ORIGINAL PAPER

THE CONSISTENCY OF THE KERNEL ESTIMATION OF THE FUNCTION CONDITIONAL DENSITY FOR ASSOCIATED DATA IN HIGH-DIMENSIONAL STATISTICS

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Manuscript received: 10.01.2022; Accepted paper: 19.05.2022;
Published online: 30.06.2022.

Abstract.*The purpose of the present paper is to investigate by the kernel method a nonparametric estimate of the conditional density function of a scalar response variable given a random variable taking values in a separable real Hilbert space when the observations are quasi-associated dependent. Under some general conditions, we establish the pointwise almost complete consistencies with rates of this estimator. The principal aim is to investigate of the convergence rate of the proposed estimator.*

Keywords:*non parameter kernel estimation; small ball probabilities; quasi associated data.*

1. INTRODUCTION

In statistics, (FDA) has received much attention in the feild of applied mathematcs. This type of data is collected from a lot of fields, such as econometrics study, epidemiology control, environmental and ecological sciences, and many others sections.

Functional Statistics was produced by [1], these authors obtained some properties in the case i.i.d. Since this article, an abundant literature has developed on the estimation of the conditional density and its derivatives, in particular to utilize it to estimate the conditional mode.

Considering mixing observtions. [2] established the convergence (a.co) of the kernel estimator of the conditional mode defined by the random variable maximizing the conditional density.

Alteratively, [3-4] estimated the conditional mode by the point that cancels the derivative of the kernel density estimator.In deferent discriplines, including analyse of reliability, theoretical physics, MVA, and biological sciences the associated random variables are crucial.

Many work used positive and negative dependent random variables.The association case is a type of weak dependence introduced by [5] for stochastic processes in R.

It was generalized by [4] to real random fields, and it provides a unified approach to studying families of both positive dependence and negative dependence random

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variables.There are few number of articles dealing with nonparametric estmation of quasi-associated data.

We can cite, for quasi-associated hilberytian random variables [6] studied its limit theorem, [7] studied Asymptotic properties for Regression M-estimator, for functional single index structure, [8] studied the simulation and estimation part of conditional risk function in the quasi-associated data case.

Same model, the asymptotic normality was studied by [9].In case of relative regression, [10] studied the nonparametric estimation for assciated r.v.For the asymptotic normality of the non-parametric conditional cumulative. Function estimate studied [11]

The intervention of this work is to check the estimator properties [1] in case of associated data the a.co convergence is established (rate) of a kernel estimate for the hazard function when the variable is real random conditioned by a functional explanatory variable.

2. THE MODEL

Consider $Z_i = (X_i, Y_i)_{\overline{1,n}}$ is a simple of n dependent (Q.A) random from $Z = (X, Y)$, where X takes values in H and the latter is a real scalable Hilbert space, and Y in$\mathbb{R}$.d is a semi-metric knonw by $\forall(x, x') \in \mathcal{H}/d(x, x') = \|x - x'\|$. In the rest of this paper, we consider $x \in \mathcal{H}$ (fixed) and $\mathcal{N}_x$mention a fixed neighborhood of x and $\mathcal{S} \subset \mathbb{R}$.

To estimate the conditional density function we consider the following functional kernel estimators:

$$\hat{f}^x(y) = \frac{h_K^{-1} \sum_{i=1}^n K(h_K^{-1} d(x, X_i)) H(h_H^{-1}(y - Y_i))}{\sum_{i=1}^n K(h_K^{-1} d(x, X_i))}, \quad \forall\, y \in \mathbb{R} \tag{1}$$

we can write:

$$\hat{f}^x(y) = \frac{\hat{f}_N(y, x)}{\hat{F}_D(x)} \tag{2}$$

where

$$\hat{f}_N(y, x) = \frac{1}{n h_H \mathbb{E}[K_1(x)]} \sum_{i=1}^n K_i(x) H_i(y)$$

and

$$\hat{F}_D(x) = \frac{1}{n \mathbb{E}[K_1(x)]} \sum_{i=1}^n K_i(x)$$

with:

