ORIGINAL PAPER

SOME SPECIAL CURVES IN THE UNIT TANGENT BUNDLES OF SURFACES

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Abstract. *The aim of this paper is to give some characterizations for N-Legendre and N-slant curves in the unit tangent bundles of surfaces endowed with natural diagonal lifted structures.*

Keywords: *Unit tangent bundle, natural diagonal structures, slant curve.*

1. INTRODUCTION

In studies on curves in the unit tangent sphere bundles, researchers generally consider the standard contact metric structure which is obtained by endowing the bundle with the induced Sasaki metric. For examples, in [1] Berndt *et al.* studied the geodesics, in [2-3] Inoguchi and Munteanu investigated the magnetic curves, in [4] Hou and Sun considered the slant geodesics and in [5] Hathout *et al.* discussed N-Legendre and N-slant curves of the unit tangent bundles with respect to this structure. However, some other contact metric structures can be defined on the unit tangent bundles. One of them is introduced by Druta-Romaniuc and Oproiu on tangent (sphere) bundles and called natural diagonal structure in [6]. In this paper, they found conditions under which the tangent sphere bundles are Einstein. In their further works, they had conditions under which the tangent sphere bundles are $\eta-$Einstein and obtained some results for curvatures of the tangent sphere bundles (see [7-8]).

In this paper, N-Legendre and N-slant curves are studied in the unit tangent bundles of surfaces with natural diagonal structures and some results are given when the surface is considered to be a sphere.

2. MATERIALS AND METHODS

In this section, we give a brief introduction to natural diagonal structures, for further information see [6]. Let (M,g) be a smooth $n-$dimensional Riemannian manifold $\pi: TM \rightarrow M$ be its tangent bundle. Let $(x^i,u^i)_{(i=1,\dots,n)}$ be the locally coordinate systems on the tangent bundle TM. The natural diagonal lift metric g^d is defined as follows:

$$g^d(X^h,Y^h) = c_1 g(X,Y) + d_1 g(X,u)g(Y,u), \qquad (1)$$

$$g^d(X^v,Y^h) = g^d(X^h,Y^v) = 0,$$

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$$g^d(X^v, Y^v) = c_2 g(X,Y) + d_2 g(X,u) g(Y,u),$$

for every vector fields X, Y on M and every tangent vector u, where $t = \frac{g(u,u)}{2}$ and c_1, $c_2, d_1\, d_2$ are smooth functions of t. The conditions for g^d to be positive are $c_1 > 0, c_2 > 0, c_1 + 2td_1 > 0, c_2 + 2td_2 > 0$ for every $t \geq 0$. Here, $X^h = X^i \frac{\partial}{\partial x^i} - X^i u^j \Gamma_{ij}^k \frac{\partial}{\partial u^k}$ and $X^v = X^i \frac{\partial}{\partial u^i}$ are the horizontal and the vertical lifts X at (x,u) with respect to the Levi-Civita connection ∇ of g respectively, where $\{\Gamma_{ij}^k\}$ are the Christoffel symbols of ∇ (for background of tangent bundles, see for example [9-10]).

Let us define a $(1,1)$ −tensor field on TM as follows:

$$JX^h = a_1 X^v + b_1 g(X,u) u^v, \tag{2}$$

$$JX^v = -a_2 X^h - b_2 g(X,u) u^h,$$

for every vector fields X on M, where $a_1, a_2, b_1\, b_2$ are smooth functions of t. We note that the $(1,1)$ −tensor field J given by the relations (2) defines an almost complex structure on the tangent bundle if and only if $a_2 = 1/a_1$ and $b_2 = -b_1/[a_1(a_1 + 2tb_1)]$ (see [7]).

