ORIGINAL PAPER

A STUDY ON BINOMIAL TRANSFORM OF THE GENERALIZED PADOVAN SEQUENCE

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Abstract. *In this paper, we define the binomial transform of the generalized Padovan sequence and as special cases, the binomial transform of the Padovan, Perrin, Padovan-Perrin, modified Padovan, adjusted Padovan sequences will be introduced. We investigate their properties in details.*

Keywords: *Binomial transform; Padovan sequence; Perrin sequence; binomial transform of Padovan sequence; generalized Tribonacci sequence.*

1. INTRODUCTION AND PRELIMINARIES

The sequence of Fibonacci numbers $\{F_n\}$ is defined by

$$F_n = F_{n-1} + F_{n-2}, n \geq 2, F_0 = 0, F_1 = 1.$$

Fibonacci numbers has delighted mathematicians and amateurs alike for centuries with their beauty and over the past centuries, there has been a strong interest in the research related to Fibonacci sequence. It is well known that Fibonacci sequence and its generalizations play an important role in many research area such as mathematics, physics, computer science, biology, statistics, engineering, architecture, nature and art.

In this paper, we introduce the binomial transform of the generalized Padovan sequence and we investigate, in detail, five special cases which we call them the binomial transform of the Padovan, Perrin, Padovan-Perrin, modified Padovan, adjusted Padovan sequences. We investigate their properties in the next sections. In this section, we present some properties of the generalized Tribonacci sequence which is a generalization of Fibonacci sequence.

The generalized Tribonacci sequence

$$\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$$

(or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, n \geq 3 \tag{1}$$

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where W_0, W_1, W_2 are arbitrary complex (or real) numbers and r,s,t are real numbers.

This sequence has been studied by many authors, see for example [1-13].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1,2,3, \ldots$ when $t \neq 0$. Therefore, recurrence (1) holds for all integer n.

As $\{W_n\}$ is a third order recurrence sequence (difference equation), it's characteristic equation is

$$x^3 - rx^2 - sx - t = 0 \tag{2}$$

whose roots are

$$\alpha = \alpha(r,s,t) = \frac{r}{3} + A + B,$$
$$\beta = \beta(r,s,t) = \frac{r}{3} + \omega A + \omega^2 B,$$
$$\gamma = \gamma(r,s,t) = \frac{r}{3} + \omega^2 A + \omega B$$

where

$$A = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{\frac{1}{3}},$$
$$B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}\right)^{\frac{1}{3}},$$
$$\Delta = \Delta(r,s,t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4},$$
$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp\left(\frac{2\pi i}{3}\right).$$

Note that we have the following identities

$$\alpha + \beta + \gamma = r,$$
$$\alpha\beta + \alpha\gamma + \beta\gamma = -s,$$
$$\alpha\beta\gamma = t.$$

If $\Delta(r,s,t) > 0$, then the Equ. (2) has one real (α) and two non-real solutions with the latter being conjugate complex. So, in this case, it is well known that the generalized Tribonacci numbers can be expressed, for all integers n, using Binet's formula

$$W_n = \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{3}$$

where

$$p_1 = W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0,$$
$$p_2 = W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0,$$
$$p_3 = W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0.$$

(3) can be written in the following form:

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n$$

where

$$A_1 = \frac{W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0}{(\alpha-\beta)(\alpha-\gamma)},$$
$$A_2 = \frac{W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0}{(\beta-\alpha)(\beta-\gamma)},$$
$$A_3 = \frac{W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0}{(\gamma-\alpha)(\gamma-\beta)}.$$

Note that the Binet form of a sequence satisfying (2) for non-negative integers is valid for all integers n, for a proof of this result see [14]. This result of Howard and Saidak [14] is even true in the case of higher-order recurrence relations.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

Lemma 1. [11] Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized Tribonacci sequence) $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2}{1 - rx - sx^2 - tx^3}. \tag{4}$$

We next find Binet's formula of the generalized Tribonacci sequence $\{W_n\}$ by the use of generating function for W_n.

