sampling

$$K_j = \phi \bar{y}_{st}\left[\frac{a_{st}\bar{X} + b_{st}}{\alpha\left(a_{st}\bar{x}_{st} + b_{st}\right) + (1-\alpha)\left(a_{st}\bar{X} + b_{st}\right)}\right]^g \qquad \text{for } j=1, 2, 3, \ldots, 9.$$

For the family of estimators the bias and MSE can be expressed as

$$bias(\text{K}) = \phi\bar{Y}\left[\frac{g(g+1)}{2}\alpha^2 v^2 \eta_1 - g\alpha v \eta_{01}\right] + \bar{Y}(\phi - 1),$$

and

$$\text{MSE(K)} = \bar{Y}^2 \left\{ \begin{array}{l} \phi^2\eta_0 + \left(\phi^2\left(2g^2+g\right) - \phi\left(g^2+g\right)\right)\alpha^2 v^2 \eta_1 \\ -2g\alpha v\left(2\phi^2 - \phi\right)\eta_{01} + (\phi-1)^2 \end{array} \right\}$$

where

$$A = \left(g^2 + g\right)\alpha^2 v^2 \eta_1 - 2g\alpha v \eta_{01} + 2$$

$$B = \eta_0 + \left(2g^2 + g\right)\alpha^2 v^2 \eta_1 - 4g\alpha v \eta_{01} + 1$$

Table 1. Family of ratio estimators of Koyuncu & Kadilar [2] in Stratified random Sampling

Ratio estimators *g=1, α =1*	*ast*	*bst*
K_1	1	0
K_2	1	Ω_2
K_3	Ω_4	Ω_2
K_4	Ω_2	Ω_4
K_5	1	Ω_1
K_6	Ω_3	Ω_1
K_7	Ω_4	Ω_1
K_8	1	Ω_5
K_9	1	Ω_4

The minimum MSE is

$$MSE_{\min}\left(K\right)=\bar{Y}^2\left[1-\frac{A^2}{4B}\right] \tag{5}$$

Family of estimators of Singh and Solanki [3] in Stratified random Sampling

$$T_{SS\,r}=\left[\phi_1\bar{y}_{st}\left\{\frac{\alpha\left(a_{st}\bar{x}_{st}+b_{st}\right)+\left(1-\alpha\right)\left(a_{st}\bar{X}+b_{st}\right)}{\left(a_{st}\bar{X}+b_{st}\right)}\right\}^{\delta}+\phi_2\bar{y}_{st}\left\{\frac{\left(a_{st}\bar{X}+b_{st}\right)}{\alpha\left(a_{st}\bar{x}_{st}+b_{st}\right)+\left(1-\alpha\right)\left(a_{st}\bar{X}+b_{st}\right)}\right\}^{g}\right]$$

for r =1, 2, 3, …, 17 and *(g=1, δ=0, α=1).*

For the family of estimators the bias and MSE can be expressed as,

$$bias(\mathrm{T}_{SS})=\bar{Y}\left[\phi_1\left\{1+\alpha\delta\upsilon\eta_{01}+\frac{\delta\left(\delta-1\right)}{2}\alpha^2\upsilon^2\eta_1\right\}+\phi_2\left\{1-\alpha\delta\upsilon\eta_{01}+\frac{g\left(g+1\right)}{2}\alpha^2\upsilon^2\eta_1\right\}-1\right]$$

where $\upsilon=\dfrac{a_{st}\bar{X}}{a_{st}\bar{X}+b_{st}}$

$$MSE\left(T_{SS}\right)=\bar{Y}^2\left[1+\phi_1^2C+\phi_2^2B+2\phi_1\phi_2D-2\phi_1E-2\phi_2A\right]$$

where

$$A_{SS}=\left[1-\alpha g\upsilon\eta_{01}+\frac{g\left(g+1\right)}{2}\alpha^2\upsilon^2\eta_1\right],$$

$$B_{SS}=\left[1+\eta_0-4\alpha g\upsilon\eta_{01}+g\left(2g+1\right)\alpha^2\upsilon^2\eta_1\right],$$

$$C_{SS}=\left[1+\eta_0+4\alpha g\upsilon\eta_{01}+g\left(2g-1\right)\alpha^2\upsilon^2\eta_1\right],$$

$$D_{SS}=\left[1+\eta_0+2\alpha\left(\delta-g\right)\upsilon\eta_{01}+\left(\frac{\alpha^2\upsilon^2}{2}\right)\left(\delta-g\right)\left(\delta-g-1\right)\eta_1\right],$$

$$E_{SS}=\left[1-\alpha\delta\upsilon\eta_{01}+\frac{\delta\left(\delta-1\right)}{2}\alpha^2\upsilon^2\eta_1\right].$$

Differentiating MSE partially, with respect to $\emptyset_1$ and $\emptyset_2$ and equating to zero, we get the following optimum values of $\emptyset_1$ and $\emptyset_2$

$$\phi_{1(opt)} = \frac{\left(B_{SS}E_{SS} - A_{SS}D_{SS}\right)}{\left(B_{SS}C_{SS} - D_{ss}^2\right)},\ \phi_{2(opt)} = \frac{\left(A_{SS}C_{SS} - D_{SS}E_{SS}\right)}{\left(B_{SS}C_{SS} - D_{ss}^2\right)}$$

Table 2. Family of estimators of Singh and Solanki [3] in stratified random sampling

Estimators	*ast*	*bst*	Estimators	*ast*	*bst*
T_{ss1}	1	0	T_{ss10}	1	Ω_6
T_{ss2}	1	Ω_2	T_{ss11}	Ω_6	Ω_2
T_{ss3}	Ω_4	Ω_2	T_{ss12}	Ω_2	Ω_6
T_{ss4}	Ω_2	Ω_4	T_{ss13}	Ω_6	Ω_1
T_{ss5}	1	Ω_1	T_{ss14}	Ω_5	Ω_6
T_{ss6}	Ω_3	Ω_1	T_{ss15}	Ω_6	Ω_5
T_{ss7}	Ω_4	Ω_1	T_{ss16}	Ω_5	Ω_1
T_{ss8}	1	Ω_5	T_{ss17}	Ω_2	Ω_1
T_{ss9}	1	Ω_4			

The minimum MSE is:

$$MSE_{\min}\left(T_{SS}\right) = \overline{Y}^2\left[1 - \frac{\left(B_{SS}E_{ss}^2 - 2A_{SS}D_{SS}E_{SS} + A_{ss}^2C_{SS}\right)}{\left(B_{SS}C_{SS} - D_{ss}^2\right)}\right]. \quad (6)$$

Family of estimators of Singh and Solanki [4] in stratified random sampling

$$T_{SKq} = \phi_1\overline{y}_{st}\left[\frac{\overline{X}^*}{\alpha\overline{x}_{st}^* + (1-\alpha)\overline{X}^*}\right]^g + \phi_2\overline{y}_{st}\exp\left[\frac{\delta\left(\overline{X}^* - \overline{x}_{st}^*\right)}{\left(\overline{X}^* + \overline{x}_{st}^*\right)}\right]$$

where $\overline{X}^* = \left(a_{st}\overline{X} + b_{st}\right)$, $\overline{x}_{st}^* = \left(a_{st}\overline{x}_{st} + b_{st}\right)$

For q=1, 2, 3, …, 17 and *(g=0, δ=1)*. For the family of estimators the bias and MSE can be expressed as

$$bias(\mathrm{T}_{SK}) = \overline{Y}\left[\phi_1\left\{1 + \left(\frac{g\alpha\upsilon}{2}\right)\eta_1\left(\alpha\upsilon(g+1) - 2\frac{\eta_{01}}{\eta_1}\right)\right\} + \phi_2\left\{1 + \left(\frac{\delta\upsilon}{8}\right)\eta_1\left(\upsilon(\delta+2) - 4\frac{\eta_{01}}{\eta_1}\right)\right\} - 1\right],$$

$$MSE\left(T_{SK}\right) = \overline{Y}^2\left[1 + \phi_1^2A_{SK} + \phi_2^2C_{SK} + 2\phi_1\phi_2D_{SK} - 2\phi_1B_{SK} - 2\phi_2E_{SK}\right].$$

Table 3. Family of estimators of Singh and Solanki [4] in SRS

Estimators	***ast***	***bst***	**Estimators**	***ast***	***bst***
T_{sk1}	1	0	T_{sk7}	1	Ω_5
T_{sk2}	1	Ω_2	T_{sk8}	1	Ω_4
T_{sk3}	Ω_4	Ω_2	T_{sk9}	Ω_2	Ω_5
T_{sk4}	Ω_2	Ω_4	T_{sk10}	Ω_5	Ω_2
T_{sk5}	1	Ω_1	T_{sk11}	Ω_4	Ω_5
T_{sk6}	Ω_4	Ω_1	T_{sk12}	Ω_5	Ω_4

where

$$A_{SK}=\left[1+\eta_0+\alpha^2\upsilon^2\left(2g^2+g\right)\eta_1-4\alpha g\upsilon\eta_{01}\right],$$

$$B_{SK}=\left[1+\frac{\alpha^2\upsilon^2\left(g^2+g\right)}{2}\eta_1-\alpha g\upsilon\eta_{01}\right],$$

$$C_{SK}=\left[1+\eta_0+\frac{\upsilon^2\left(\delta^2+\delta\right)}{2}\eta_1-2\delta\upsilon\eta_{01}\right],$$

$$D_{SK}=\left[1+\eta_0+\left(\frac{A^*\upsilon^2}{8}\right)\eta_1-2\upsilon\left(2\alpha g+\delta\right)\eta_{01}\right],$$

$$E_{SK}=\left[1-\left\{\frac{\left(\delta^2+2\delta\right)}{8}\upsilon^2\right\}\eta_1+\left(\frac{\delta\upsilon}{2}\right)\eta_{01}\right],$$

where $A^*=\left[\left(2\alpha g+\delta\right)^2+2\left(2\alpha^2 g+\delta\right)\right]$.

Differentiating MSE partially, with respect to $\emptyset_1$ and $\emptyset_2$ and equating to zero, we get the following optimum values of $\emptyset_1$ and $\emptyset_2$

$$\phi_{1(opt)}=\frac{\left(B_{SK}C_{SK}-D_{SK}E_{SK}\right)}{\left(A_{SK}C_{SK}-D^2_{SK}\right)},\ \phi_{2(opt)}=\frac{\left(A_{SK}E_{SK}-B_{SK}D_{SK}\right)}{\left(B_{SK}C_{SK}-D^2_{SK}\right)}.$$

Minimum MSE is

$$MSE_{\min}\left(T_{SK}\right)=\bar{Y}^2\left[1-\frac{\left(B^2_{SK}C_{SK}-2B_{SK}D_{SK}E_{SK}+A_{SK}E^2_{SK}\right)}{\left(A_{SK}C_{SK}-D^2_{SK}\right)}\right]. \quad (7)$$

2. A PROPOSED CLASS OF ESTIMATORS

In this section, some improved estimators of finite population mean in stratified sampling are proposed, which are based on first order approximation. Formulations of the proposed estimators are explained step-by-step as follows.

