ORIGINAL PAPER

NEIGHBORHOOD PROPERTIES OF CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS DEFINED BY GENERALIZED MITTAG-LEFFLER FUNCTION

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Abstract. *In this paper, we introduce new subclasses* $\mathcal{M}_{\lambda,\mu}^{n}(\alpha,\beta,\gamma)$ *and* $\mathcal{R}_{\lambda,\mu}^{n}(\alpha,\beta,\gamma;\vartheta)$ *of analytic functions in the open unit disk* $\mathcal{U}$ *with negative coefficients defined by generalized Mittag-Leffler function. The object of the present paper is to determine coefficient inequalities, inclusion relations and neighborhoods properties for functions* $f(z)$ *belonging to these subclasses.*

Keywords: *Analytic function; starlike and convex functions; Mittag-Leffler function; neighborhoods; coefficient inequality.*

1. INTRODUCTION

Let $\mathcal{A}$ be a class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Denote by $\mathrm{T}(n)$ the class of functions consisting of functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0) \tag{2}$$

which are analytic in $\mathcal{U}$.

We recall that the convolution (or Hadamard product) of two functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

is given by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z), \quad (z \in \mathcal{U}).$$

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Note that $f * g \in \mathcal{A}$.

Next, following the earlier investigations by Altıntaş et al. [1, 2], Goodman [3], Ruscheweyh [4], Silverman [5], and Srivastava and Bulut [6] (see also [7-17]), we define the (n,δ) −neighborhood of a function $f \in \mathrm{T}(n)$ by

$$\mathcal{N}_{n,\delta}(f) = \left\{ g \in \mathrm{T}(n) : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \text{ and } \sum_{n=2}^{\infty} n|a_n - b_n| \leq \delta \right\}. \tag{3}$$

For $e(z) = z$, we have

$$\mathcal{N}_{n,\delta}(e) = \left\{ g \in \mathrm{T}(n) : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \text{ and } \sum_{n=2}^{\infty} n|b_n| \leq \delta \right\}. \tag{4}$$

A function $f \in \mathrm{T}(n)$ is α −starlike of complex order γ, denoted by $f \in \mathcal{S}_n^*(\alpha,\gamma)$ if it satisfies the following condition

$$Re\left\{1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > \alpha, \qquad (\gamma \in \mathbb{C}\backslash\{0\}, 0 \leq \alpha < 1, z \in \mathcal{U})$$

and a function $f \in \mathrm{T}(n)$ is α −convex of complex order γ, denoted by $f \in \mathcal{C}_n(\alpha,\gamma)$ if it satisfies the following condition

$$Re\left\{1 + \frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right\} > \alpha, \qquad (\gamma \in \mathbb{C}\backslash\{0\}, 0 \leq \alpha < 1, z \in \mathcal{U}).$$

The Mittag-Leffler [18] function $E_\lambda(z)$, defined by

$$E_\lambda(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n + 1)}, \qquad (\lambda \in \mathbb{C}, Re(\lambda) > 0, z \in \mathcal{U}). \tag{5}$$

Prabhakar [19] considered a new Mittag-Leffler type function $E_{\lambda,\mu}^{\beta}(z)$ of the form

$$E_{\lambda,\mu}^{\beta}(z) = \sum_{n=0}^{\infty} \frac{(\beta)_n}{\Gamma(\lambda n + \mu)} \frac{z^n}{n!}, (z \in \mathcal{U}), \tag{6}$$

where $\lambda, \mu, \beta \in \mathbb{C}$, $Re(\lambda) > 0$, $Re(\mu) > 0$, $Re(\beta) > 0$ and

$$(\beta)_n = \begin{cases} 1, & n = 0 \\ \beta(\beta+1)\cdots(\beta+n-1), & n \in \mathbb{N} \end{cases}$$

is the well-known Pochhammer symbol.

Note that $E_\lambda(z) = E_{\lambda,1}^{1}(z)$ and $E_{\lambda,\mu}(z) = E_{\lambda,\mu}^{1}(z)$.