Theorem 1. [11] (Binet's formula of the generalized Tribonacci numbers) For all integers n, we have

$$W_n = \frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{5}$$

where

$$q_1 = W_0\alpha^2 + (W_1 - rW_0)\alpha + (W_2 - rW_1 - sW_0),$$
$$q_2 = W_0\beta^2 + (W_1 - rW_0)\beta + (W_2 - rW_1 - sW_0),$$
$$q_3 = W_0\gamma^2 + (W_1 - rW_0)\gamma + (W_2 - rW_1 - sW_0).$$

Note that from (3) and (5) we have

$$W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0 = W_0\alpha^2 + (W_1 - rW_0)\alpha + (W_2 - rW_1 - sW_0),$$
$$W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0 = W_0\beta^2 + (W_1 - rW_0)\beta + (W_2 - rW_1 - sW_0),$$
$$W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0 = W_0\gamma^2 + (W_1 - rW_0)\gamma + (W_2 - rW_1 - sW_0).$$

In this paper, we consider the case $r = 0$, $s = t = 1$ and in this case we write $V_n = W_n$. So, the generalized Padovan sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = V_{n-2} + V_{n-3} \tag{6}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} + V_{-(n-3)}$$

for $n = 1,2,3,\ldots$. Therefore, recurrence (6) holds for all integer n.

(3) can be used to obtain Binet's formula of generalized Padovan numbers. Binet's formula of generalized Padovan numbers can be given as

$$V_n = \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{pb_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$p_1 = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0 = V_0\alpha^2 + V_1\alpha + (V_2 - V_0) = q_1, \tag{7}$$

$$p_2 = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0 = V_0\beta^2 + V_1\beta + (V_2 - V_0) = q_2, \tag{8}$$

$$p_3 = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0 = V_0\gamma^2 + V_1\gamma + (V_2 - V_0) = q_3. \tag{9}$$

Here, α, β and γ are the roots of the cubic equation $x^3 - x - 1 = 0$. Moreover

$$\alpha = \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} = 1.324\,717\,957\,24,$$

$$\beta = \omega\left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \omega^2\left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3},$$

$$\gamma = \omega^2\left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \omega\left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\alpha + \beta + \gamma = 0,$$
$$\alpha\beta + \alpha\gamma + \beta\gamma = -1,$$
$$\alpha\beta\gamma = 1.$$

Now, we present five special cases of the generalized Padovan sequence $\{V_n\}$. Padovan sequence $\{P_n\}_{n\geq 0}$, Perrin sequence $\{E_n\}_{n\geq 0}$, Padovan-Perrin sequence $\{S_n\}_{n\geq 0}$, modified Padovan sequence $\{A_n\}_{n\geq 0}$, adjusted Padovan sequence $\{U_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$P_{n+3} = P_{n+1} + P_n, P_0 = 1, P_1 = 1, P_2 = 1, \tag{10}$$

$$E_{n+3} = E_{n+1} + E_n, E_0 = 3, E_1 = 0, E_2 = 2, \tag{11}$$

$$S_{n+3} = S_{n+1} + S_n, S_0 = 0, S_1 = 0, S_2 = 1, \tag{12}$$

$$A_{n+3} = A_{n+1} + A_n, A_0 = 3, A_1 = 1, A_2 = 3, \tag{13}$$

$$U_{n+3} = U_{n+1} + U_n, U_0 = 0, U_1 = 1, U_2 = 0. \tag{14}$$

Note that the case $V_n = R_n$, $R_0 = 1, R_1 = 0, R_2 = 1$ (or $V_n = R_n$, $R_0 = 0, R_1 = 1, R_2 = 0$) is called the sequence of the Van der Laan numbers, in the literature.