First proposed estimator

For the formulation of first proposed estimator our motivation comes from:
Rao introduced the following estimator

$$\bar{y}_{Rao}=\varphi_1\bar{y}+\varphi_2\left(\bar{X}-\bar{x}\right). \quad (8)$$

The average of ratio and product estimators is

$$\bar{y}_w=\frac{\bar{y}}{2}\left(\frac{\bar{X}}{\bar{x}}+\frac{\bar{x}}{\bar{X}}\right). \quad (9)$$

By replacing $\bar{y}$ with $\bar{y}_w$ in (8)

$$\bar{y}_M = \phi_1 \bar{y}_w + \phi_2 \left(\bar{X} - \bar{x} \right) \quad (10)$$

By replacing $\bar{y}_M$ in Bhal and Tuteja [1], we proposed the following estimator.

$$U_{p_1} = \bar{y}_w \exp\left(\frac{\bar{X}}{\bar{x}} + \frac{\bar{x}}{\bar{X}} \right) \quad (11)$$

$$U_{p_1} = \left[\phi_1 \frac{\overline{y_{st}}}{2} \left(\frac{\bar{X}}{\overline{x_{st}}} + \frac{\overline{x_{st}}}{\bar{X}} \right) + \phi_2 \left(\bar{X} - \overline{x_{st}} \right) \right] \exp\left[\frac{a\left(\bar{X} - \overline{x_{st}} \right)}{a\left(\bar{X} - \overline{x_{st}} \right) + 2b} \right] \quad (12)$$

We used transformation of $a = \sum_{h=1}^{L} W_h C_{xh}$, $b = \sum_{h=1}^{L} W_h \beta_{2h}(x)$

$$bias\left(U_{p_1}\right) = \phi_1 \bar{Y} \left(1 - \frac{\vartheta \eta_{01}}{2} + \alpha_1 \eta_1 \right) + \phi_2 \bar{X} \frac{\vartheta^2 \eta_1}{2} - \bar{Y} \quad (13)$$

where,

$$\alpha_1 = \frac{1}{2} + \frac{3\vartheta^2}{8} \quad \text{and} \quad \vartheta = \frac{a\bar{X}}{a\bar{X} + b}$$

$$MSE\left(U_{p_1}\right) = \bar{Y}^2 + \phi_1^2 U_{A1(i)} + \phi_2^2 U_{B1(i)} + 2\phi_1\phi_2 U_{C1(i)} - 2\phi_1 U_{D1(i)} - 2\phi_2 U_{E1(i)} \quad (14)$$

$$MSE_{\min}\left(U_{p_1}\right) = \bar{Y}^2 - \frac{U_{A1(i)} U_{E1(i)}{}^2 + U_{B1(i)} U_{D1(i)}{}^2 - 2U_{C1(i)} U_{D1(i)} U_{E1(i)}}{U_{A1(i)} U_{B1(i)} - U_{C1(i)}{}^2} \quad (15)$$

The expressions of above proof are explained in appendix A.

Second proposed estimator

For the formulation of second proposed estimator our motivation from:

The average of exponential ratio and product estimator is

$$\bar{y}_{ew} = \frac{\bar{y}}{2} \left\{ \exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right) + \exp\left(\frac{\bar{x} - \bar{X}}{\bar{X} + \bar{x}} \right) \right\}. \quad (16)$$

By replacing $\bar{y}_{ew}$ with $\bar{y}_w$, in U_{p_1}, we propose the following estimator

$$U_{p_2} = \bar{y}_{ew} \exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right) \quad (17)$$

$$U_{p_2}=\left[\phi_1\frac{\overline{y_{st}}}{2}\left\{\exp\left(\frac{\overline{X}-\overline{x_{st}}}{\overline{X}+\overline{x_{st}}}\right)+\exp\left(\frac{\overline{x_{st}}-\overline{X}}{\overline{X}+\overline{x_{st}}}\right)\right\}+\phi_2\left(\overline{X}-\overline{x_{st}}\right)\right]\exp\left[\frac{a\left(\overline{X}-\overline{x_{st}}\right)}{a\left(\overline{X}-\overline{x_{st}}\right)+2b}\right] \tag{18}$$

We used transformation of $a=\sum_{h=1}^{L}W_hC_{xh}$, $b=\sum_{h=1}^{L}W_h\beta_{2h}\left(x\right)$

$$bias\left(U_{p_2}\right)=\phi_1\overline{Y}\left(1-\frac{\vartheta\eta_{01}}{2}+\alpha_2\eta_1\right)-\phi_2\overline{X}\frac{\vartheta^2\eta_1}{2}-\overline{Y} \tag{19}$$

where,

$$\alpha_2=\frac{3}{8}+\frac{3\vartheta^2}{8}\text{ and }\vartheta=\frac{a\overline{X}}{a\overline{X}+b}$$

$$MSE\left(U_{p_2}\right)=\overline{Y}^2+\phi_1^2U_{A2(i)}+\phi_2^2U_{B2(i)}+2\phi_1\phi_2U_{C2(i)}-2\phi_1U_{D2(i)}-2\phi_2U_{E2(i)} \tag{20}$$

$$MSE_{\min}\left(U_{p_2}\right)=\overline{Y}^2-\frac{U_{A2(i)}U_{E2(i)}^{\;\;2}+U_{B2(i)}U_{D2(i)}^{\;\;2}-2U_{C2(i)}U_{D2(i)}U_{E2(i)}}{U_{A2(i)}U_{B2(i)}-U_{C2(i)}^{\;\;2}} \tag{21}$$

The expressions of above proof are explained in appendix A.

Third proposed estimator

For the formulation of third proposed estimator our motivation from, by replacing $\overline{y}$ with $\bar{y}_w$, in U_{p_2}, we propose the following estimator.

$$U_{p_3}=\left[\phi_1\frac{\overline{y_{st}}}{4}\left(\frac{\overline{X}}{\overline{x_{st}}}+\frac{\overline{x_{st}}}{\overline{X}}\right)\left\{\exp\left(\frac{\overline{X}-\overline{x_{st}}}{\overline{X}+\overline{x_{st}}}\right)+\exp\left(\frac{\overline{x_{st}}-\overline{X}}{\overline{X}+\overline{x_{st}}}\right)\right\}+\phi_2\left(\overline{X}-\overline{x_{st}}\right)\right]\exp\left[\frac{a\left(\overline{X}-\overline{x_{st}}\right)}{a\left(\overline{X}-\overline{x_{st}}\right)+2b}\right] \tag{22}$$

We used transformation of $a=\sum_{h=1}^{L}W_hC_{xh}$, $b=\sum_{h=1}^{L}W_h\beta_{2h}\left(x\right)$

$$bias\left(U_{p_3}\right)=\phi_1\overline{Y}\left(1-\frac{\vartheta\eta_{01}}{2}+\alpha_3\eta_1\right)+\phi_2\overline{X}\frac{\vartheta^2\eta_1}{2}-\overline{Y} \tag{23}$$

where,

$$\alpha_3=\frac{7}{8}+\frac{3\vartheta^2}{8}\text{ and }\vartheta=\frac{a\overline{X}}{a\overline{X}+b}$$

$$MSE\left(U_{p_3}\right)=\overline{Y}^2+\phi_1^2U_{A3(i)}+\phi_2^2U_{B3(i)}+2\phi_1\phi_2U_{C3(i)}-2\phi_1U_{D3(i)}-2\phi_2U_{E3(i)} \tag{24}$$

$$MSE_{\min}\left(U_{p_3}\right)=\overline{Y}^2-\frac{U_{A3(i)}U_{E3(i)}^{\;\;2}+U_{B3(i)}U_{D3(i)}^{\;\;2}-2U_{C3(i)}U_{D3(i)}U_{E3(i)}}{U_{A3(i)}U_{B3(i)}-U_{C3(i)}^{\;\;2}} \tag{25}$$

The expressions of above proof are explained in appendix A.

Fourth proposed estimator

For the formulation of fourth proposed estimator our motivation from Shahzad and Hanif [5] and U_{p_1}, we propose the following estimator.

$$U_{p_4} = \left[\phi_1 \frac{\overline{y_{st}}}{2}\left(\frac{\overline{X}}{\overline{x_{st}}} + \frac{\overline{x_{st}}}{\overline{X}}\right) + (1-2\phi_2)\left(\overline{X} - \overline{x_{st}}\right)\right] \exp\left[\frac{a\left(\overline{X} - \overline{x_{st}}\right)}{a\left(\overline{X} - \overline{x_{st}}\right) + 2b}\right] \tag{26}$$

for a=0 and b=1

$$bias\left(U_{p_4}\right) = \phi_1 \overline{Y}\left(1 - \frac{\vartheta\eta_{01}}{2} + \alpha_4\eta_1\right) + (1-2\phi_2)\overline{X}\frac{\vartheta^2\eta_1}{2} - \overline{Y} \tag{27}$$

where,

$$\alpha_4 = \frac{1}{2} + \frac{3\vartheta^2}{8} \text{ and } \vartheta = \frac{a\overline{X}}{a\overline{X} + b}$$

$$MSE\left(U_{p_4}\right) = \left(\overline{Y}^2 + \overline{X}^2\eta_1 - \overline{X}\,\overline{Y}\vartheta\eta_1\right) + \phi_1^2 U_{A4(i)} + \phi_2^2 U_{B4(i)} + 2\phi_1\phi_2 U_{C4(i)} - 2\phi_1 U_{D4(i)} - 2\phi_2 U_{E4(i)} \tag{28}$$

$$MSE_{\min}\left(U_{p_4}\right) = \left(\overline{Y}^2 + \overline{X}^2\eta_1 - \overline{X}\,\overline{Y}\vartheta\eta_1\right) - \frac{U_{A4(i)}U_{E4(i)}^{\ 2} + U_{B4(i)}U_{D4(i)}^{\ 2} - 2U_{C4(i)}U_{D4(i)}U_{E4(i)}}{U_{A4(i)}U_{B4(i)} - U_{C4(i)}^{\ 2}} \tag{29}$$

The expressions of above proof are explained in appendix A.