The generalized Mittag-Leffler function $E_{\lambda,\mu}^{\beta}(z)$ does not belong to the class $\mathcal{A}$. Therefore, we consider the following normalization for the function $E_{\lambda,\mu}^{\beta}(z)$:

$$\tilde{E}_{\lambda,\mu}^{\beta}(z) = \Gamma(\mu) z E_{\lambda,\mu}^{\beta}(z) = z + \sum_{n=1}^{\infty} \frac{\Gamma(\mu)(\beta)_n}{\Gamma(\lambda n + \mu)} \frac{z^{n+1}}{n!}, (z \in \mathcal{U}). \tag{7}$$

In terms of Hadamard product and $E_{\lambda,\mu}^{\beta}(z)$ given by (1.7), a new operator $\varepsilon_{\lambda,\mu}^{\beta}: \mathcal{A} \to \mathcal{A}$ can be defined as follows:

$$\varepsilon_{\lambda,\mu}^{\beta} f(z) = \left(\tilde{E}_{\lambda,\mu}^{\beta} * f\right)(z) = z + \sum_{n=1}^{\infty} \frac{\Gamma(\mu)(\beta)_n a_{n+1}}{\Gamma(\lambda n + \mu)} \frac{z^{n+1}}{n!}, (z \in \mathcal{U}). \tag{8}$$

If $f \in \mathrm{T}(n)$ is given by (1.2), then we have

$$\varepsilon_{\lambda,\mu}^{\beta} f(z) = z - \sum_{n=1}^{\infty} \frac{\Gamma(\mu)(\beta)_n a_{n+1}}{\Gamma(\lambda n + \mu)} \frac{z^{n+1}}{n!}, (z \in \mathcal{U}). \tag{9}$$

Finally, by using the differential operator defined by (1.9), we investigate the subclasses $\mathcal{M}_{\lambda,\mu}^{n}(\alpha,\beta,\gamma)$ and $\mathcal{R}_{\lambda,\mu}^{n}(\alpha,\beta,\gamma;\vartheta)$ of $\mathrm{T}(n)$ consisting of functions f as the followings:

However, throughout this paper, we restrict our attention to the case real-valued λ, μ, β with $\lambda > 0$, $\mu > 0$ and $\beta > 0$.

Definition 1.1 The subclass $\mathcal{M}_{\lambda,\mu}^{n}(\alpha,\beta,\gamma)$ of $\mathrm{T}(n)$ is defined as the class of functions f such that

$$\left|\frac{1}{\gamma}\left(\frac{z\left[\varepsilon_{\lambda,\mu}^{\beta} f(z)\right]'}{\varepsilon_{\lambda,\mu}^{\beta} f(z)} - 1\right)\right| < \alpha, (z \in \mathcal{U}), \tag{10}$$

where $\gamma \in \mathbb{C}\backslash\{0\}$ and $0 \le \alpha < 1$.

Definition 1.2 Let $\mathcal{R}_{\lambda,\mu}^{n}(\alpha,\beta,\gamma;\vartheta)$ denote the subclass of $\mathrm{T}(n)$ consisting of f which satisfy the inequality

$$\left|\frac{1}{\gamma}\left[(1-\vartheta)\frac{\varepsilon_{\lambda,\mu}^{\beta} f(z)}{z} + \vartheta\left(\varepsilon_{\lambda,\mu}^{\beta} f(z)\right)' - 1\right]\right| < \alpha, \tag{11}$$

where $\gamma \in \mathbb{C}\backslash\{0\}$, $0 \le \alpha < 1$ and $0 \le \vartheta \le 1$.

In this paper, we obtain the coefficient inequalities, inclusion relations and neighborhood properties of the subclasses $\mathcal{M}_{\lambda,\mu}^{n}(\alpha,\beta,\gamma)$ and $\mathcal{R}_{\lambda,\mu}^{n}(\alpha,\beta,\gamma;\vartheta)$.

2. COEFFICIENT INEQUALITIES FOR $\mathcal{M}_{\lambda,\mu}^{n}(\alpha,\beta,\gamma)$ AND $\mathcal{R}_{\lambda,\mu}^{n}(\alpha,\beta,\gamma;\vartheta)$

Theorem 2.1 Let $f \in \mathrm{T}(n)$. Then $f \in \mathcal{M}_{\lambda,\mu}^{n}(\alpha,\beta,\gamma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{\Gamma(\mu)(\beta)_{n-1}}{\Gamma(\lambda(n-1)+\mu)(n-1)!}[n-1+\alpha|\gamma|]a_n \leq \alpha|\gamma| \tag{12}$$

for $\gamma \in \mathbb{C}\backslash\{0\}$ and $0 \leq \alpha < 1$.