The sequences $\{P_n\}_{n\geq 0}$, $\{E_n\}_{n\geq 0}$, $\{S_n\}_{n\geq 0}$ and $\{A_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}
P_{-n} &= -P_{-(n-1)} + P_{-(n-3)}, \\
E_{-n} &= -E_{-(n-1)} + E_{-(n-3)}, \\
S_{-n} &= -S_{-(n-1)} + S_{-(n-3)}, \\
A_{-n} &= -A_{-(n-1)} + A_{-(n-3)}, \\
U_{-n} &= -U_{-(n-1)} + U_{-(n-3)}.
\end{aligned}$$

for $n = 1,2,3,\ldots$ respectively. Therefore, recurrences (10)-(14) hold for all integer n.

For more details on the generalized Padovan numbers, see Soykan [15]. For all integers n, Padovan, Perrin, Padovan-Perrin, modified Padovan, adjusted Padovan numbers (using initial conditions in (7)-(9)) can be expressed using Binet's formulas as

$$\begin{aligned}
P_n &= \frac{\alpha^{n+4}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+4}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+4}}{(\gamma-\alpha)(\gamma-\beta)}, \\
E_n &= \alpha^n + \beta^n + \gamma^n, \\
S_n &= \frac{\alpha^{n}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n}}{(\gamma-\alpha)(\gamma-\beta)}, \\
A_n &= \frac{(3\alpha+1)\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(3\beta+1)\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{(3\gamma+1)\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \\
U_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},
\end{aligned}$$

respectively, see, Soykan [15] for more details.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} V_n x^n$ of the generalized Padovan sequence V_n (see, Soykan [15] for more details.).

Lemma 2. Suppose that $f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized Padovan sequence $\{V_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} V_n x^n$ is given by

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + V_1 x + (V_2 - V_0)x^2}{1 - x^2 - x^3}. \tag{15}$$

Proof: Take $r = 0$, $s = t = 1$ in Lemma 1.

The previous lemma gives the following results as particular examples.

Corollary 1. Generating functions of Padovan, Perrin, Padovan-Perrin modified Padovan and adjusted Padovan numbers are

$$\sum_{n=0}^{\infty} P_n x^n = \frac{1 + x}{1 - x^2 - x^3},$$

$$\sum_{n=0}^{\infty} E_n x^n = \frac{3-x^2}{1-x^2-x^3},$$
$$\sum_{n=0}^{\infty} S_n x^n = \frac{x^2}{1-x^2-x^3},$$
$$\sum_{n=0}^{\infty} A_n x^n = \frac{3+x}{1-x^2-x^3},$$
$$\sum_{n=0}^{\infty} U_n x^n = \frac{x}{1-x^2-x^3},$$

respectively.

2. BINOMIAL TRANSFORM OF THE GENERALIZED PADOVAN SEQUENCE V_n

In [16, p. 137], Knuth introduced the idea of the binomial transform. Given a sequence of numbers (a_n), its binomial transform $(\hat{a}_n)$ may be defined by the rule

$$\hat{a}_n = \textstyle\sum_{i=0}^{n} \binom{n}{i} a_i, \text{ with inversion } a_n = \textstyle\sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \hat{a}_i,$$

or, in the symmetric version

$$\hat{a}_n = \textstyle\sum_{i=0}^{n} \binom{n}{i} (-1)^{i+1} a_i, \text{ with inversion } a_n = \textstyle\sum_{i=0}^{n} \binom{n}{i} (-1)^{i+1} \hat{a}_i.$$

For more information on binomial transform, see, for example [17-20] and references therein. In this section, we define the binomial transform of the generalized Padovan sequence V_n and as special cases the binomial transform of the Padovan, Perrin, Padovan-Perrin, modified Padovan, adjusted Padovan sequences will be introduced.