$$K_i(x) = K(h_K^{-1} d(x, X_i)) \text{ and } H_i(y) = H'(h_H^{-1}(y - Y_i))$$

3. ASSUMPTIONS

3.1. Definition

A sequence $(X_n)_{n\in\mathbb{N}}$ of real random vectors variables be Quasi-Association (QA), if for any disjoint subsets I and J of $\mathbb{N}$ and all bounded Lipschitz functions $f: \mathbb{R}^{|I|d} \to \mathbb{R}$ and $g: \mathbb{R}^{|J|d} \to \mathbb{R}$ satisfying :

$$Cov\left(f\,(X_i, i \in I), g\big(X_j, j \in J\big)\right) \leq Lip(f)Lip(g)\sum_{i\in I}\sum_{j\in J}\sum_{k=1}^{d}\sum_{l=1}^{d}\left|Cov\big(X_i^k, X_j^l\big)\right| \tag{3}$$

where X_i^k imply the k^{th} composent of X_i

$$Lip(f) = sup_{x\neq y}\frac{|f(x)-f(y)|}{\|x-y\|_1}\ with\ \|(x_1, \dots, x_k)\|_1 = |x_1| + \ \dots + |x_k| \tag{4}$$

Along the paper, when there is no possible confusion, we will refer by $\mathcal{C}$ or/and $\mathcal{C}'$ to some completely positive global constants whose values are allowed to be changed.Suppose the coefficient of covariance is defined as:

$$\lambda_k = sup_{s\geq k}\sum_{|i-j|\geq s}\lambda_{ij}$$

where

$$\lambda_{ij} = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left|cov(X_i^k, X_j^l)\right| + \sum_{k=1}^{\infty}\left|cov(X_i^k, Y_j)\right| + \sum_{l=1}^{\infty}\left|cov(Y_i, X_j^l)\right| + \left|cov\big(Y_i, Y_j\big)\right|$$

X_i^kimply the $k\ ^{th}$ composent of $X_i$specified as $X_i^k := < X_i, e^k >$.

With $h_K > 0$, let $B_\theta(x, h_K) := \{x' \in \mathcal{H}/d(x', x) < h_K\}$ a ball with its center x and radius h_K. We now mention the assumptions that help us reach the desired results:

(H1) $\mathbb{P}(|< X - x >| < h_K) = \mathbb{P}\big(X \in B(x, h_K)\big) =: \phi_x(h_K) > 0.$

(H2) The conditional density $f^x(y)$ satisfies the Hölder condition, that is: $\forall(x_1, x_2) \in \mathcal{N}_x \times \mathcal{N}_x$, $\forall(y_1, y_2) \in \mathcal{S}^2$

$$|f^{x_1}(y_1) - f^{x_2}(y_2)| \leq C(\|x_1 - x_2\|^{b_1} + |y_1 - y_2|^{b_2}), \quad b_1 > 0,\ b_2 > 0.$$

(H3) $\int H\,(t)dt = 1, \quad \int |t|^{b_2} H(t)dt < \infty$ and $\int H^2(t)dt < \infty.$

(H4) K is a bounded continuous Lipschitz function such that:

$$\mathcal{C}\mathbb{1}_{\overline{0,1}}(.) < K(.) < \mathcal{C}'\mathbb{1}_{\overline{0,1}}(.),$$

where $\mathbb{1}_{\overline{0,1}}$ is a indicator function.

(H5) The sequence of random pairs (X_i, Y_i), $i \in \mathbb{N}$ is quasi-associated with covariance coefficient λ_K, $K \in \mathbb{N}$:

$$\alpha > 0\,,\ \exists\, \mathcal{C} > 0,\ \lambda_k \leq \mathcal{C}e^{-\alpha k}.$$

(H6) $\psi_{\mathrm{i,j}}(h) = \mathbb{P}\left[\left(X_i, X_j\right) \in \beta(x, h_K) \times \beta(x, h_K)\right] = O\left(\phi_x^2(h_K)\right)$, satisfy:

$$\sup_{\mathrm{i}\neq \mathrm{j}} \psi_{\mathrm{i,j}}(h) = O\left(\phi_x^2(h_K)\right) > 0.$$

(H7) The bandwidths h_K and h_H are a sequences of positive numbers satisfying For $\mathrm{j} = 0, 1$

i) $\lim_{n\to\infty} h_K = 0$ *and* $\lim_{n\to\infty} h_H = 0$.

ii) $\lim_{n\to\infty} \frac{log(n)}{nh_H^j\phi_x(h_K)} = 0$ *and* $\lim_{n\to\infty} \frac{log^5(n)}{nh_H^j\phi_x(h_K)} = 0$