Fifth proposed estimator

For the formulation of fifth proposed estimator our motivation from Shahzad and Hanif [5] and U_{p_2}, we propose the following estimator.

$$U_{p_5} = \left[\phi_1 \frac{\overline{y_{st}}}{2}\left\{\exp\left(\frac{\overline{X} - \overline{x_{st}}}{\overline{X} + \overline{x_{st}}}\right) + \exp\left(\frac{\overline{x_{st}} - \overline{X}}{\overline{X} + \overline{x_{st}}}\right)\right\} + (1-2\phi_2)\left(\overline{X} - \overline{x_{st}}\right)\right] \exp\left[\frac{a\left(\overline{X} - \overline{x_{st}}\right)}{a\left(\overline{X} - \overline{x_{st}}\right) + 2b}\right] \tag{30}$$

for a=0 and b=1

$$bias\left(U_{p_5}\right) = \phi_1 \overline{Y}\left(1 - \frac{\vartheta\eta_{01}}{2} + \alpha_2\eta_1\right) + (1-2\phi_2)\overline{X}\frac{\vartheta^2\eta_1}{2} - \overline{Y}$$

where,

$$\alpha_5 = \frac{3}{8} + \frac{3\vartheta^2}{8} \text{ and } \vartheta = \frac{a\overline{X}}{a\overline{X} + b}$$

$$MSE\left(U_{p_5}\right) = \left(\overline{Y}^2 + \overline{X}^2\eta_1 - \overline{X}\,\overline{Y}\vartheta\eta_1\right) + \phi_1^2 U_{A5(i)} + \phi_2^2 U_{B5(i)} + 2\phi_1\phi_2 U_{C5(i)} - 2\phi_1 U_{D5(i)} - 2\phi_2 U_{E5(i)} \tag{31}$$

$$MSE_{\min}\left(U_{p_5}\right) = \left(\overline{Y}^2 + \overline{X}^2\eta_1 - \overline{X}\,\overline{Y}\vartheta\eta_1\right) - \frac{U_{A5(i)}U_{E5(i)}^{\ 2} + U_{B5(i)}U_{D5(i)}^{\ 2} - 2U_{C5(i)}U_{D5(i)}U_{E5(i)}}{U_{A5(i)}U_{B5(i)} - U_{C5(i)}^{\ 2}} \tag{32}$$

The expressions of above proof are explained in appendix A.

Sixth proposed estimator

For the formulation of sixth proposed estimator our motivation from Shahzad and Hanif [5] and U_{p_3}, we propose the following estimator

$$U_{p_6}=\left[\phi_1\frac{\overline{y_{st}}}{4}\left(\frac{\overline{X}}{\overline{x_{st}}}+\frac{\overline{x_{st}}}{\overline{X}}\right)\left\{\exp\left(\frac{\overline{X}-\overline{x_{st}}}{\overline{X}+\overline{x_{st}}}\right)+\exp\left(\frac{\overline{x_{st}}-\overline{X}}{\overline{X}+\overline{x_{st}}}\right)\right\}+\left(1-2\phi_2\right)\left(\overline{X}-\overline{x_{st}}\right)\right]$$
$$\exp\left[\frac{a\left(\overline{X}-\overline{x_{st}}\right)}{a\left(\overline{X}-\overline{x_{st}}\right)+2b}\right] \tag{33}$$

for a=0 and b=1

$$bias\left(U_{p_6}\right)=\phi_1\overline{Y}\left(1-\frac{\vartheta\eta_{01}}{2}+\alpha_6\eta_1\right)+\left(1-2\phi_2\right)\overline{X}\,\frac{\vartheta^2\eta_1}{2}-\overline{Y} \tag{34}$$

where,

$$\alpha_6=\frac{7}{8}+\frac{3\vartheta^2}{8}\ \text{ and }\ \vartheta=\frac{a\overline{X}}{a\overline{X}+b}$$

$$MSE\left(U_{p_6}\right)=\left(\overline{Y}^2+\overline{X}^2\eta_1-\overline{X}\,\overline{Y}\vartheta\eta_1\right)+\phi_1^2U_{A6(i)}+\phi_2^2U_{B6(i)}+2\phi_1\phi_2U_{C6(i)}-2\phi_1U_{D6(i)}-2\phi_2U_{E6(i)} \tag{35}$$

$$MSE_{\min}\left(U_{p_6}\right)=\left(\overline{Y}^2+\overline{X}^2\eta_1-\overline{X}\,\overline{Y}\vartheta\eta_1\right)-\frac{U_{A6(i)}U_{E6(i)}{}^2+U_{B6(i)}U_{D6(i)}{}^2-2U_{C6(i)}U_{D6(i)}U_{E6(i)}}{U_{A6(i)}U_{B6(i)}-U_{C6(i)}{}^2} \tag{36}$$

The expressions of above proof are explained in appendix A.

Seventh proposed estimator

For the formulation of seventh proposed estimator our motivation from by adding $\bar{y}_w$ in $\hat{\bar{y}}_{Rao}$, we develop the following class of estimator.

$$\overline{y}_{WR}=\overline{y}_w+\overline{y}_{Rao} \tag{37}$$

$$\overline{y}_{WR}=\left[\frac{\overline{y}}{2}\left(\frac{\overline{X}}{\overline{x}}+\frac{\overline{x}}{\overline{X}}\right)+\phi_1\overline{y}+\phi_2\left(\overline{X}-\overline{x}\right)\right] \tag{38}$$

By replacing $\bar{y}_w$ in $\hat{\bar{y}}_{WR}$, in Bhal and Tuteja [1], we proposed the following estimator.

$$U_{p_7}=\overline{y}_{WR}\exp\left(\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}}\right) \tag{39}$$

$$U_{p_7}=\left[\frac{\overline{y_{st}}}{2}\left(\frac{\overline{X}}{\overline{x_{st}}}+\frac{\overline{x_{st}}}{\overline{X}}\right)+\phi_1\overline{y_{st}}+\phi_2\left(\overline{X}-\overline{x_{st}}\right)\right]\exp\left[\frac{a\left(\overline{X}-\overline{x_{st}}\right)}{a\left(\overline{X}-\overline{x_{st}}\right)+2b}\right] \tag{40}$$

We used transformation of $a=\sum_{h=1}^{L}W_hC_{xh}$, $b=\sum_{h=1}^{L}W_h\beta_{2h}\left(x\right)$

$$bias\left(U_{p_7}\right)=\bar{Y}\left(-\frac{\vartheta\eta_1}{2}+\frac{3\vartheta^2\eta_1}{8}-\frac{\vartheta\eta_{01}}{2}+\frac{\eta_1}{2}\right)+\phi_1\bar{Y}\left(1-\frac{\vartheta\eta_{10}}{2}+\frac{\vartheta\eta_1}{2}+\frac{3\vartheta^2\eta_1}{8}\right)+\phi_2\bar{X}\frac{\vartheta\eta_1}{2} \quad (41)$$

where,

$$\alpha_7=\frac{1}{2}+\frac{3\vartheta^2}{8} \text{ and } \vartheta=\frac{a\bar{X}}{a\bar{X}+b}$$

$$MSE\left(U_{p_7}\right)=\bar{Y}^2\left(\eta_0+\frac{\vartheta^2\eta_1}{4}-\vartheta\eta_{01}\right)+\phi_1^2U_{A7(i)}+\phi_2^2U_{B7(i)}+2\phi_1\phi_2U_{C7(i)}-2\phi_1U_{D7(i)}-2\phi_2U_{E7(i)} \quad (42)$$

$$MSE_{\min}\left(U_{p_7}\right)=\bar{Y}^2\left(\eta_0+\frac{\vartheta^2\eta_1}{4}-\vartheta\eta_{01}\right)-\frac{U_{A7(i)}U_{E7(i)}{}^2+U_{B7(i)}U_{D7(i)}{}^2-2U_{C7(i)}U_{D7(i)}U_{E7(i)}}{U_{A7(i)}U_{B7(i)}-U_{C7(i)}{}^2} \quad (43)$$

The expressions of above proof are explained in appendix A.

Eighth proposed estimator

For the formulation of eighth proposed estimator our motivation comes from, by replacing $\bar{y}_w$ with $\bar{y}_{ew}$, in U_{p_7}, we propose the following estimator.

$$U_{p_8}=\left[\frac{\bar{y}_{st}}{2}\left\{\exp\left(\frac{\bar{X}-\bar{x}_{st}}{\bar{X}+\bar{x}_{st}}\right)+\exp\left(\frac{\bar{x}_{st}-\bar{X}}{\bar{X}+\bar{x}_{st}}\right)\right\}+\phi_1\bar{y}_{st}+\phi_2\left(\bar{X}-\bar{x}_{st}\right)\right]\exp\left[\frac{a\left(\bar{X}-\bar{x}_{st}\right)}{a\left(\bar{X}-\bar{x}_{st}\right)+2b}\right] \quad (44)$$

We used transformation of $a=\sum_{h=1}^{L}W_hC_{xh}$, $b=\sum_{h=1}^{L}W_h\beta_{2h}(x)$

$$bias\left(U_{p_8}\right)=\bar{Y}\left(1-\frac{\vartheta\eta_{01}}{2}+\alpha_8\eta_1\right)+\phi_1\bar{Y}\left(1+\frac{3\vartheta^2\eta_1}{8}-\frac{\vartheta\eta_{01}}{2}\right)+\phi_2\bar{X}\frac{\vartheta^2\eta_1}{2}-\bar{Y} \quad (45)$$

where,

$$\alpha_8=\frac{3}{8}+\frac{3\vartheta^2}{8} \text{ and } \vartheta=\frac{a\bar{X}}{a\bar{X}+b}$$

$$MSE\left(U_{p_8}\right)=\bar{Y}^2\left(\eta_0+\frac{\vartheta^2\eta_1}{4}-\vartheta\eta_{01}\right)+\phi_1^2U_{A8(i)}+\phi_2^2U_{B8(i)}+2\phi_1\phi_2U_{C8(i)}-2\phi_1U_{D8(i)}-2\phi_2U_{E8(i)} \quad (46)$$

$$MSE_{\min}\left(U_{p_8}\right)=\bar{Y}^2\left(\eta_0+\frac{\vartheta^2\eta_1}{4}-\vartheta\eta_{01}\right)-\frac{U_{A8(i)}U_{E8(i)}{}^2+U_{B8(i)}U_{D8(i)}{}^2-2U_{C8(i)}U_{D8(i)}U_{E8(i)}}{U_{A8(i)}U_{B8(i)}-U_{C8(i)}{}^2} \quad (47)$$

The expressions of above proof are explained in appendix A.