Definition 1. The binomial transform of the generalized Padovan sequence V_n is defined by

$$b_n = \hat{V}_n = \sum_{i=0}^{n} \binom{n}{i} V_i.$$

The few terms of b_n are

$$b_0 = \sum_{i=0}^{0} \binom{0}{i} V_i = V_0,$$
$$b_1 = \sum_{i=0}^{1} \binom{1}{i} V_i = V_0 + V_1,$$
$$b_2 = \sum_{i=0}^{2} \binom{2}{i} V_i = V_0 + 2V_1 + V_2.$$

Translated to matrix language, b_n has the nice (lower-triangular matrix) form

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ \vdots \end{pmatrix}.$$

As special cases of $b_n = \widehat{V}_n$, the binomial transforms of the Padovan, Perrin, Padovan-Perrin, modified Padovan, adjusted Padovan sequences are defined as follows: The binomial transform of the Padovan sequence P_n is

$$\widehat{P}_n = \sum_{i=0}^{n} \binom{n}{i} P_i,$$

the binomial transform of the Perrin sequence E_n is

$$\widehat{E}_n = \sum_{i=0}^{n} \binom{n}{i} E_i,$$

the binomial transform of the Padovan-Perrin sequence S_n is

$$\hat{S}_n = \sum_{i=0}^{n} \binom{n}{i} S_i,$$

the binomial transform of the modified Padovan sequence A_n is

$$\hat{A}_n = \sum_{i=0}^{n} \binom{n}{i} A_i,$$

the binomial transform of the adjusted Padovan sequence U_n is

$$\widehat{U}_n = \sum_{i=0}^{n} \binom{n}{i} U_i.$$

Lemma 3. For $n \geq 0$, the binomial transform of the generalized Padovan sequence V_n satisfies the following relation:

$$b_{n+1} = \sum_{i=0}^{n} \binom{n}{i} (V_i + V_{i+1}).$$

Proof: We use the following well-known identity:

$$\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}.$$

Note also that

$$\binom{n+1}{0} = \binom{n}{0} = 1 \text{ and } \binom{n}{n+1} = 0.$$

Then

$$\begin{aligned} b_{n+1} &= V_0 + \sum_{i=1}^{n+1} \binom{n+1}{i} V_i \\ &= V_0 + \sum_{i=1}^{n+1} \binom{n}{i} V_i + \sum_{i=1}^{n+1} \binom{n}{i-1} V_i \\ &= V_0 + \sum_{i=1}^{n} \binom{n}{i} V_i + \sum_{i=0}^{n} \binom{n}{i} V_{i+1} \\ &= \sum_{i=0}^{n} \binom{n}{i} V_i + \sum_{i=0}^{n} \binom{n}{i} V_{i+1} \\ &= \sum_{i=0}^{n} \binom{n}{i} (V_i + V_{i+1}). \end{aligned}$$

This completes the Proof:

Remark 1. From the last Lemma, we see that

$$b_{n+1} = b_n + \sum_{i=0}^{n} \binom{n}{i} V_{i+1}.$$

The following theorem gives recurrent relations of the binomial transform of the generalized Padovan sequence.

Theorem 2. For $n \geq 0$, the binomial transform of the generalized Padovan sequence V_n satisfies the following recurrence relation:

$$b_{n+3} = 3b_{n+2} - 2b_{n+1} + b_n. \tag{16}$$

Proof: To show (16), writing

$$b_{n+3} = r_1 \times b_{n+2} + s_1 \times b_{n+1} + t_1 \times b_n$$

and taking the values $n = 0,1,2$ and then solving the system of equations

$$\begin{aligned} b_3 &= r_1 \times b_2 + s_1 \times b_1 + t_1 \times b_0 \\ b_4 &= r_1 \times b_3 + s_1 \times b_2 + t_1 \times b_1 \\ b_5 &= r_1 \times b_4 + s_1 \times b_3 + t_1 \times b_2 \end{aligned}$$

we find that $r_1 = 3, s_1 = -2, t_1 = 1$.

The sequence $\{b_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$b_{-n} = 2b_{-n+1} - 3b_{-n+2} + b_{-n+3}$$

for $n = 1,2,3, \ldots$. Therefore, recurrence (16) holds for all integer n.