4. MAIN RESULTS AND PROOFS

4.1. MAIN RESULTS

Theorem 4.1. Under assumptions **(H1)-(H7)**, we have, for any $x \in \mathcal{A}$:

$$\hat{f}^x(y) - f^x(y) = O\left(h_K^{b_1} + h_H^{b_2}\right) + O_{\mathrm{a.co}}\left(\sqrt{\frac{\log(\mathrm{n})}{nh_H\phi_x(h_K)}}\right)$$

Proof: **Theorem 4.1**: Is based on the following decomposition,

$$\hat{f}^x(y) - f^x(y) = \frac{\hat{f}_N^x(y) - \hat{f}^x(y)\hat{F}_D(x)}{\hat{F}_D(x)}$$

$$= \frac{1}{\hat{F}_D^x(y)}\left(\left\{\hat{f}_N^x(y) - \mathbb{E}\left[\hat{f}_N^x(y)\right]\right\} - \left\{f^x(y) - \mathbb{E}\left[\hat{f}_N^x(y)\right]\right\}\right.$$

$$\left. + \frac{\hat{f}^x(y)}{\hat{F}_D(x)}\left\{\mathbb{E}\left[\hat{F}_D(x)\right] - \hat{F}_D(x)\right\}\right)$$

Lemma 4.1. Under assumptions **(H1)-(H4)-(H7)** we get:

$$\hat{f}_N^x(y) - \mathbb{E}\left[\hat{f}_N^x(y)\right] = O_{\mathrm{a.co}}\left(\sqrt{\frac{\log(\mathrm{n})}{nh_H\phi_x(h_K)}}\right) \tag{5}$$

Corollary 4.1. Under assumptions **(H1)-(H4)-(H6)**, we get:

$$\sum_{i=1}^{\infty} \mathbb{P}\left(\left|\hat{\mathrm{F}}_{\mathrm{D}}^{\mathrm{x}}\right| < \frac{1}{2}\right) < \infty \tag{6}$$

Lemma 4.2. Under assumptoins**(H1)-(H3)-(H7)**, we get:

$$\hat{\mathrm{F}}_{\mathrm{D}}^{\mathrm{x}}(\mathrm{y}) - \mathbb{E}\left[\hat{\mathrm{F}}_{\mathrm{D}}^{\mathrm{x}}(\mathrm{y})\right] = \mathrm{O}_{\mathrm{a.co}}\left(\sqrt{\frac{\log(\mathrm{n})}{nh_H\phi_x(h_K)}}\right) \tag{7}$$

Lemma 4.3. Under hypotheses **(H1)-(H5)**, we get:

$$f^x(y) - \mathbb{E}\left[\hat{f}_N^x(y)\right] = O\left(h_K^{b_1} + h_H^{b_2}\right) \tag{8}$$

4.2. PROOFS OF TECHNICAL LEMMAS

Proof : **Lemma 4.1.**We put

$$\delta_{\mathrm{i}} = \frac{1}{nh_H\mathbb{E}[K_1(x)]}\chi(\mathrm{X_i}, \mathrm{Y_i}), 1 \le \mathrm{i} \le \mathrm{n},$$

where:

$$\chi(\mathrm{X_i}, \mathrm{Y_i}) = K(h_K^{-1}d(x, X_i))\,\mathrm{H}(y - Y_i) - \mathbb{E}[K_1H_1], \qquad 1 \le \mathrm{i} \le \mathrm{n}, \tag{9}$$

$\mathrm{X_i} \in \mathrm{H}, \mathrm{Y_i} \in \mathbb{R}$. Clearly, we have: $\mathbb{E}[\delta_{\mathrm{i}}] = 0$ and,

$$\left|\hat{f}_N^x(y) - \mathbb{E}\left[\hat{f}_N^x(y)\right]\right| = \sum_{\mathrm{i}=1}^{\mathrm{n}} \delta_{\mathrm{i}},$$

we also write:

$$||\chi||_\infty^2 \le 2\,\mathcal{C}\,||\mathrm{K}||_\infty||\mathrm{H}||_\infty,$$

and

$$\mathrm{Lip}(\chi) \le \mathcal{C}(\,h_K^{-1}||\mathrm{H}||_\infty\mathrm{Lip}(\mathrm{K}) + h_H^{-1}||\mathrm{K}||_\infty\mathrm{Lip}(\mathrm{H})\,.$$