We know that the unit tangent bundle $T_1M = \{u \in TM: g(u,u) = 1\}$ of a Riemannian manifold M is a $(2n-1)$ −dimensional submanifold of TM. The canonical vector field u^v is normal to T_1M. The horizontal lift of any vector field on M is tangent to T_1M, but the vertical lift is not always tangent to T_1M. The tangential lift of a vector field X of M is defined by $X^t = X^v - g(X,u)u^v$. Hence, we write the Lie algebra of C^∞ vector fields on T_1M as $\chi(T_1M) = \{X^h + Y^t : X, Y \in \chi(M)\}$ [11]. The induced Riemannian metric g_1^d on T_1M from (1) is uniquely determined by

$$g_1^d(X^h, Y^h) = c_1 g(X,Y) + d_1 g(X,u) g(Y,u), \tag{3}$$

$$g_1^d(X^t, Y^h) = g^d(X^h, Y^t) = 0,$$

$$g_1^d(X^t, Y^t) = c_2[g(X,Y) - g(X,u) g(Y,u)],$$

for every vector fields X, Y on M and every tangent vector u, where c_1, d_1, c_2 are constants. The conditions for g_1^d to be positive are $c_1 > 0, c_2 > 0, c_1 + d_1 > 0$ [6].

Remark that the functions c_1, d_1, c_2 become constant, since in the case of unit tangent bundle, the function t becomes a constant equal to 1/2.

In [8], it is proved that there is a contact metric structure $(\phi_1, \xi_1, \eta_1, g_1)$ on T_1M given by

$$\phi_1(X^h) = a_1 X^t, \phi_1(X^t) = -a_2 X^h + a_2 g(X,u) u^h, \tag{4}$$

$$\xi_1 = \frac{1}{2\lambda\alpha} u^h, \eta_1(X^t) = 0, \eta_1(X^h) = 2\alpha\lambda g(X,u), g_1 = \alpha g_1^d,$$

for every vector fields X, Y on M and every tangent vector u, where λ is a scalar, $\alpha = \frac{c_1 + d_1}{4\lambda^2}$ and a_1 and a_2 are the functions defined in (2). This contact metric structure is called natural diagonal structure. Furthermore, $(T_1M, \phi_1, \xi_1, \eta_1, g_1)$ is Sasakian if and only if M has constant sectional curvature $K = {a_1}^2$ [8].

The Levi-Civita connection ∇_1 of (T_1M, g_1) satisfies the following relations:

$$\nabla_{1X^h}Y^h = (\nabla_X Y)^h - \frac{1}{2}(R(X,Y)u)^t - \frac{d_1}{2c_2}[g(X,u)Y^t - g(Y,u)X^t], \tag{5}$$

$$\begin{aligned}\nabla_{1X^h}Y^t = (\nabla_X Y)^t - \frac{c_2}{2c_1}(R(Y,u)X)^h + \frac{d_1}{2c_1}g(X,u)Y^h + \frac{d_1}{2(c_1+d_1)}g(X,Y)u^h \\ - \frac{d_1(2c_1+d_1)}{2c_1(c_1+d_1)}g(X,u)g(Y,u)u^h - \frac{c_2 d_1}{2c_1(c_1+d_1)}g(Y,R(X,u)u)u^h,\end{aligned}$$

$$\begin{aligned}\nabla_{1X^t}Y^h = -\frac{c_2}{2c_1}(R(X,u)Y)^h + \frac{d_1}{2c_1}g(Y,u)X^h + \frac{d_1}{2(c_1+d_1)}g(X,Y)u^h \\ - \frac{d_1(2c_1+d_1)}{2c_1(c_1+d_1)}g(X,u)g(Y,u)u^h - \frac{c_2 d_1}{2c_1(c_1+d_1)}g(X,R(Y,u)u)u^h,\end{aligned}$$

$$\nabla_{1X^t}Y^t = -g(Y,u)X^t,$$

for every vector fields X, Y on M and every tangent vector u, where ∇ and R denote the Levi-Civita connection and the curvature tensor of (M, g), respectively [8].