Ninth proposed estimator

For the formulation of ninth proposed estimator our motivation from, by repeating substitutions in U_{p_7}, and U_{p_8}, we propose the following estimator.

$$U_{p_9}=\left[\frac{\overline{y_{st}}}{4}\left(\frac{\overline{X}}{\overline{x_{st}}}+\frac{\overline{x_{st}}}{\overline{X}}\right)\left\{\exp\left(\frac{\overline{X}-\overline{x_{st}}}{\overline{X}+\overline{x_{st}}}\right)+\exp\left(\frac{\overline{x_{st}}-\overline{X}}{\overline{X}+\overline{x_{st}}}\right)\right\}+\phi_1\overline{y_{st}}+\phi_2\left(\overline{X}-\overline{x_{st}}\right)\right]$$
$$\exp\left[\frac{a\left(\overline{X}-\overline{x_{st}}\right)}{a\left(\overline{X}-\overline{x_{st}}\right)+2b}\right] \tag{48}$$

We used transformation of $a=\sum_{h=1}^{L}W_hC_{xh}$, $b=\sum_{h=1}^{L}W_h\beta_{2h}(x)$

$$bias\left(U_{p_9}\right)=\overline{Y}\left(1-\frac{\vartheta\eta_{01}}{2}+\alpha_9\eta_1\right)+\phi_1\overline{Y}\left(1+\frac{3\vartheta^2\eta_1}{8}-\frac{\vartheta\eta_{10}}{2}\right)+\phi_2\overline{X}\frac{\vartheta^2\eta_1}{2}-\overline{Y} \tag{49}$$

where,

$\alpha_9=\frac{7}{8}+\frac{3\vartheta^2}{8}$ and $\vartheta=\frac{a\overline{X}}{a\overline{X}+b}$

$$MSE\left(U_{p_9}\right)=\overline{Y}^2\left(\eta_0+\frac{\vartheta^2\eta_1}{4}-\vartheta\eta_{01}\right)+\phi_1^2U_{A9(i)}+\phi_2^2U_{B9(i)}+2\phi_1\phi_2U_{C9(i)}-2\phi_1U_{D9(i)}-2\phi_2U_{E9(i)} \tag{50}$$

$$MSE_{\min}\left(U_{p_9}\right)=\overline{Y}^2\left(\eta_0+\frac{\vartheta^2\eta_1}{4}-\vartheta\eta_{01}\right)-\frac{U_{A9(i)}U_{E9(i)}^2+U_{B9(i)}U_{D9(i)}^2-2U_{C9(i)}U_{D9(i)}U_{E9(i)}}{U_{A9(i)}U_{B9(i)}-U_{C9(i)}^2} \tag{51}$$

The expressions of above proof are explained in appendix A.

3. NUMERICAL INVESTIGATION

For assessing the merits of the proposed class of estimators over existing ones, we investigate numerical results by utilizing the following real data set

Source of data: Murthy [6], p. 228

The total sample size is n=45 and strata are as under

Strata 1= x is less than 100, Strata 2= x is between 100 to 200, Strata 3= x is between 200 to 500, Strata 4= x is greater than 500

X = Data on number of workers,

Y = Output for 80 factories in a region

Table 4. Descriptive Data

Stratum	1st	2nd	3rd	4th	Total
N_h	25.00	23.00	16.00	16.00	80.00
n_h	14.00	13.00	9.00	9.00	45
$\overline{X}_h$	71.00	140.69	362.93	749.50	284.75
$\overline{Y}_h$	3156.64	4766.22	6334.19	7795.31	5182.64
$\beta_{2(x)h}$	1.75	2.19	1.61	1.90	3.53
C_{xh}	0.20	0.19	0.25	0.23	0.94
S_{xh}	14.61	28.03	91.38	174.46	270.49
S_{yh}	740.01	515.69	501.39	653.09	1835.66
S_{xy}	8830.78	11900.60	43903.70	111718.00	454033.30
ρ_h	0.81	0.8231	0.95	0.98	0.91
λ_h	0.03	0.03	0.04	0.04	
W_h	0.31	0.28	0.20	0.20	

Table 5. MSE values of unbiased, ratio, exponential and regression estimators under stratified random sampling

Estimators	MSE
$\bar{y}_{st}$	6093.681
$G_{R(st)}$	5550.412
$G_{B(st)}$	2676.871
$G_{Reg(st)}$	2671.006

Table 6. MSE values of Koyuncu and Kadilar [2] estimators under stratified random sampling

Estimators	MSE	Estimators	MSE
K_1	5549.451	K_6	5550.044
K_2	5550.173	K_7	5549.493
K_3	5549.815	K_8	5552.181
K_4	5584.171	K_9	5556.219
K_5	5549.533		

Table 7. MSE values of Singh and Solanki [3] estimators under stratified random sampling

Estimators	MSE	Estimators	MSE
T_{ss1}	2669.749	T_{ss10}	2669.758
T_{ss2}	2669.751	T_{ss11}	2669.752
T_{ss3}	2669.75	T_{ss12}	2669.787
T_{ss4}	2669.835	T_{ss13}	2670.285
T_{ss5}	2670.225	T_{ss14}	2669.759
T_{ss6}	2670.895	T_{ss15}	2669.76
T_{ss7}	2670.028	T_{ss16}	2670.27
T_{ss8}	2669.758	T_{ss17}	2670.784
T_{ss9}	2669.769		

Table 8. MSE values of Singh and Solanki [4] estimators under stratified random sampling

Estimators	MSE	Estimators	MSE
T_{sk1}	2670.359	T_{sk7}	2670.369
T_{sk2}	2670.36	T_{sk8}	2670.359
T_{sk3}	2670.359	T_{sk9}	2670.405
T_{sk4}	2670.362	T_{sk10}	2670.359
T_{sk5}	2670.613	T_{sk11}	2670.429
T_{sk6}	2670.961	T_{sk12}	2670.359

Table 9. MSE values of proposed estimators under stratified random sampling

Estimators	MSE	Estimators	MSE
U_{p1}	2666.951	U_{p6}	2669.2001
U_{p2}	2668.081	U_{p7}	2667.267
U_{p3}	2662.461	U_{p8}	2668.348
U_{p4}	2669.4532	U_{p9}	2662.918
U_{p5}	2669.3021		

4. CONCLUDING REMARKS

Ratio estimators are biased estimators, which are used in under stratified random sampling. Many researchers conducted case studies to improve ratio estimators using various transformations. Many modifications have been made to impose the population mean in stratified random sampling.

Our present study is also focused on modifying the combination of combined ratio and product estimators to estimate the population mean under stratified sampling scheme. For this purpose, different new estimators were proposed and compared with the existing estimators in stratified random sampling.

Our numerical results of the suggested and existing estimators based on the MSE. All results indicate that the mean square error of the suggested modified estimators is lower than the MSE of existing estimators. We conclude, therefore, that the suggested modified estimators are the better and more efficient estimators as compared to the existing ones.

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Appendix A

The explanation of the expressions used in above proof of proposed estimators.

1st Estimator

Substituting values of $\overline{y_{st}}$ and $\overline{x_{st}}$ in (11)

$$=\left[\phi_1\bar{Y}\left(\frac{1}{2}+\frac{\lambda_0}{2}\right)\left(\left(1+\lambda_1\right)^{-1}+\left(1+\lambda_1\right)\right)+\phi_2\left(-\bar{X}\lambda_1\right)\right]\left(1-\frac{\vartheta\lambda_1}{2}+\frac{3\vartheta^2\lambda_1^{\ 2}}{8}\right) \tag{52}$$

$$U_{p_1}-\bar{Y}=\phi_1\bar{Y}\left(1+\lambda_0-\frac{\vartheta\lambda_1}{2}-\frac{\vartheta\lambda_0\lambda_1}{2}+\alpha_1\lambda_1^{\ 2}\right)-\phi_2\bar{X}\left(\lambda_1-\frac{\vartheta^2\lambda_1^{\ 2}}{2}\right)-\bar{Y} \tag{53}$$

Applying expectation

$$bias\left(U_{p_1}\right)=\phi_1\bar{Y}\left(1-\frac{\vartheta\eta_{01}}{2}+\alpha_1\eta_1\right)+\phi_2\bar{X}\frac{\vartheta^2\eta_1}{2}-\bar{Y} \tag{54}$$

$$\begin{aligned}\left(U_{p_1}-\bar{Y}\right)^2&=\bar{Y}^2+\phi_1^{\ 2}\bar{Y}^2\left(1+\lambda_0^{\ 2}+\frac{\vartheta^2\lambda_1^{\ 2}}{4}-2\vartheta\lambda_0\lambda_1+2\alpha_1\lambda_1^{\ 2}\right)+\phi_2^{\ 2}\bar{X}^2\lambda_1^{\ 2}\\&+2\phi_1\phi_2\bar{Y}\bar{X}\left(\vartheta\lambda_1^{\ 2}-\lambda_0\lambda_1\right)-2\phi_1\bar{Y}^2\left(1-\frac{\vartheta\lambda_0\lambda_1}{2}+\alpha_1\lambda_1^{\ 2}\right)-2\phi_2\frac{\bar{Y}\bar{X}\vartheta\lambda_1^{\ 2}}{2}\end{aligned} \tag{55}$$

Applying expectation

$$MSE\left(U_{p_1}\right)=\bar{Y}^2+\phi_1^2U_{A1(i)}+\phi_2^2U_{B1(i)}+2\phi_1\phi_2U_{C1(i)}-2\phi_1U_{D1(i)}-2\phi_2U_{E1(i)} \tag{54}$$

where

$$U_{A1(i)}=\bar{Y}^2\left(1+\eta_0+\frac{\vartheta^2\eta_1}{4}-2\vartheta\eta_{01}+2\alpha_1\eta_1\right)$$

$$U_{B1(i)}=\bar{X}^2\eta_1$$

$$U_{C1(i)}=\bar{Y}\bar{X}\left(\vartheta\eta_1-\eta_{01}\right)$$

$$U_{D1(i)}=\bar{Y}^2\left(1-\frac{\vartheta\eta_{01}}{2}+\alpha_1\eta_1\right)$$

$$U_{E1(i)}=\frac{\bar{Y}\bar{X}\vartheta\eta_1}{2}$$

Differentiating MSE partially, with respect to $\emptyset_1$ and $\emptyset_2$ and equating to zero, we get the following optimum values of $\emptyset_1$ and $\emptyset_2$

$$\phi_{1(opt)}=\frac{U_{B1(i)}U_{D1(i)}-U_{C1(i)}U_{E1(i)}}{U_{A1(i)}U_{B1(i)}-U_{C1(i)}^{\ 2}},\ \phi_{2(opt)}=\frac{U_{A1(i)}U_{E1(i)}-U_{C1(i)}U_{D1(i)}}{U_{A1(i)}U_{B1(i)}-U_{C1(i)}^{\ 2}}$$