Now, to evaluate the variance $\mathrm{Var}(\sum_{\mathrm{i}=1}^{\mathrm{n}}\delta_{\mathrm{i}})$ (as first term) and the covarince(as a second term)$Cov\left(\prod_{i=1}^{u}\delta_{s_i}, \prod_{j=1}^{v}\delta_{t_j}\right)$ and for all $(s_1, \ldots, s_u) \in \mathbb{N}^u$, $(t_1, \ldots, t_v) \in \mathbb{N}^v$ with $1 \le s_1 \le \cdots \le s_u \le t_1 \le \cdots \le t_v \le \mathrm{n}$.

At first, we study the second terme,let's begin with the first case: if $t_1 = s_u$

$$\left|Cov\left(\prod_{i=1}^{u}\delta_{s_i}, \prod_{j=1}^{v}\delta_{t_j}\right)\right| \le \left(\frac{\mathcal{C}}{nh_H\mathbb{E}[K_1(x)]}\right)^{u+v}\mathbb{E}_\chi|\mathrm{X}_1, \mathrm{Y}_1|^{u+v}$$

$$\leq \left(\frac{\mathcal{C}\left||\mathrm{K}|\right|_{\infty}\left||\mathrm{H}|\right|_{\infty}}{nh_H\mathbb{E}[K_1(x)]}\right)^{u+v} \mathbb{E}[K_1H_1]$$

$$\leq h_H\phi_x(h_K)\left(\frac{\mathcal{C}}{nh_H\phi_x(h_K)}\right)^{u+v}.$$

If $t_1 > s_u$, we use the quasi-association, by **(H5)**, we get:

$$\begin{aligned}
&\left|Cov\left(\prod_{i=1}^{u}\delta_{s_i},\prod_{j=1}^{v}\delta_{t_j}\right)\right| \\
&\leq \left(\frac{h_K^{-1}\,\mathrm{Lip(K)} \,+\, h_H^{-1}\,\mathrm{Lip(H)}}{nh_H\mathbb{E}[K_1(x)]}\right)^2\left(\frac{\mathcal{C}}{nh_H\mathbb{E}[K_1(x)]}\right)^{u+v}\sum_{i=1}^{u}\sum_{j=1}^{v}\lambda_{s_it_j} \qquad (10)\\
&\leq (h_K^{-1}\,\mathrm{Lip(K)} \,+\, h_H^{-1}\,\mathrm{Lip(H)})^2\left(\frac{\mathcal{C}}{nh_H\mathbb{E}[K_1(x)]}\right)^{u+v} v\lambda_{t_1-s_u}\\
&\leq (h_K^{-1}\,\mathrm{Lip(K)} \,+\, h_H^{-1}\,\mathrm{Lip(H)})^2\left(\frac{\mathcal{C}}{h_H\phi_x(h_K)}\right)^{u+v} ve^{-\alpha(t_1-s_u)}
\end{aligned}$$

On the other hand, by **(H6)** we get:

$$\begin{aligned}
\left|Cov\left(\prod_{i=1}^{u}\delta_{s_i},\prod_{j=1}^{v}\delta_{t_j}\right)\right| &\leq \left(\frac{\mathcal{C}\left||\mathrm{K}|\right|_{\infty}\left||\mathrm{H}|\right|_{\infty}}{nh_H\mathbb{E}[K_1(x)]}\right)^{u+v-2}\left(\mathbb{E}\left|\Delta_{s_u},\Delta_{t_1}\right| + \mathbb{E}\left|\Delta_{s_u}\right|\mathbb{E}\left|\Delta_{t_1}\right|\right)\\
&\leq \left(\frac{\mathcal{C}\left||\mathrm{K}|\right|_{\infty}\left||\mathrm{H}|\right|_{\infty}}{nh_H\mathbb{E}[K_1(x)]}\right)^{u+v-2}\left(\frac{\mathcal{C}}{nh_H\mathbb{E}[K_1(x)]}\right)h_H\left(\sup_{\mathrm{i}\neq\mathrm{j}}\mathbb{P}\left[(X_i,X_j)\right.\right. \qquad (11)\\
&\qquad \left.\left.\in \beta(x,h_K)\times\beta_\theta(x,h_K)\right] + \mathbb{P}[X_1\in\beta(x,h_K)]^2\right)\\
&\leq \left(\frac{\mathcal{C}}{h_H\phi_x(h_K)}\right)^{u+v}\left(h_H\phi_x(h_K)\right)^2
\end{aligned}$$