Proof: Let $f \in \mathrm{T}(n)$. Then, by *(1.10)* we can write

$$Re\left\{\frac{z\left[\varepsilon_{\lambda,\mu}^{\beta} f(z)\right]'}{\varepsilon_{\lambda,\mu}^{\beta} f(z)} - 1\right\} > -\alpha|\gamma|, (z \in \mathcal{U}). \tag{13}$$

Using (1.2) and (1.9), we have,

$$Re\left\{\frac{-\sum_{n=2}^{\infty} \frac{\Gamma(\mu)(\beta)_{n-1}}{\Gamma(\lambda(n-1)+\mu)(n-1)!}[n-1]a_n z^n}{z-\sum_{n=2}^{\infty} \frac{\Gamma(\mu)(\beta)_{n-1}}{\Gamma(\lambda(n-1)+\mu)(n-1)!}a_n z^n}\right\} > -\alpha|\gamma|, (z \in \mathcal{U}). \tag{14}$$

Since (2.3) is true for all $z \in \mathcal{U}$, choose values of z on the real axis. Letting $z \to 1$, through the real values, the inequality (2.3) yields the desired inequality

$$\sum_{n=2}^{\infty} \frac{\Gamma(\mu)(\beta)_{n-1}}{\Gamma(\lambda(n-1)+\mu)(n-1)!}[n-1+\alpha|\gamma|]a_n \leq \alpha|\gamma|.$$

Conversely, supposed that the inequality (2.1) holds true and $|z| = 1$, then we obtain

$$\left|\frac{z\left[\varepsilon_{\lambda,\mu}^{\beta} f(z)\right]'}{\varepsilon_{\lambda,\mu}^{\beta} f(z)} - 1\right| \leq \left|\frac{\sum_{n=2}^{\infty} \frac{\Gamma(\mu)(\beta)_{n-1}}{\Gamma(\lambda(n-1)+\mu)(n-1)!}[n-1]a_n z^n}{z-\sum_{n=2}^{\infty} \frac{\Gamma(\mu)(\beta)_{n-1}}{\Gamma(\lambda(n-1)+\mu)(n-1)!}a_n z^n}\right|$$

$$\leq \frac{\sum_{n=2}^{\infty} \frac{\Gamma(\mu)(\beta)_{n-1}}{\Gamma(\lambda(n-1)+\mu)(n-1)!}[n-1]a_n}{1-\sum_{n=2}^{\infty} \frac{\Gamma(\mu)(\beta)_{n-1}}{\Gamma(\lambda(n-1)+\mu)(n-1)!}a_n}$$
$$\leq \alpha|\gamma|.$$

Hence, by the maximum modulus theorem, we have $f(z) \in \mathcal{M}_{\lambda,\mu}^{n}(\alpha,\beta,\gamma)$, which establishes the required result.

Theorem 2.2 Let $f \in \mathrm{T}(n)$. Then $f \in \mathcal{R}_{\lambda,\mu}^{n}(\alpha,\beta,\gamma;\vartheta)$ if and only if

$$\sum_{n=2}^{\infty} \frac{\Gamma(\mu)(\beta)_{n-1}}{\Gamma(\lambda(n-1)+\mu)(n-1)!}[1+\vartheta(n-1)]a_n \leq \alpha|\gamma| \tag{15}$$

for $\gamma \in \mathbb{C}\backslash\{0\}$, $0 \leq \alpha < 1$ and $0 \leq \vartheta \leq 1$.

Proof: We omit the proofs since it is similar to Theorem 2.1.