The minimum MSE of U_{p_1}

$$MSE_{\min}\left(U_{p_1}\right)=\overline{Y}^2-\frac{U_{A1(i)}U_{E1(i)}{}^2+U_{B1(i)}U_{D1(i)}{}^2-2U_{C1(i)}U_{D1(i)}U_{E1(i)}}{U_{A1(i)}U_{B1(i)}-U_{C1(i)}{}^2} \tag{57}$$

2nd Estimator

Substituting values of $\overline{y_{st}}$ and $\overline{x_{st}}$ in (17)

$$=\left[\phi_1\overline{Y}\left(\frac{1}{2}+\frac{\lambda_0}{2}\right)\left\{\exp\left(-\lambda_1\right)\left(2+\lambda_1\right)^{-1}+\exp\left(\lambda_1\right)\left(2+\lambda_1\right)^{-1}\right\}+\phi_2\left(-\overline{X}\lambda_1\right)\right]\left(1-\frac{\vartheta\lambda_1}{2}+\frac{3\vartheta^2\lambda_1{}^2}{8}\right) \tag{58}$$

$$U_{p_2}-\overline{Y}=\phi_1\overline{Y}\left(1+\lambda_0-\frac{\vartheta\lambda_1}{2}-\frac{\vartheta\lambda_0\lambda_1}{2}+\alpha_2\lambda_1{}^2\right)+\phi_2\overline{X}\left(\lambda_1-\frac{\vartheta^2\lambda_1{}^2}{2}\right)-\overline{Y} \tag{59}$$

Applying expectation

$$bias\left(U_{p_2}\right)=\phi_1\overline{Y}\left(1-\frac{\vartheta\eta_{01}}{2}+\alpha_2\eta_1\right)-\phi_2\overline{X}\frac{\vartheta^2\eta_1}{2}-\overline{Y} \tag{60}$$

$$\begin{aligned}\left(U_{p_2}-\overline{Y}\right)^2&=\overline{Y}^2+\phi_1{}^2\overline{Y}^2\left(1+\lambda_0{}^2+\frac{\vartheta^2\lambda_1{}^2}{4}-2\vartheta\lambda_0\lambda_1+2\alpha_2\lambda_1{}^2\right)+\phi_2{}^2\overline{X}^2\lambda_1{}^2\\&+2\phi_1\phi_2\overline{Y}\,\overline{X}\left(\vartheta\lambda_1{}^2-\lambda_0\lambda_1\right)-2\phi_1\overline{Y}^2\left(1-\frac{\vartheta\lambda_0\lambda_1}{2}+\alpha_2\lambda_1{}^2\right)-2\phi_2\frac{\overline{Y}\,\overline{X}\vartheta\lambda_1{}^2}{2}\end{aligned} \tag{61}$$

Applying expectation

$$MSE\left(U_{p_2}\right)=\overline{Y}^2+\phi_1{}^2U_{A2(i)}+\phi_2{}^2U_{B2(i)}+2\phi_1\phi_2U_{C2(i)}-2\phi_1U_{D2(i)}-2\phi_2U_{E2(i)} \tag{62}$$

where

$$U_{A2(i)}=\overline{Y}^2\left(1+\eta_0+\frac{\vartheta^2\eta_1}{4}-2\vartheta\eta_{01}+2\alpha_1\eta_1\right)$$

$$U_{B2(i)}=\overline{X}^2\eta_1$$

$$U_{C2(i)}=\overline{Y}\,\overline{X}\left(\vartheta\eta_1-\eta_{01}\right)$$

$$U_{D2(i)}=\overline{Y}^2\left(1-\frac{\vartheta\eta_{01}}{2}+\alpha_1\eta_1\right)$$

$$U_{E2(i)}=\frac{\overline{Y}\,\overline{X}\vartheta\eta_1}{2}$$

Differentiating MSE partially, with respect to $\emptyset_1$ and $\emptyset_2$ and equating to zero, we get the following optimum values of $\emptyset_1$ and $\emptyset_2$

$$\phi_{1(opt)} = \frac{U_{B2(i)}U_{D2(i)} - U_{C2(i)}U_{E2(i)}}{U_{A2(i)}U_{B2(i)} - U_{C2(i)}^2}, \ \phi_{2(opt)} = \frac{U_{A2(i)}U_{E2(i)} - U_{C2(i)}U_{D2(i)}}{U_{A2(i)}U_{B2(i)} - U_{C2(i)}^2}$$

The minimum MSE of U_{p_2}

$$MSE_{\min}\left(U_{p_2}\right) = \bar{Y}^2 - \frac{U_{A2(i)}U_{E2(i)}^2 + U_{B2(i)}U_{D2(i)}^2 - 2U_{C2(i)}U_{D2(i)}U_{E2(i)}}{U_{A2(i)}U_{B2(i)} - U_{C2(i)}^2} \quad (63)$$

3rd Estimator

Substituting values of $\overline{y_{st}}$ and $\overline{x_{st}}$ in (21)

$$= \phi_1\bar{Y}\left(1 + \lambda_0 + \frac{7\lambda_1^2}{8}\right)\left(1 - \frac{\vartheta\lambda_1}{2} + \frac{3\vartheta^2\lambda_1^2}{8}\right) - \phi_2\bar{X}\lambda_1\left(1 - \frac{\vartheta\lambda_1}{2} + \frac{3\vartheta^2\lambda_1^2}{8}\right) \quad (64)$$

$$U_{p_3} - \bar{Y} = \phi_1\bar{Y}\left(1 + \lambda_0 - \frac{\vartheta\lambda_1}{2} - \frac{\vartheta\lambda_0\lambda_1}{2} + \alpha_3\lambda_1^2\right) - \phi_2\bar{X}\left(\lambda_1 - \frac{\vartheta^2\lambda_1^2}{2}\right) - \bar{Y} \quad (65)$$

Applying expectation

$$bias\left(U_{p_3}\right) = \phi_1\bar{Y}\left(1 - \frac{\vartheta\eta_{01}}{2} + \alpha_3\eta_1\right) + \phi_2\bar{X}\frac{\vartheta^2\eta_1}{2} - \bar{Y} \quad (66)$$

$$\left(U_{p_3} - \bar{Y}\right)^2 = \bar{Y}^2 + \phi_1^2\bar{Y}^2\left(1 + \lambda_0^2 + \frac{\vartheta^2\lambda_1^2}{4} - 2\vartheta\lambda_0\lambda_1 + 2\alpha_3\lambda_1^2\right) + \phi_2^2\bar{X}^2\lambda_1^2$$
$$+2\phi_1\phi_2\bar{Y}\bar{X}\left(\vartheta\lambda_1^2 - \lambda_0\lambda_1\right) - 2\phi_1\bar{Y}^2\left(1 - \frac{\vartheta\lambda_0\lambda_1}{2} + \alpha_3\lambda_1^2\right) - 2\phi_2\frac{\bar{Y}\bar{X}\vartheta\lambda_1^2}{2} \quad (67)$$

Applying expectation

$$MSE\left(U_{p_3}\right) = \bar{Y}^2 + \phi_1^2U_{A3(i)} + \phi_2^2U_{B3(i)} + 2\phi_1\phi_2U_{C3(i)} - 2\phi_1U_{D3(i)} - 2\phi_2U_{E3(i)} \quad (68)$$

where

$$U_{A3(i)} = \bar{Y}^2\left(1 + \eta_0 + \frac{\vartheta^2\eta_1}{4} - 2\vartheta\eta_{01} + 2\alpha_1\eta_1\right)$$

$$U_{B3(i)} = \bar{X}\eta_1$$

$$U_{C3(i)} = \bar{Y}\bar{X}\left(\vartheta\eta_1 - \eta_{01}\right)$$

$$U_{D3(i)} = \bar{Y}^2\left(1 - \frac{\vartheta\eta_{01}}{2} + \alpha_1\eta_1\right)$$

$$U_{E3(i)} = \frac{\bar{Y}\bar{X}\vartheta\eta_1}{2}$$

Differentiating MSE partially, with respect to $\emptyset_1$and $\emptyset_2$ and equating to zero, we get the following optimum values of $\emptyset_1$and $\emptyset_2$

$$\phi_{1(opt)}=\frac{U_{B3(i)}U_{D3(i)}-U_{C3(i)}U_{E3(i)}}{U_{A3(i)}U_{B3(i)}-U_{C3(i)}{}^{2}},\ \phi_{2(opt)}=\frac{U_{A3(i)}U_{E3(i)}-U_{C3(i)}U_{D3(i)}}{U_{A3(i)}U_{B3(i)}-U_{C3(i)}{}^{2}}$$

The minimum MSE of U_{p_3}

$$MSE_{\min}\left(U_{p_3}\right)=\overline{Y}^{2}-\frac{U_{A3(i)}U_{E3(i)}{}^{2}+U_{B3(i)}U_{D3(i)}{}^{2}-2U_{C3(i)}U_{D3(i)}U_{E3(i)}}{U_{A3(i)}U_{B3(i)}-U_{C3(i)}{}^{2}} \tag{69}$$

4th Estimator

Substituting values of $\overline{y_{st}}$ and $\overline{x_{st}}$ in (25)

$$=\left[\phi_1\overline{Y}\left(\frac{1}{2}+\frac{\lambda_0}{2}\right)\left((1+\lambda_1)^{-1}+(1+\lambda_1)\right)+(1-2\phi_2)\left(-\overline{X}\lambda_1\right)\right]\left(1-\frac{\vartheta\lambda_1}{2}+\frac{3\vartheta^2\lambda_1{}^2}{8}\right) \tag{70}$$

$$U_{p_4}-\overline{Y}=\phi_1\overline{Y}\left(1+\lambda_0-\frac{\vartheta\lambda_1}{2}-\frac{\vartheta\lambda_0\lambda_1}{2}+\alpha_4\lambda_1{}^2\right)-(1-2\phi_2)\,\overline{X}\left(\lambda_1-\frac{\vartheta^2\lambda_1{}^2}{2}\right)-\overline{Y} \tag{71}$$