Furthermore, taking a γ−power of (11), (1 − γ)−power of (12), we obtain an upper-bound of the tree terms as follows, for: $1 \leq s_1 \leq \cdots \leq s_u \leq t_1 \leq \cdots \leq t_v \leq \mathrm{n}$,

$$\left|Cov\left(\prod_{i=1}^{u}\delta_{s_i},\prod_{j=1}^{v}\delta_{t_j}\right)\right| \leq h_H\phi_x(h_K)\left(\frac{\mathcal{C}}{nh_H\phi_x(h_K)}\right)^{u+v}.$$

Secondly, for the variance term $\mathrm{Var}(\sum_{\mathrm{i}=1}^{\mathrm{n}}\delta_{\mathrm{i}})$, we put for all $1 \leq \mathrm{i} \leq \mathrm{n}$:

$$\begin{aligned}
\left|Var\left(\prod_{i=1}^{u}\delta_{s_i},\prod_{j=1}^{v}\delta_{t_j}\right)\right| &= \left(\frac{1}{nh_H\mathbb{E}[K_1(x)]}\right)^2\sum_{i=1}^{u}\sum_{j=1}^{v}Cov(K_iH_i,K_jH_j)\\
&= \underbrace{\left(\frac{1}{nh_H\mathbb{E}[K_1(x)]}\right)^2\sum_{i=1}^{u}\sum_{j=1}^{v}Var(K_1H_1)}_{\mathrm{T}_1} \qquad (12)
\end{aligned}$$

$$+\underbrace{\left(\frac{1}{nh_H\mathbb{E}[K_1(x)]}\right)^2\sum_{i=1}^{n}\sum_{j=1,i\neq j}^{n}Cov\left(K_iH_i,K_jH_j\right)}_{\mathrm{T}_2}.$$

For the first term **T1**, we have:

$$Var(K_1H_1)=\mathbb{E}[K_1^2H_1^2\ ]-(\mathbb{E}[K_1H_1])^2,$$

then,

$$\mathbb{E}[K_1^2H_1^2]=\mathbb{E}\left[K_1^2\mathbb{E}[H_1^2/X_1]\right],$$

thus, under **(H2) − (H3)**, and by integration on the real composent y we get,

$$\mathbb{E}[H_1^2/X_1]=O(h_H\phi_x(h_K)),$$

as, for all j $\geq$ 1, $\mathbb{E}\left[K_1^j\right]=O(\phi_x(h_K))$, then:

$$\mathbb{E}[K_1^2H_1^2]=O(h_H\phi_x(h_K)),$$

it follows that,

$$\left(\frac{1}{n(h_H\mathbb{E}[K_1(x)])^2}\right)Var(K_1H_1)=O\left(nh_H\phi_x(h_K)\right), \tag{13}$$

Now, we deal with the part**T2**from equation (13), in the same way that Masry[12] developed it,we need the following decomposition:

$$\left|Cov\left(\prod_{i=1}^{u}\delta_{s_i},\prod_{j=1}^{v}\delta_{t_j}\right)\right|=\underbrace{\sum_{i=1}^{u}\sum_{\substack{j=1\\0\leq|i-j|\leq m_n}}^{v}Cov(K_1H_1)}_{\mathbf{P_1}}$$

$$+\underbrace{\sum_{i=1}^{n}\sum_{\substack{j=1\\|i-j|>m_n}}^{n}Cov\left(K_iH_i,K_jH_j\right)}_{\mathbf{P_2}}.$$

where$m_n\xrightarrow[n\to\infty]{}\infty$ and from assumptions **(H1)-(H3)-(H6)**, we have, for $i\neq j$:

$$\begin{aligned}P_1&\leq nm_n\left(\max_{i\neq j}\left|\mathbb{E}\left(K_iH_i,K_jH_j\right)\right|+(\mathbb{E}[K_1H_1])^2\right)\\&\leq Cnm_n\left(h_H^2\phi_X^2(h_K)\ +\left(h_H\phi_x(h_K)\right)^2\right)\\&\leq Cnm_n\left(h_H^2\phi_X^2(h_K)\right),\end{aligned} \tag{14}$$