3. RESULTS

Let (M, g) be a surface and let $\gamma: I \subset \mathbb{R} \to M$ be a curve on M. Assume that $\tilde{\gamma}(s) = (\gamma(s), X(s))$ is a curve in $(T_1M, g_1, \phi_1, \xi_1, \eta_1)$, where the contact metric structure is given by (4). We have four kinds of curves which are defined below.

Definition 1. [4] Let γ be a curve in an almost contact metric manifold (M, g, ϕ, ξ, η). The curve γ is called Legendrian (resp. slant) if the angle between the tangent vector field T of γ and ξ is (π/2) (resp. [0,π]-{π/2}), i.e. $g(T, \xi) = 0$ (resp. $g(T, \xi) = c$), where c is a non-zero constant.

Definition 2. [5] Let γ be a curve in an almost contact metric manifold (M, g, ϕ, ξ, η). The curve γ is called N-Legendre (resp. N-slant) if the angle between the normal vector field N of γ and ξ is π/2 (resp. [0,π]-{π/2}), i.e. $g(N, \xi) = 0$ (resp. $g(N, \xi) = c$), where c is a non-zero constant.

Suppose that $\tilde{\gamma}$ is parameterized by the arc-length and denote the Frenet apparatus of $\tilde{\gamma}$ by $(\tilde{T}, \tilde{N}, \tilde{B}, \tilde{\kappa}, \tilde{\tau})$. Then,

$$\begin{aligned}\tilde{T}(s) &= \frac{d\gamma^i}{ds}\frac{\partial}{\partial x^i} + \frac{dX^i}{ds}\frac{\partial}{\partial u^i} \\ &= \frac{d\gamma^i}{ds}\left(\frac{\partial}{\partial x^i}\right)^h(\tilde{\gamma}(s)) + \left(\frac{dX^i}{ds} + \frac{d\gamma^j}{ds}X^k\Gamma_{ij}^k\right)\frac{\partial}{\partial u^i}(\tilde{\gamma}(s)) \\ &= (E^h + (\nabla_E X)^t)(\tilde{\gamma}(s)),\end{aligned} \tag{6}$$

where $E = \gamma'(s)$.

Let θ be the angle between $\tilde{T}$ and ξ_1. From equations (3) and (4), we have

$$\frac{g_1(\tilde{T},\xi_1)}{|\tilde{T}||\xi_1|} = cos\theta = \sqrt{c_1+d_1}g(E,X). \tag{7}$$

If we differentiate both side of equation (7) with respect to s, and use equations (4), (5) and (6), we have

$$\begin{aligned}\frac{d}{ds}g_1(\tilde{T},\xi_1) &= g_1(\nabla_{1\tilde{T}}\tilde{T},\xi_1)+g_1(\tilde{T},\nabla_{1\tilde{T}}\xi_1)\\ &= \tilde{\kappa}g_1(\tilde{N},\xi_1)+\frac{1}{2\lambda\alpha}g_1(\tilde{T},\nabla_{1E^h}X^h+\nabla_{1(\nabla_E X)^t}X^h)\\ &= \tilde{\kappa}g_1(\tilde{N},\xi_1)+\frac{1}{2\lambda\alpha^2}[(c_1+d_1)g(E,\nabla_E X)-c_2R(E,X,X,\nabla_E X)]\\ &= -|\xi_1|\theta' sin\theta.\end{aligned}$$

So,

$$g_1(\tilde{N},\xi_1) = \frac{1}{2\lambda\alpha^2\tilde{\kappa}}\left(c_2R(E,X,X,\nabla_E X)-(c_1+d_1)g(E,\nabla_E X)\right)-|\xi_1|\frac{\theta' sin\theta}{\tilde{\kappa}}, \tag{8}$$

where $\theta' = \frac{d\theta}{ds}$ and $\xi_1 = \frac{1}{2\lambda\alpha}X^h$.