Applying expectation

$$bias\left(U_{p_4}\right)=\phi_1\overline{Y}\left(1-\frac{\vartheta\eta_{01}}{2}+\alpha_4\eta_1\right)+(1-2\phi_2)\,\overline{X}\,\frac{\vartheta^2\eta_1}{2}-\overline{Y} \tag{72}$$

$$\begin{aligned}\left(U_{p_4}-\overline{Y}\right)^2&=\left(\overline{Y}^2+\overline{X}^2\lambda_1{}^2-\overline{X}\overline{Y}\vartheta\lambda_1{}^2\right)+\phi_1{}^2\overline{Y}^2\left(1+\lambda_0{}^2+\frac{\vartheta^2\lambda_1{}^2}{4}-2\vartheta\lambda_0\lambda_1+2\alpha_4\lambda_1{}^2\right)+\phi_2{}^2 4\overline{X}^2\lambda_1{}^2\\&+2\phi_1\phi_2\overline{Y}\overline{X}\left(2\lambda_0\lambda_1-2\vartheta\lambda_1{}^2\right)-2\phi_1\overline{Y}\left(\overline{Y}-\frac{\overline{Y}\vartheta\lambda_0\lambda_1}{2}+\overline{Y}\alpha_4\lambda_1{}^2+\overline{X}\lambda_0\lambda_1-\overline{X}\vartheta\lambda_1{}^2\right)-2\phi_2\overline{Y}\overline{X}\vartheta\lambda_1{}^2\end{aligned} \tag{73}$$

Applying expectation

$$\begin{aligned}MSE\left(U_{p_4}\right)&=\left(\overline{Y}^2+\overline{X}^2\eta_1-\overline{X}\overline{Y}\vartheta\eta_1\right)+\phi_1^2U_{A4(i)}+\phi_2{}^2U_{B4(i)}+2\phi_1\phi_2U_{C4(i)}\\&-2\phi_1U_{D4(i)}-2\phi_2U_{E4(i)}\end{aligned} \tag{74}$$

where

$$U_{A4(i)}=\overline{Y}^2\left(1+\eta_0+\frac{\vartheta^2\eta_1}{4}-2\vartheta\eta_{01}+2\alpha_4\eta_1\right)$$

$$U_{B4(i)}=4\overline{X}^2\eta_1$$

$$U_{C4(i)}=\overline{Y}\overline{X}\left(2\eta_{01}-2\vartheta\eta_1\right)$$

$$U_{D4(i)}=\overline{Y}\left(\overline{Y}-\frac{\overline{Y}\vartheta\eta_{01}}{2}+\overline{Y}\alpha_4\eta_1+\overline{X}\eta_{01}-\overline{X}\vartheta\eta_1\right)$$

$$U_{E4(i)}=\overline{Y}\overline{X}\vartheta\eta_1$$

Differentiating MSE partially, with respect to $\emptyset_1$and $\emptyset_2$ and equating to zero, we get the following optimum values of $\emptyset_1$and $\emptyset_2$

$$\phi_{1(opt)} = \frac{U_{B4(i)}U_{D4(i)} - U_{C4(i)}U_{E4(i)}}{U_{A4(i)}U_{B4(i)} - U_{C4(i)}^{\ 2}},\ \phi_{2(opt)} = \frac{U_{A4(i)}U_{E4(i)} - U_{C4(i)}U_{D4(i)}}{U_{A4(i)}U_{B4(i)} - U_{C4(i)}^{\ 2}}$$

the minimum MSE of U_{p_4}

$$MSE_{\min}\left(U_{p_4}\right) = \left(\overline{Y}^2 + \overline{X}^2\eta_1 - \overline{X}\overline{Y}\vartheta\eta_1\right) - \frac{U_{A4(i)}U_{E4(i)}^{\ 2} + U_{B4(i)}U_{D4(i)}^{\ 2} - 2U_{C4(i)}U_{D4(i)}U_{E4(i)}}{U_{A4(i)}U_{B4(i)} - U_{C4(i)}^{\ 2}} \tag{75}$$

5th Estimator

Substituting values of $\overline{y_{st}}$ and $\overline{x_{st}}$ in (29)

$$= \phi_1\overline{Y}\left(1 + \lambda_0 + \frac{3\lambda_1^{\ 2}}{8}\right)\left(1 - \frac{\vartheta\lambda_1}{2} + \frac{3\vartheta^2\lambda_1^{\ 2}}{8}\right) - \left(1 - 2\phi_2\right)\overline{X}\lambda_1\left(1 - \frac{\vartheta\lambda_1}{2} + \frac{3\vartheta^2\lambda_1^{\ 2}}{8}\right) \tag{76}$$

$$U_{p_5} - \overline{Y} = \phi_1\overline{Y}\left(1 + \lambda_0 - \frac{\vartheta\lambda_1}{2} - \frac{\vartheta\lambda_0\lambda_1}{2} + \alpha_2\lambda_1^{\ 2}\right) - \left(1 - 2\phi_2\right)\overline{X}\left(\lambda_1 - \frac{\vartheta^2\lambda_1^{\ 2}}{2}\right) - \overline{Y} \tag{77}$$

Applying expectation

$$bias\left(U_{p_5}\right) = \phi_1\overline{Y}\left(1 - \frac{\vartheta\eta_{01}}{2} + \alpha_2\eta_1\right) + \left(1 - 2\phi_2\right)\overline{X}\,\frac{\vartheta^2\eta_1}{2} - \overline{Y} \tag{78}$$

$$\begin{aligned}\left(U_{p_5} - \overline{Y}\right)^2 &= \left(\overline{Y}^2 + \overline{X}^2\lambda_1^{\ 2} - \overline{X}\overline{Y}\vartheta\lambda_1^{\ 2}\right) + \phi_1^{\ 2}\overline{Y}^2\left(1 + \lambda_0^{\ 2} + \frac{\vartheta^2\lambda_1^{\ 2}}{4} - 2\vartheta\lambda_0\lambda_1 + 2\alpha_5\lambda_1^{\ 2}\right) + \phi_2^{\ 2}4\overline{X}^2\lambda_1^{\ 2} \\ &+ 2\phi_1\phi_2\overline{Y}\overline{X}\left(2\lambda_0\lambda_1 - 2\vartheta\lambda_1^{\ 2}\right) - 2\phi_1\overline{Y}\left(\overline{Y} - \frac{\overline{Y}\vartheta\lambda_0\lambda_1}{2} + \overline{Y}\alpha_5\lambda_1^{\ 2} + \overline{X}\lambda_0\lambda_1 - \overline{X}\vartheta\lambda_1^{\ 2}\right) - 2\phi_2\overline{X}\left(\overline{Y}\vartheta\lambda_1^{\ 2} - 2\overline{X}\lambda_1^{\ 2}\right.\end{aligned} \tag{79}$$

Applying expectation

$$MSE\left(U_{p_5}\right) = \left(\overline{Y}^2 + \overline{X}^2\eta_1 - \overline{X}\overline{Y}\vartheta\eta_1\right) + \phi_1^2 U_{A5(i)} + \phi_2^2 U_{B5(i)} + 2\phi_1\phi_2 U_{C5(i)} - 2\phi_1 U_{D5(i)} - 2\phi_2 U_{E5(i)} \tag{80}$$

where

$$U_{A5(i)} = \overline{Y}^2\left(1 + \eta_0 + \frac{\vartheta^2\eta_1}{4} - 2\vartheta\eta_{01} + 2\alpha_5\eta_1\right)$$

$$U_{B5(i)} = 4\overline{X}^2\eta_1$$

$$U_{C5(i)} = \overline{Y}\overline{X}\left(2\eta_{01} - 2\vartheta\eta_1\right)$$

$$U_{D5(i)} = \overline{Y}\left(\overline{Y} - \frac{\overline{Y}\vartheta\eta_{01}}{2} + \overline{Y}\alpha_5\eta_1 + \overline{X}\eta_{01} - \overline{X}\vartheta\eta_1\right)$$

$$U_{E5(i)} = \overline{X}\left(\overline{Y}\vartheta\eta_1 - 2\overline{X}\eta_1\right)$$

Differentiating MSE partially, with respect to $\emptyset_1$ and $\emptyset_2$ and equating to zero, we get the following optimum values of $\emptyset_1$ and $\emptyset_2$

$$\phi_{1(opt)} = \frac{U_{B5(i)}U_{D5(i)} - U_{C5(i)}U_{E5(i)}}{U_{A5(i)}U_{B5(i)} - U_{C5(i)}{}^2}, \quad \phi_{2(opt)} = \frac{U_{A5(i)}U_{E5(i)} - U_{C5(i)}U_{D5(i)}}{U_{A5(i)}U_{B5(i)} - U_{C5(i)}{}^2}$$

The minimum MSE of U_{p_5}

$$MSE_{\min}\left(U_{p_5}\right) = \left(\overline{Y}^2 + \overline{X}^2\eta_1 - \overline{X}\,\overline{Y}\vartheta\eta_1\right) - \frac{U_{A5(i)}U_{E5(i)}{}^2 + U_{B5(i)}U_{D5(i)}{}^2 - 2U_{C5(i)}U_{D5(i)}U_{E5(i)}}{U_{A5(i)}U_{B5(i)} - U_{C5(i)}{}^2} \tag{81}$$

6th Estimator

Substituting values of $\overline{y_{st}}$ and $\overline{x_{st}}$ in (32)

$$= \left[\phi_1\overline{Y}\left(\frac{1}{4} + \frac{\lambda_0}{4}\right)\left(2 + \lambda_1^2\right)\left\{\exp\left(\frac{-\lambda_1}{2}\right)\left(1 - \frac{\lambda_1}{2} + \frac{\lambda_1^2}{4}\right) + \exp\left(\frac{\lambda_1}{2}\right)\left(1 - \frac{\lambda_1}{2} + \frac{\lambda_1^2}{4}\right)\right\} + \left(1 - 2\phi_2\right)\left(-\overline{X}\lambda_1\right)\right.$$
$$\left(1 - \frac{\vartheta\lambda_1}{2} + \frac{3\vartheta^2\lambda_1^2}{8}\right) \tag{82}$$

$$U_{p_6} - \overline{Y} = \phi_1\overline{Y}\left(1 + \lambda_0 - \frac{\vartheta\lambda_1}{2} - \frac{\vartheta\lambda_0\lambda_1}{2} + \alpha_6\lambda_1^2\right) - \left(1 - 2\phi_2\right)\overline{X}\left(\lambda_1 - \frac{\vartheta^2\lambda_1^2}{2}\right) - \overline{Y} \tag{83}$$