andfor $\mathbf{P_2}$**:**

$$P_2 \leq (h_K^{-1}\,\mathrm{Lip(K)} \, + \, h_H^{-1}\,\mathrm{Lip(H)})^2 \sum_{i=1}^{u} \sum_{\substack{j=1 \\ |i-j|>m_n}}^{v} \lambda_{i,j}$$

$$\leq \mathcal{C}(h_K^{-1}\,\mathrm{Lip(K)} \, + \, h_H^{-1}\,\mathrm{Lip(H)})^2 \sum_{i=1}^{u} \sum_{\substack{j=1 \\ |i-j|>m_n}}^{v} \lambda_{i,j} \tag{15}$$

$$\leq \mathcal{C}n(h_K^{-1}\,\mathrm{Lip(K)} \, + \, h_H^{-1}\,\mathrm{Lip(H)})^2 \lambda_{m_n}$$

$$\leq \mathcal{C}n(h_K^{-1}\,\mathrm{Lip(K)} \, + \, h_H^{-1}\,\mathrm{Lip(H)})^2 e^{-\alpha m_n},$$

then, by (15)- (16), we get:

$$\sum_{j=1, i\neq j}^{n} Cov\big(K_i H_i, K_j H_j\big) \leq \mathcal{C}n\Big(m_n\big(h_H^2 \phi_X^2(h_K)\big) + (h_K^{-1}\,\mathrm{Lip(K)} \, + \, h_H^{-1}\,\mathrm{Lip(H)})^2 e^{-\alpha m_n}\Big),$$

by choosing:

$$m_{n=} \log\left(\frac{(h_K^{-1}\,\mathrm{Lip(K)} \, + \, h_H^{-1}\,\mathrm{Lip(H)})^2}{\alpha h_H^2 \phi_X^2(h_K)}\right),$$

we get:

$$\frac{1}{h_H \phi_x(h_K)} \sum_{j=1, i\neq j}^{n} Cov\big(K_i H_i, K_j H_j\big) \to 0, as\ n \to \infty \tag{16}$$

At last, by collecting the results (13)-(14) -(17), we obtain:

$$Var\left(\sum_{\mathrm{i}=1}^{\mathrm{n}} \delta_{\mathrm{i}}\right) = O\left(\frac{1}{n h_H \phi_x(h_K)}\right)$$

thus, the variables $\delta_{\mathrm{i}}, \mathrm{i} = 1, ...,$n be content with the assumptions**(H1)-(H4)-(H6)**:

$$K_n = \frac{\mathcal{C}}{n h_H \sqrt{\phi_x(h_K)}},\ M_n = \frac{\mathcal{C}}{n h_H \phi_x(h_K)} and\ A_n = Var(\textstyle\sum_{\mathrm{i}=1}^{\mathrm{n}} \delta_{\mathrm{i}}).$$

So,

$$\mathbb{P}\left(\left|\hat{f}_N^x(y) - \mathbb{E}\big[\hat{f}_N^x(y)\big]\right| > \eta\sqrt{\frac{\log(\mathrm{n})}{n h_H \phi_x(h_K)}}\right) = \mathbb{P}\left(\left|\sum_{\mathrm{i}=1}^{\mathrm{n}} \delta_{\mathrm{i}}\right| > \eta\sqrt{\frac{\log(\mathrm{n})}{n h_H \phi_x(h_K)}}\right)$$

$$\leq exp\left\{-\eta^2 \frac{\log(n)}{n h_H \phi_x(h_K) Var(\sum_{\mathrm{i}=1}^{\mathrm{n}} \delta_{\mathrm{i}}) + \sqrt[6]{\frac{log^5(n)}{\big(n h_H \phi_x(h_K)\big)^7}}}\right\} \tag{17}$$

$$\leq exp\left\{-\eta^2 \frac{\log(n)}{\mathcal{C} + \sqrt[6]{\frac{log^5(n)}{n h_H \phi_x(h_K)}}}\right\}$$

$$\leq \mathcal{C}' exp\left\{-\eta^2 \frac{\log(n)}{C + \sqrt[6]{\frac{log^5(n)}{nh_H\phi_x(h_K)}}}\right\}$$