Let (T,N) be a Frenet frame on γ. From equation (7), we get the following

$$X = \frac{1}{r\sqrt{c_1+d_1}}cos\theta T+\beta N, \tag{9}$$

for a smooth function β, where $r = \|E\|$. Since X is a unit vector, we have

$$\frac{\lambda^2}{(c_1+d_1)r^2}cos^2\theta+\beta^2 = 1,$$

and

$$\beta = \pm\frac{1}{r}\sqrt{r^2-(\frac{\lambda}{\sqrt{c_1+d_1}})^2cos^2\theta} \tag{10}$$

Differentiating equation (9) with respect to s, we derive

$$\begin{aligned}\nabla_E X &= \frac{1}{\sqrt{c_1+d_1}}\left(\frac{cos\theta}{r}\right)'T+\frac{\kappa cos\theta}{\sqrt{c_1+d_1}}N+\beta' N-r\beta\kappa T\\ &= \left(\left(\frac{cos\theta}{r\sqrt{c_1+d_1}}\right)'-r\beta\kappa\right)T+\left(\frac{\kappa cos\theta}{\sqrt{c_1+d_1}}+\beta'\right)N.\end{aligned} \tag{11}$$

Equations (7) and (11), and orthogonality of the vectors X and $\nabla_E X$ give us

$$E = \frac{cos\theta}{\sqrt{c_1 + d_1}} X + \frac{r}{g(\nabla_E X, \nabla_E X)} \left(\left(\frac{cos\theta}{r\sqrt{c_1 + d_1}} \right)' - r\beta\kappa \right) \nabla_E X. \tag{12}$$

Using the last expression, we can write

$$R(E, X, X, \nabla_E X) = r \left(\left(\frac{cos\theta}{r\sqrt{c_1 + d_1}} \right)' - r\beta\kappa \right) \frac{R(\nabla_E X, X, X, \nabla_E X)}{g(\nabla_E X, \nabla_E X)} \tag{13}$$

$$= r \left(\left(\frac{cos\theta}{r\sqrt{c_1 + d_1}} \right)' - r\beta\kappa \right) K(s),$$

where $K(s)$ is the sectional curvature of M. Putting the equations (10)-(13) in (8) we state the following equation

$$g_1(\widetilde{N}, \xi_1) = \frac{r}{2\lambda\alpha^2\tilde{\kappa}} (c_2 K(s) - (c_1 + d_1)) \left(\left(\frac{cos\theta}{r\sqrt{c_1 + d_1}} \right)' \pm r\kappa \sqrt{r^2 - \left(\frac{\lambda}{\sqrt{c_1 + d_1}}\right)^2 cos^2\theta} \right) \tag{14}$$
$$- |\xi_1| \frac{\theta' sin\theta}{\tilde{\kappa}}.$$

Now we can prove the following propositions.

Proposition 1. Let T_1S^2 be the unit tangent bundle of the unit sphere S^2 with the natural diagonal metric structure given by (4) such that $c_2 = c_1 + d_1$. Then all Legendre and slant curves are $\widetilde{N}$-Legendre curves.

Proof: Let $\tilde{\gamma}(s) = (\gamma(s), X(s))$ be a Legendre or a slant curve with arc-parameter in the contact metric manifold T_1S^2. Since the sectional curvature of the unit sphere K is equal to 1, from Definition 1, equation (14) and under the assumption $c_2 = c_1 + d_1$, we get

$$g_1(\widetilde{N}, \xi_1) = 0.$$

This completes the proof.

Proposition 2. Let T_1S^2 be the unit tangent bundle of the unit sphere S^2 with the natural diagonal metric structure given by (4) such that $c_2 = c_1 + d_1$ and let $\tilde{\gamma}$ be a non-slant curve on T_1S^2. Then $\tilde{\gamma}$ is an $\widetilde{N}$-slant curve if the angle θ satisfies the equation

$$\theta = arccos\, c \int \tilde{\kappa},$$

where c is a non-zero constant.