3. INCLUSION RELATIONS INVOLVING $\boldsymbol{\mathcal{N}_{n,\delta}(e)}$ OF $\boldsymbol{\mathcal{M}_{\lambda,\mu}^{n}(\alpha,\beta,\gamma)}$ AND $\boldsymbol{\mathcal{R}_{\lambda,\mu}^{n}(\alpha,\beta,\gamma;\vartheta)}$

Theorem 3.1 *If*

$$\delta = \frac{2\alpha|\gamma|\Gamma(\lambda+\mu)}{\beta(1+\alpha|\gamma|)\Gamma(\mu)}, (|\gamma| < 1), \tag{16}$$

then $\mathcal{M}_{\lambda,\mu}^{n}(\alpha,\beta,\gamma) \subset \mathcal{N}_{n,\delta}(e)$.

Proof: Let $f(z) \in \mathcal{M}_{\lambda,\mu}^{n}(\alpha,\beta,\gamma)$. By Theorem 2.1, we have

$$\frac{\beta\Gamma(\mu)}{\Gamma(\lambda+\mu)}(1+\alpha|\gamma|)\sum_{n=2}^{\infty} a_n \leq \alpha|\gamma|,$$

which implies

$$\sum_{n=2}^{\infty} a_n \leq \frac{\alpha|\gamma|}{\frac{\beta\Gamma(\mu)}{\Gamma(\lambda+\mu)}(1+\alpha|\gamma|)}. \tag{17}$$

Using (2.1) and (3.2), we get

$$\frac{\beta\Gamma(\mu)}{\Gamma(\lambda+\mu)}\sum_{n=2}^{\infty} na_n \leq \alpha|\gamma| + \frac{\beta\Gamma(\mu)}{\Gamma(\lambda+\mu)}(1-\alpha|\gamma|)\sum_{n=2}^{\infty} a_n$$

$$\leq \frac{2\alpha|\gamma|}{(1+\alpha|\gamma|)} = \delta.$$

That is,

$$\sum_{n=2}^{\infty} na_n \leq \frac{2\alpha|\gamma|}{\frac{\beta\Gamma(\mu)}{\Gamma(\lambda+\mu)}(1+\alpha|\gamma|)} = \delta.$$

Thus, by the definition given by (1.4), $f(z) \in \mathcal{N}_{n,\delta}(e)$, which completes the proof.

Theorem 3.2 If

$$\delta = \frac{2\alpha|\gamma|\Gamma(\lambda+\mu)}{\beta(1+\vartheta)\Gamma(\mu)}, (|\gamma| < 1), \tag{18}$$

then $\mathcal{R}_{\lambda,\mu}^{n}(\alpha,\beta,\gamma;\vartheta) \subset \mathcal{N}_{n,\delta}(e)$.

Proof: For $f(z) \in \mathcal{R}_{\lambda,\mu}^{n}(\alpha,\beta,\gamma;\vartheta)$ and making use of the condition (2.4), we obtain

$$\frac{\beta\Gamma(\mu)}{\Gamma(\lambda+\mu)}(1+\vartheta)\sum_{n=2}^{\infty} a_n \le \alpha|\gamma|$$

so that

$$\sum_{n=2}^{\infty} a_n \le \frac{\alpha|\gamma|}{\frac{\beta\Gamma(\mu)}{\Gamma(\lambda+\mu)}(1+\vartheta)}. \tag{19}$$

Thus, using (2.4) along with (3.4), we also get

$$\vartheta\frac{\beta\Gamma(\mu)}{\Gamma(\lambda+\mu)}\sum_{n=2}^{\infty} na_n \le \alpha|\gamma| + (\vartheta-1)\frac{\Gamma(\mu)(\beta)}{\Gamma(\lambda+\mu)}\sum_{n=2}^{\infty} a_n$$

$$\le \alpha|\gamma| + \frac{\beta(\vartheta-1)\Gamma(\mu)}{\Gamma(\lambda+\mu)}\frac{\alpha|\gamma|\Gamma(\lambda+\mu)}{\beta(\vartheta+1)\Gamma(\mu)}$$

$$\le \frac{2\alpha|\gamma|}{(1+\vartheta)} = \delta.$$

Hence,

$$\sum_{n=2}^{\infty} na_n \le \frac{2\alpha|\gamma|}{\frac{\beta\Gamma(\mu)}{\Gamma(\lambda+\mu)}(1+\vartheta)} = \delta$$

which in view of (1.4), completes the proof of theorem.