Taking expectation

$$bias\left(U_{p_6}\right) = \phi_1\overline{Y}\left(1 - \frac{\vartheta\eta_{01}}{2} + \alpha_6\eta_1\right) + \left(1 - 2\phi_2\right)\overline{X}\,\frac{\vartheta^2\eta_1}{2} - \overline{Y} \tag{84}$$

$$\left(U_{p_6} - \overline{Y}\right)^2 = \left(\overline{Y}^2 + \overline{X}^2\lambda_1^2 - \overline{X}\,\overline{Y}\vartheta\lambda_1^2\right) + \phi_1^2\overline{Y}^2\left(1 + \lambda_0^2 + \frac{\vartheta^2\lambda_1^2}{4} - 2\vartheta\lambda_0\lambda_1 + 2\alpha_6\lambda_1^2\right) + \phi_2^2 4\overline{X}^2\lambda$$
$$+2\phi_1\phi_2\overline{Y}\,\overline{X}\left(2\lambda_0\lambda_1 - 2\vartheta\lambda_1^2\right) - 2\phi_1\overline{Y}\left(\overline{Y} - \frac{\overline{Y}\vartheta\lambda_0\lambda_1}{2} + \overline{Y}\alpha_6\lambda_1^2 + \overline{X}\lambda_0\lambda_1 - \overline{X}\vartheta\lambda_1^2\right) \tag{85}$$
$$-2\phi_2\overline{X}\left(\overline{Y}\vartheta\lambda_1^2 - 2\overline{X}\lambda_1^2\right)$$

Taking expectation

$$MSE\left(U_{p_6}\right) = \left(\overline{Y}^2 + \overline{X}^2\eta_1 - \overline{X}\,\overline{Y}\vartheta\eta_1\right) + \phi_1^2U_{A6(i)} + \phi_2^2U_{B6(i)} + 2\phi_1\phi_2U_{C6(i)} - 2\phi_1U_{D6(i)} - 2\phi_2U_{E6(i)} \tag{86}$$

where

$$U_{A6(i)} = \bar{Y}^2\left(1+\eta_0+\frac{\vartheta^2\eta_1}{4}-2\vartheta\eta_{01}+2\alpha_6\eta_1\right)$$

$$U_{B6(i)} = 4\bar{X}^2\eta_1$$

$$U_{C6(i)} = \bar{Y}\bar{X}\left(2\eta_{01}-2\vartheta\eta_1\right)$$

$$U_{D6(i)} = \bar{Y}\left(\bar{Y}-\frac{\bar{Y}\vartheta\eta_{01}}{2}+\bar{Y}\alpha_6\eta_1+\bar{X}\eta_{01}-\bar{X}\vartheta\eta_1\right)$$

$$U_{E6(i)} = \bar{X}\left(\bar{Y}\vartheta\eta_1-2\bar{X}\eta_1\right)$$

Differentiating MSE partially, with respect to $\emptyset_1$ and $\emptyset_2$ and equating to zero, we get the following optimum values of $\emptyset_1$ and $\emptyset_2$

$$\phi_{1(opt)} = \frac{U_{B6(i)}U_{D6(i)}-U_{C6(i)}U_{E6(i)}}{U_{A6(i)}U_{B6(i)}-U_{C6(i)}^2},\ \phi_{2(opt)} = \frac{U_{A6(i)}U_{E6(i)}-U_{C6(i)}U_{D6(i)}}{U_{A6(i)}U_{B6(i)}-U_{C6(i)}^2}$$

the minimum MSE of U_{p_6}

$$MSE_{\min}\left(U_{p_6}\right) = \left(\bar{Y}^2+\bar{X}^2\eta_1-\bar{X}\bar{Y}\vartheta\eta_1\right)-\frac{U_{A6(i)}U_{E6(i)}^2+U_{B6(i)}U_{D6(i)}^2-2U_{C6(i)}U_{D6(i)}U_{E6(i)}}{U_{A6(i)}U_{B6(i)}-U_{C6(i)}^2} \tag{87}$$

7th Estimator

Substituting values of $\overline{y_{st}}$ and $\overline{x_{st}}$ in (39)

$$= \left[\bar{Y}\left(\frac{1}{2}+\frac{\lambda_0}{2}\right)\left(\left(1+\lambda_1\right)^{-1}+\left(1+\lambda_1\right)\right)+\phi_1\bar{Y}\left(1+\lambda_0\right)+\phi_2\left(-\bar{X}\lambda_1\right)\right]\left(1-\frac{\vartheta\lambda_1}{2}+\frac{3\vartheta^2\lambda_1^2}{8}\right) \tag{88}$$

$$\begin{aligned} U_{p_7}-\bar{Y} = {} & \bar{Y}\left(\lambda_0-\frac{\vartheta\lambda_1}{2}+\frac{3\vartheta^2\lambda_1^2}{8}-\frac{\vartheta\lambda_0\lambda_1}{2}+\frac{\lambda_1^2}{2}\right)+\phi_1\bar{Y}\left(1+\lambda_0-\frac{\vartheta\lambda_1}{2}-\frac{\vartheta\lambda_1\lambda_0}{2}+\frac{\vartheta\lambda_1^2}{2}+\frac{3\vartheta^2\lambda_1^2}{8}\right) \\ & -\phi_2\bar{X}\lambda_1\left(1-\frac{\vartheta\lambda_1}{2}+\frac{3\vartheta^2\lambda_1^2}{8}\right) \end{aligned} \tag{89}$$

Applying expectation

$$bias\left(U_{p_7}\right) = \bar{Y}\left(-\frac{\vartheta\eta_1}{2}+\frac{3\vartheta^2\eta_1}{8}-\frac{\vartheta\eta_{01}}{2}+\frac{\eta_1}{2}\right)+\phi_1\bar{Y}\left(1-\frac{\vartheta\eta_{10}}{2}+\frac{\vartheta\eta_1}{2}+\frac{3\vartheta^2\eta_1}{8}\right)+\phi_2\bar{X}\frac{\vartheta\eta_1}{2} \tag{90}$$

$$\begin{aligned} \left(U_{p_7}-\bar{Y}\right)^2 = {} & \bar{Y}^2\left(\lambda_0^2+\frac{\vartheta^2\lambda_1^2}{4}-\vartheta\lambda_0\lambda_1\right)+\phi_1^2\bar{Y}^2\left(1+\lambda_0^2+\frac{\vartheta^2\lambda_1^2}{4}-2\vartheta\lambda_0\lambda_1+\frac{3}{4}\vartheta^2\lambda_1^2\right)+\phi_2^2\bar{X}^2\lambda_1^2 \\ & +2\phi_1\phi_2\bar{Y}\bar{X}\left(\vartheta\lambda_1^2-\lambda_0\lambda_1\right)-2\phi_1\bar{Y}^2\left(\lambda_0^2-\frac{\vartheta\lambda_0\lambda_1}{2}-\frac{\vartheta\lambda_0\lambda_1}{2}+\frac{\vartheta^2\lambda_1^2}{4}-\frac{\vartheta\lambda_0\lambda_1}{2}+\alpha_7\lambda_1^2\right)-2\phi_2\bar{X}\bar{Y}\left(\lambda_0\lambda_1-\frac{\vartheta\lambda_1^2}{2}\right) \end{aligned} \tag{91}$$

Applying expectation

$$MSE\left(U_{p_7}\right)=\overline{Y}^2\left(\eta_0+\frac{\vartheta^2\eta_1}{4}-\vartheta\eta_{01}\right)+\phi_1^2U_{A7(i)}+\phi_2^{\,2}U_{B7(i)}+2\phi_1\phi_2U_{C7(i)}-2\phi_1U_{D7(i)}-2\phi_2U_{E7(i)} \quad (92)$$

where

$$U_{A7(i)}=\overline{Y}^2\left(1+\eta_0+\frac{\vartheta^2\eta_1}{4}-2\vartheta\eta_{01}+\frac{3}{4}\vartheta^2\eta_1\right)$$

$$U_{B7(i)}=\overline{X}^2\eta_1$$

$$U_{C7(i)}=\overline{Y}\,\overline{X}\left(\vartheta\eta_1-\eta_{01}\right)$$

$$U_{D7(i)}=\overline{Y}^2\left(\eta_0-\frac{\vartheta\eta_{01}}{2}-\frac{\vartheta\eta_{01}}{2}+\frac{\vartheta^2\eta_1}{4}-\frac{\vartheta\eta_{01}}{2}+\alpha_7\eta_1\right)$$

$$U_{E7(i)}=\overline{X}\,\overline{Y}\left(\eta_{01}-\frac{\vartheta\eta_1}{2}\right)$$

Differentiating MSE partially, with respect to $\emptyset_1$ and $\emptyset_2$ and equating to zero, we get the following optimum values of $\emptyset_1$ and $\emptyset_2$

$$\phi_{1(opt)}=\frac{U_{B7(i)}U_{D7(i)}-U_{C7(i)}U_{E7(i)}}{U_{A7(i)}U_{B7(i)}-U_{C7(i)}{}^2},\ \phi_{2(opt)}=\frac{U_{A7(i)}U_{E7(i)}-U_{C7(i)}U_{D7(i)}}{U_{A7(i)}U_{B7(i)}-U_{C7(i)}{}^2}$$

The minimum MSE of U_{p_7}

$$MSE_{\min}\left(U_{p_7}\right)=\overline{Y}^2\left(\eta_0+\frac{\vartheta^2\eta_1}{4}-\vartheta\eta_{01}\right)-\frac{U_{A7(i)}U_{E7(i)}{}^2+U_{B7(i)}U_{D7(i)}{}^2-2U_{C7(i)}U_{D7(i)}U_{E7(i)}}{U_{A7(i)}U_{B7(i)}-U_{C7(i)}{}^2} \quad (93)$$

8th Estimator

Substituting values of $\overline{y_{st}}$ and $\overline{x_{st}}$ in (43)

$$=\left[\phi_1\overline{Y}\left(\frac{1}{2}+\frac{\lambda_0}{2}\right)\left\{\exp\left(-\lambda_1\right)\left(2+\lambda_1\right)^{-1}+\exp\left(\lambda_1\right)\left(2+\lambda_1\right)^{-1}\right\}+\phi_1\overline{Y}\left(1+\lambda_0\right)+\phi_2\left(-\overline{X}\lambda_1\right)\right]$$
$$\left(1-\frac{\vartheta\lambda_1}{2}+\frac{3\vartheta^2\lambda_1{}^2}{8}\right) \quad (94)$$

$$U_{p_8}-\overline{Y}=\overline{Y}\left(1+\lambda_0-\frac{\vartheta\lambda_1}{2}-\frac{\vartheta\lambda_0\lambda_1}{2}+\alpha_8\lambda_1{}^2\right)+\phi_1\overline{Y}\left(1-\frac{\vartheta\lambda_1}{2}+\frac{3\vartheta^2\lambda_1{}^2}{8}+\lambda_0-\frac{\vartheta\lambda_1\lambda_0}{2}\right)$$
$$-\phi_2\overline{X}\left(\lambda_1-\frac{\vartheta^2\lambda_1{}^2}{2}\right)-\overline{Y} \quad (95)$$