Proof: Let $\tilde{\gamma}(s) = (\gamma(s), X(s))$ be a non-slant curve with arc-parameter in the contact metric manifold T_1S^2. Since the sectional curvature of the unit sphere K is equal to 1, under the assumption $c_2 = c_1 + d_1$, Definition 1 and equation (14) give us

$$g_1(\widetilde{N},\xi_1) = -\frac{\theta' sin\theta}{\tilde{\kappa}} = c = constant.$$

So,

$$g_1(\widetilde{N},\xi_1) = (cos\theta)' = c\tilde{\kappa}$$

By solving the last differential equation, we get

$$\theta = arccos\, c\int \tilde{\kappa},$$

which completes the proof.

Proposition 3. Let M be a non-unit sphere whose constant sectional curvature is $K = a_1^2$. Suppose that $\tilde{\gamma}(s) = (\gamma(s), X(s))$ is a slant curve in $(T_1M, g_1, \phi_1, \xi_1, \eta_1)$ such that $c_2 = c_1 + d_1$ and γ is a curve with constant velocity r_0. If the torsion $\tilde{\tau}$ of $\tilde{\gamma}$ equals to sectional curvature K of M, then $\tilde{\gamma}$ is an $\widetilde{N}$-Legendre (resp. $\widetilde{N}$-slant) curve if and only if γ is a geodesic (resp. has a non-zero constant curvature κ).

Proof: Let M be a sphere with constant sectional curvature $K = a_1^2$ and let $c_2 = c_1 + d_1$. If the curve $\tilde{\gamma}(s) = (\gamma(s), X(s))$ is a slant curve in $(T_1M, g_1, \phi_1, \xi_1, \eta_1)$ where γ has constant velocity of r_0, then from (14) we have

$$g_1(\widetilde{N},\xi_1) = \frac{r_0 c_2 (K(s)-1)}{2\lambda\alpha^2\tilde{\kappa}}\left(\pm\kappa r_0\sqrt{{r_0}^2 - (\frac{\lambda}{\sqrt{c_1+d_1}})^2 cos^2\theta}\right)$$

We know that in a Sasakian 3-manifold, a curve σ is slant if and only if $(\tau_\sigma \pm 1)/\kappa_\sigma$ is a non-zero constant, where τ_σ and κ_σ are torsion and curvature of σ, respectively (see [12]). If we assume $\tau = K$, from the above equation, we have

$$g_1(\widetilde{N},\xi_1) = \frac{\tilde{\tau}-1}{\tilde{\kappa}}\frac{r_0 c_2}{2\lambda\alpha^2}\left(\pm r_0\sqrt{{r_0}^2 - (\frac{\lambda}{\sqrt{c_1+d_1}})^2 cos^2\theta}\right)\kappa = \bar{c}\kappa$$

where $\bar{c}$ is a constant. So, it is clear that $\tilde{\gamma}$ is an $\widetilde{N}$-Legendre (resp. $\widetilde{N}$-slant) curve if and only if $\kappa = 0$ (resp. non-zero constant). This ends the proof.

Example 1. Let S^2 be a non-unit sphere with radius R. In this case, the sectional curvature (Gaussian curvature) of S^2 equals to $\frac{1}{R^2}$. Under the assumptions in Proposition 3, the projection curves γ of all slant and $\widetilde{N}$ −slant curves $\tilde{\gamma}$ in T_1S^2 are circles in S^2 when their Frenet apparatus are $(T, N, B, \kappa, \tau = \frac{1}{R^2})$.

Proposition 4. Let M be a non-unit sphere and T_1M be the unit tangent bundle of M with the natural diagonal metric structure given by (4) such that $c_2 = c_1 + d_1$. Suppose that $\tilde{\gamma}(s) = (\gamma(s), X(s))$ is a slant curve on T_1M and γ is a curve with constant velocity r_0. Then the curve $\tilde{\gamma}$ is $\widetilde{N}$-slant if and only if

$$\frac{(K-1)\kappa}{\tilde{\kappa}}$$

is a non-zero constant.