4. NEIGHBORHOOD PROPERTIES FOR THE CLASSES $\mathcal{M}_{\lambda,\mu}^{\eta}(\alpha,\beta,\gamma)$ AND $\mathcal{R}_{\lambda,\mu}^{\eta}(\alpha,\beta,\gamma;\vartheta)$

Definition 4.1 For $0 \le \eta < 1$ and $z \in \mathcal{U}$, a function $f(z) \in \mathcal{M}_{\lambda,\mu}^{\eta}(\alpha,\beta,\gamma)$ if there exists a function $g(z) \in \mathcal{M}_{\lambda,\mu}^{n}(\alpha,\beta,\gamma)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1-\eta. \tag{20}$$

For $0 \le \eta < 1$ and $z \in \mathcal{U}$, a function $f(z) \in \mathcal{R}_{\lambda,\mu}^{\eta}(\alpha,\beta,\gamma;\vartheta)$ if there exists a function $g(z) \in \mathcal{R}_{\lambda,\mu}^{\eta}(\alpha,\beta,\gamma;\vartheta)$ such that the inequality (4.1) holds true.

Theorem 4.1 If $g(z) \in \mathcal{M}_{\lambda,\mu}^{\eta}(\alpha,\beta,\gamma)$ and

$$\eta = 1 - \frac{\beta\delta(1+\alpha|\gamma|)\Gamma(\mu)}{2[\beta(1+\alpha|\gamma|)\Gamma(\mu) - \alpha|\gamma|\Gamma(\lambda+\mu)]}, \tag{21}$$

then $\mathcal{N}_{n,\delta}(g) \subset \mathcal{M}_{\lambda,\mu}^{\eta}(\alpha,\beta,\gamma)$.

Proof: Let $f(z) \in \mathcal{N}_{n,\delta}(g)$. Then,

$$\sum_{n=2}^{\infty} n|a_n - b_n| \leq \delta, \tag{22}$$

which yields the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n - b_n| \leq \frac{\delta}{2}, \qquad (n \in \mathbb{N}).$$

Since $g(z) \in \mathcal{M}_{\lambda,\mu}^{\eta}(\alpha,\beta,\gamma)$ by (3.2), we have

$$\sum_{n=2}^{\infty} b_n \leq \frac{\alpha|\gamma|}{\frac{\beta\Gamma(\mu)}{\Gamma(\lambda+\mu)}(1+\alpha|\gamma|)}, \tag{23}$$

and so

$$\begin{aligned}\left|\frac{f(z)}{g(z)} - 1\right| &< \frac{\sum_{n=2}^{\infty} |a_n - b_n|}{1 - \sum_{n=2}^{\infty} b_n} \\ &\leq \frac{\delta}{2}\frac{\frac{\beta\Gamma(\mu)}{\Gamma(\lambda+\mu)}(1+\alpha|\gamma|)}{\frac{\beta\Gamma(\mu)}{\Gamma(\lambda+\mu)}(1+\alpha|\gamma|) - \alpha|\gamma|} \\ &= 1 - \eta.\end{aligned}$$

Thus, by definition, $f(z) \in \mathcal{M}_{\lambda,\mu}^{\eta}(\alpha,\beta,\gamma)$ for η given by (4.2), which establishes the desired result.

Theorem 4.2 If $g(z) \in \mathcal{R}_{\lambda,\mu}^{\eta}(\alpha,\beta,\gamma;\vartheta)$ and

$$\eta = 1 - \frac{\beta\delta(1+\vartheta)\Gamma(\mu)}{2[\beta(1+\vartheta)\Gamma(\mu) - \alpha|\gamma|\Gamma(\lambda+\mu)]}, \tag{24}$$

then $\mathcal{N}_{n,\delta}(g) \subset \mathcal{R}_{\lambda,\mu}^{\eta}(\alpha,\beta,\gamma;\vartheta)$.

Proof: We omit the proofs since it is similar to Theorem 4.1.

5. CONCLUSION

In this study, we obtain the coefficient inequalities, inclusion relations and neighborhood properties of the subclasses $\mathcal{M}_{\lambda,\mu}^{n}(\alpha,\beta,\gamma)$ and $\mathcal{R}_{\lambda,\mu}^{n}(\alpha,\beta,\gamma;\vartheta)$.

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