Finally, by **(H7)** and for a favorable choice of η, Borel-Cantellis lemma allows to finish the proof of **Lemma4.1.**

*Proof:***Corollary 4.1**.We have,

$$\left\{\left|\widehat{\mathrm{F}}_{\mathrm{D}}^{\mathrm{x}}\right| < \tfrac{1}{2}\right\} \subseteq \left\{\left|\widehat{\mathrm{F}}_{\mathrm{D}}^{\mathrm{x}} - 1\right| > \tfrac{1}{2}\right\},$$

therefor,

$$\mathbb{P}\left\{\left|\widehat{\mathrm{F}}_{\mathrm{D}}^{\mathrm{x}}\right| < \tfrac{1}{2}\right\} \leq \mathbb{P}\left\{\left|\widehat{\mathrm{F}}_{\mathrm{D}}^{\mathrm{x}} - 1\right| > \tfrac{1}{2}\right\},$$

$$\leq \mathbb{P}\left\{\left|\widehat{\mathrm{F}}_{\mathrm{D}}^{\mathrm{x}} - \mathbb{E}\left[\widehat{\mathrm{F}}_{\mathrm{D}}^{\mathrm{x}}\right]\right| > \tfrac{1}{2}\right\},$$

for$\mathbb{E}\left[\widehat{\mathrm{F}}_{\mathrm{D}}^{\mathrm{x}}\right] = 1$, we apply the result of **Lemma4.1**we show that:

$$\mathbb{P}\left\{\left|\widehat{\mathrm{F}}_{\mathrm{D}}^{\mathrm{x}}\right| < \frac{1}{2}\right\} \leq \infty.$$

*Proof:***Lemma 4.2***:* The proof is the same method of **Lemma 4.1**but we replace χ(., .) in equation (10) by:

$$\chi(\mathrm{X_i}, \mathrm{Y_i}) = K(h_K^{-1}d(x,X_i)) - \mathbb{E}[K_1], \qquad \forall X_i \in \mathcal{H} \tag{18}$$

Proof: **Lemma 4.3***:* We have,

$$\mathbb{E}\left[\hat{f}_N^x(y)\right] - f^x(y) = \frac{1}{nh_H\mathbb{E}[K_1(x)]}\sum_{i=1}^n \mathbb{E}[K_i(x)H_i(y)] - h_H f^X(y)$$

$$= \frac{1}{h_H\mathbb{E}[K_1(x)]}\ \mathbb{E}[K_1(x)H_1(y) - h_H f^X(y)]$$

$$= \frac{1}{h_H\mathbb{E}[K_1(x)]}\mathbb{E}(K_1[\mathbb{E}(H_1(y)/X) - h_H f^X(y)]),$$

using the stationarity of the observations, the conditioning by the explanatory variable and the usual change of variable $t = \frac{y-u}{h_H}$, we obtain:

$$\mathbb{E}(H'(h_H^{-1}(y - Y_i))/X) = \int_{-\infty}^{+\infty} H'(h_H^{-1}(y - Y_i))\, f^X(u)du$$
$$= \int_{-\infty}^{+\infty} H'(h_H^{-1}(y - Y_i))\, f^X(u)du$$
$$= h_H \int_{-\infty}^{+\infty} H'(t)\, f^X(y - h_H)dt.$$

and we deduce,

$$|\mathbb{E}(H(h_H^{-1}(y-Y_i))/X)-f^X(y)| = h_H\int_{-\infty}^{+\infty} H'(t)\,|f^X(y-h_Ht)-f^X(y)|dt,$$

under**(H3)**: $\forall\, y \in S$,

$$|\mathbb{E}(H(h_H^{-1}(y-Y_i))/X)-h_Hf^X(y)| \leq A_X\int_{-\infty}^{+\infty} H'(h_K^{b_1}+|t|h_H^{b_2})\,dt.$$

Hypothesis **(H4)** and **Corollary 4.1** completes the proof of **Lemma 4.2**.

5. CONCLUSION

In this paper, we established the consistency properties (with rates) of the conditional density function in a scalar response variable given a random variable taking values in a separable real Hilbert space when the observations are quasi-associated dependent; the pointwise almost complete convergence (with rates) of the kernel estimate of this model is obtained.

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