Proof: Let M be a non-unit sphere ($K \neq 1$). Assume that the curve $\tilde{\gamma}(s) = (\gamma(s), X(s))$ is a slant curve in $(T_1M, g_1, \phi_1, \xi_1, \eta_1)$ such that $c_2 = c_1 + d_1$, where γ has constant velocity r_0. Then from (14) we get

$$g_1(\widetilde{N}, \xi_1) = \frac{K-1}{\tilde{\kappa}} \frac{r_0 c_2}{2\lambda\alpha^2}\left(\pm r_0 \sqrt{{r_0}^2 - (\frac{\lambda}{\sqrt{c_1+d_1}})^2 cos^2\theta} \right)\kappa = \bar{c}\frac{(K-1)\kappa}{\tilde{\kappa}},$$

where $\bar{c}$ is a non-zero constant. The proof follows from the Definition 2.

Example 2. Let $\tilde{\gamma}$ be an arbitrary slant curve in $T_1\mathbb{R}^2$ and its projection curve γ be a geodesic in $\mathbb{R}^2$. Then under the assumptions in Proposition 4, $\tilde{\gamma}$ is an $\widetilde{N}$ −Legendre curve. Clearly, if γ is not geodesic, then $\tilde{\gamma}$ is an $\widetilde{N}$-slant curve if and only if $\frac{\kappa}{\tilde{\kappa}}$ is a non-zero constant.

Proposition 5. Let M be a non-unit sphere and T_1M be the unit tangent bundle of M with the natural diagonal metric structure given by (4) such that $c_2 = c_1 + d_1$. Suppose that $\tilde{\gamma}(s) = (\gamma(s), X(s))$ is a non-slant curve in T_1M and γ is a curve with constant velocity $\frac{\lambda}{\sqrt{c_1+d_1}}$. If the angle θ is linear (i.e. $\theta = es + f$, e and f are constants), then

(1) $\tilde{\gamma}(s)$ is a $\widetilde{N}$-Legendre curve if and only if

$$\frac{c_2(K-1)}{2\alpha^2|\xi_1|\sqrt{c_1+d_1}}\left(\frac{-e}{\lambda} \pm \frac{\lambda\kappa}{c_1+d_1}\right) = e$$

(2) $\tilde{\gamma}(s)$ is a $\widetilde{N}$-slant curve if and only if

$$\theta = \arcsin\left(\frac{\bar{c}\tilde{\kappa}}{-e\left(\frac{c_2(K-1)}{2\alpha^2\lambda\sqrt{c_1+d_1}} + 1\right) \pm \frac{\lambda c_2 \kappa(K-1)}{2\alpha^2(c_1+d_1)^{3/2}}}\right)$$

where $\bar{c}$ is a non-zero constant.

Proof: 1) If the angle θ is linear, under the assumptions and from equation (14), we have

$$g_1(\widetilde{N}, \xi_1) = \frac{c_2(K-1)}{2\alpha^2\sqrt{c_1+d_1}\tilde{\kappa}}\left(-\frac{esin\theta}{\lambda} \pm \frac{\lambda\kappa}{c_1+d_1}sin\theta\right) - \frac{esin\theta}{\tilde{\kappa}}|\xi_1|.$$

The proof follows from the condition $g_1(\widetilde{N}, \xi_1) = 0$ and direct computations.

(2) The above equation, the $\widetilde{N}$ −slant condition $g_1(\widetilde{N}, \xi_1) = \bar{c}$ and direct computations give the proof.

4. CONCLUSION

In this paper, we examine the $\widetilde{N}$-Legendre and $\widetilde{N}$-slant curves in the unit tangent sphere bundles of surfaces. The unit tangent sphere bundles are considered with the general natural metric structures which are introduced and studied by Druta and Oproiu in [6-8]. By using these structures, we obtain a generalization for the results in [5].

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