Applying expectation

$$bias\left(U_{p_8}\right)=\overline{Y}\left(1-\frac{\vartheta\eta_{01}}{2}+\alpha_8\eta_1\right)+\phi_1\overline{Y}\left(1+\frac{3\vartheta^2\eta_1}{8}-\frac{\vartheta\eta_{01}}{2}\right)+\phi_2\overline{X}\frac{\vartheta^2\eta_1}{2}-\overline{Y} \tag{96}$$

$$\begin{aligned}\left(U_{p_8}-\overline{Y}\right)^2&=\overline{Y}^2\left(\lambda_0^2+\frac{\vartheta^2\lambda_1^2}{4}-\vartheta\lambda_0\lambda_1\right)+\phi_1^2\overline{Y}^2\left(1+\lambda_0^2+\frac{\vartheta^2\lambda_1^2}{4}-2\vartheta\lambda_0\lambda_1+\frac{3}{4}\vartheta^2\lambda_1^2\right)+\phi_2^2\overline{X}^2\lambda_1^2\\&+2\phi_1\phi_2\overline{Y}\,\overline{X}\left(\vartheta\lambda_1^2-\lambda_0\lambda_1\right)-2\phi_1\overline{Y}^2\left(\lambda_0^2-\frac{\vartheta\lambda_0\lambda_1}{2}-\frac{\vartheta\lambda_0\lambda_1}{2}+\frac{\vartheta^2\lambda_1^2}{4}-\frac{\vartheta\lambda_0\lambda_1}{2}+\alpha_8\lambda_1^2\right)-2\phi_2\overline{X}\,\overline{Y}\left(\lambda_0\lambda_1-\frac{\vartheta\lambda_1^2}{2}\right)\end{aligned} \tag{97}$$

Applying expectation

$$MSE\left(U_{p_8}\right)=\overline{Y}^2\left(\eta_0+\frac{\vartheta^2\eta_1}{4}-\vartheta\eta_{01}\right)+\phi_1^2U_{A8(i)}+\phi_2^2U_{B8(i)}+2\phi_1\phi_2U_{C8(i)}-2\phi_1U_{D8(i)}-2\phi_2U_{E8(i)} \tag{98}$$

where

$$U_{A8(i)}=\overline{Y}^2\left(1+\eta_0+\frac{\vartheta^2\eta_1}{4}-2\vartheta\eta_{01}+\frac{3}{4}\vartheta^2\eta_1\right)$$

$$U_{B8(i)}=\overline{X}^2\eta_1$$

$$U_{C8(i)}=\overline{Y}\,\overline{X}\left(\vartheta\eta_1-\eta_{01}\right)$$

$$U_{D8(i)}=\overline{Y}^2\left(\eta_0-\frac{\vartheta\eta_{01}}{2}-\frac{\vartheta\eta_{01}}{2}+\frac{\vartheta^2\eta_1}{4}-\frac{\vartheta\eta_{01}}{2}+\alpha_8\eta_1\right)$$

$$U_{E8(i)}=\overline{X}\,\overline{Y}\left(\eta_{01}-\frac{\vartheta\eta_1}{2}\right)$$

Differentiating MSE partially, with respect to $\emptyset_1$and $\emptyset_2$ and equating to zero, we get the following optimum values of $\emptyset_1$and $\emptyset_2$

$$\phi_{1(opt)}=\frac{U_{B8(i)}U_{D8(i)}-U_{C8(i)}U_{E8(i)}}{U_{A8(i)}U_{B8(i)}-U_{C8(i)}^2},\ \phi_{2(opt)}=\frac{U_{A8(i)}U_{E8(i)}-U_{C8(i)}U_{D8(i)}}{U_{A8(i)}U_{B8(i)}-U_{C8(i)}^2}$$

The minimum MSE of U_{p_8}

$$MSE_{\min}\left(U_{p_8}\right)=\overline{Y}^2\left(\eta_0+\frac{\vartheta^2\eta_1}{4}-\vartheta\eta_{01}\right)-\frac{U_{A8(i)}U_{E8(i)}^2+U_{B8(i)}U_{D8(i)}^2-2U_{C8(i)}U_{D8(i)}U_{E8(i)}}{U_{A8(i)}U_{B8(i)}-U_{C8(i)}^2} \tag{99}$$

9th Estimator

Substituting values of $\overline{y_{st}}$ and $\overline{x_{st}}$ in (47)

$$\begin{aligned}&=\left[\overline{Y}\left(\frac{1}{4}+\frac{\lambda_0}{4}\right)\left(2+\lambda_1^2\right)\left\{\exp\left(\frac{-\lambda_1}{2}\right)\left(1-\frac{\lambda_1}{2}+\frac{\lambda_1^2}{4}\right)+\exp\left(\frac{\lambda_1}{2}\right)\left(1-\frac{\lambda_1}{2}+\frac{\lambda_1^2}{4}\right)\right\}+\phi_1\overline{Y}\left(1+\lambda_0\right)+\phi_2\left(-\overline{X}\lambda_1\right)\right]\\&\left(1-\frac{\vartheta\lambda_1}{2}+\frac{3\vartheta^2\lambda_1^2}{8}\right)\end{aligned} \tag{100}$$

$$U_{p_9}-\overline{Y}=\overline{Y}\left(1+\lambda_0-\frac{\vartheta\lambda_1}{2}-\frac{\vartheta\lambda_0\lambda_1}{2}+\alpha_9{\lambda_1}^2\right)+\phi_1\overline{Y}\left(1-\frac{\vartheta\lambda_1}{2}+\frac{3\vartheta^2{\lambda_1}^2}{8}+\lambda_0-\frac{\vartheta\lambda_1\lambda_0}{2}\right)$$
$$-\phi_2\overline{X}\left(\lambda_1-\frac{\vartheta^2{\lambda_1}^2}{2}\right)-\overline{Y} \tag{101}$$

Applying expectation

$$bias\left(U_{p_9}\right)=\overline{Y}\left(1-\frac{\vartheta\eta_{01}}{2}+\alpha_9\eta_1\right)+\phi_1\overline{Y}\left(1+\frac{3\vartheta^2\eta_1}{8}-\frac{\vartheta\eta_{10}}{2}\right)+\phi_2\overline{X}\,\frac{\vartheta^2\eta_1}{2}-\overline{Y} \tag{102}$$

$$\left(U_{p_9}-\overline{Y}\right)^2=\overline{Y}^2\left({\lambda_0}^2+\frac{\vartheta^2{\lambda_1}^2}{4}-\vartheta\lambda_0\lambda_1\right)+{\phi_1}^2\overline{Y}^2\left(1+{\lambda_0}^2+\frac{\vartheta^2{\lambda_1}^2}{4}-2\vartheta\lambda_0\lambda_1+\frac{3}{4}\vartheta^2{\lambda_1}^2\right)+{\phi_2}^2\overline{X}^2{\lambda_1}^2$$
$$+2\phi_1\phi_2\overline{Y}\,\overline{X}\left(\vartheta{\lambda_1}^2-\lambda_0\lambda_1\right)-2\phi_1\overline{Y}^2\left({\lambda_0}^2-\frac{\vartheta\lambda_0\lambda_1}{2}-\frac{\vartheta\lambda_0\lambda_1}{2}+\frac{\vartheta^2{\lambda_1}^2}{4}-\frac{\vartheta\lambda_0\lambda_1}{2}+\alpha_9{\lambda_1}^2\right)-2\phi_2\overline{X}\,\overline{Y}\left(\lambda_0\lambda_1-\frac{\vartheta{\lambda_1}^2}{2}\right) \tag{103}$$

Applying expectation

$$MSE\left(U_{p_9}\right)=\overline{Y}^2\left(\eta_0+\frac{\vartheta^2\eta_1}{4}-\vartheta\eta_{01}\right)+{\phi_1}^2U_{A9(i)}+{\phi_2}^2U_{B9(i)}+2\phi_1\phi_2U_{C9(i)}-2\phi_1U_{D9(i)}$$
$$-2\phi_2U_{E9(i)} \tag{104}$$

where

$$U_{A9(i)}=\overline{Y}^2\left(1+\eta_0+\frac{\vartheta^2\eta_1}{4}-2\vartheta\eta_{01}+\frac{3}{4}\vartheta^2\eta_1\right)$$
$$U_{B9(i)}=\overline{X}^2\eta_1$$
$$U_{C9(i)}=\overline{Y}\,\overline{X}\left(\vartheta\eta_1-\eta_{01}\right)$$
$$U_{D9(i)}=\overline{Y}^2\left(\eta_0-\frac{\vartheta\eta_{01}}{2}-\frac{\vartheta\eta_{01}}{2}+\frac{\vartheta^2\eta_1}{4}-\frac{\vartheta\eta_{01}}{2}+\alpha_9\eta_1\right)$$
$$U_{E9(i)}=\overline{X}\,\overline{Y}\left(\eta_{01}-\frac{\vartheta\eta_1}{2}\right)$$

Differentiating MSE partially, with respect to $\emptyset_1$ and $\emptyset_2$ and equating to zero, we get the following optimum values of $\emptyset_1$ and $\emptyset_2$

$$\phi_{1(opt)}=\frac{U_{B9(i)}U_{D9(i)}-U_{C9(i)}U_{E9(i)}}{U_{A9(i)}U_{B9(i)}-{U_{C9(i)}}^2},\ \phi_{2(opt)}=\frac{U_{A9(i)}U_{E9(i)}-U_{C9(i)}U_{D9(i)}}{U_{A9(i)}U_{B9(i)}-{U_{C9(i)}}^2}$$

The minimum MSE of U_{p_9}

$$MSE_{\min}\left(U_{p_9}\right)=\overline{Y}^2\left(\eta_0+\frac{\vartheta^2\eta_1}{4}-\vartheta\eta_{01}\right)-\frac{U_{A9(i)}{U_{E9(i)}}^2+U_{B9(i)}{U_{D9(i)}}^2-2U_{C9(i)}U_{D9(i)}U_{E9(i)}}{U_{A9(i)}U_{B9(i)}-{U_{C9(i)}}^2} \tag{105}$$