Note that the recurence relation (16) is independent from initial values. So,

$$\widehat{P}_{n+3}=3\widehat{P}_{n+2}-2\widehat{P}_{n+1}+\widehat{P}_{n},$$
$$\widehat{E}_{n+3}=3\widehat{E}_{n+2}-2\widehat{E}_{n+1}+\widehat{E}_{n},$$
$$\widehat{S}_{n+3}=3\widehat{S}_{n+2}-2\widehat{S}_{n+1}+\widehat{S}_{n},$$
$$\widehat{A}_{n+3}=3\widehat{A}_{n+2}-2\widehat{A}_{n+1}+\widehat{A}_{n},$$
$$\widehat{U}_{n+3}=3\widehat{U}_{n+2}-2\widehat{U}_{n+1}+\widehat{U}_{n},$$

and

$$\widehat{P}_{-n}=2\widehat{P}_{-n+1}-3\widehat{P}_{-n+2}+\widehat{P}_{-n+3},$$
$$\widehat{E}_{-n}=2\widehat{E}_{-n+1}-3\widehat{E}_{-n+2}+\widehat{E}_{-n+3},$$
$$\widehat{S}_{-n}=2\widehat{S}_{-n+1}-3\widehat{S}_{-n+2}+\widehat{S}_{-n+3},$$
$$\widehat{A}_{-n}=2\widehat{A}_{-n+1}-3\widehat{A}_{-n+2}+\widehat{A}_{-n+3},$$
$$\widehat{U}_{-n}=2\widehat{U}_{-n+1}-3\widehat{U}_{-n+2}+\widehat{U}_{-n+3}.$$

The first few terms of the binomial transform of the generalized Padovan sequence with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few binomial transform (terms) of the generalized Padovan sequence.

n	b_n	b_{-n}
0	V_0	…
1	V_0+V_1	V_2-V_1
2	$V_0+2V_1+V_2$	$2V_2-V_1-2V_0$
3	$2V_0+4V_1+3V_2$	$V_1-3V_0+V_2$
4	$5V_0+9V_1+7V_2$	$4V_1-3V_2$
5	$12V_0+21V_1+16V_2$	$7V_0+4V_1-7V_2$
6	$28V_0+49V_1+37V_2$	$11V_0-3V_1-4V_2$
7	$65V_0+114V_1+86V_2$	$V_0-14V_1+10V_2$
8	$151V_0+265V_1+200V_2$	$25V_2-15V_1-24V_0$
9	$351V_0+616V_1+465V_2$	$9V_1-40V_0+16V_2$
10	$816V_0+1432V_1+1081V_2$	$49V_1-7V_0-33V_2$
11	$1897V_0+3329V_1+2513V_2$	$82V_0+56V_1-89V_2$
12	$4410V_0+7739V_1+5842V_2$	$145V_0-26V_1-63V_2$
13	$10\,252V_0+17\,991V_1+13\,581V_2$	$37V_0-171V_1+108V_2$

The first few terms of the binomial transform numbers of the Padovan, Perrin, Padovan-Perrin, modified Padovan, adjusted Padovan sequences with positive subscript and negative subscript are given in the following Table 2.

Table 2. A few binomial transform (terms).

n	0	1	2	3	4	5	6	7	8	9	10
$\widehat{P}_n$	1	2	4	9	21	49	114	265	616	1432	3329
$\widehat{P}_{-n}$		0	−1	−1	1	4	4	−3	−14	−15	9
$\widehat{E}_n$	3	3	5	12	29	68	158	367	853	1983	4610
$\widehat{E}_{-n}$		2	−2	−7	−6	7	25	23	−22	−88	−87
$\widehat{S}_n$	0	0	1	3	7	16	37	86	200	465	1081
$\widehat{S}_{-n}$		1	2	1	−3	−7	−4	10	25	16	−33
$\widehat{A}_n$	3	4	8	19	45	105	244	567	1318	3064	7123
$\widehat{A}_{-n}$		2	−1	−5	−5	4	18	19	−12	−63	−71
$\widehat{U}_n$	0	1	2	4	9	21	49	114	265	616	1432
$\widehat{U}_{-n}$		−1	−1	1	4	4	−3	−14	−15	9	49

(3) can be used to obtain Binet's formula of the binomial transform of generalized Padovan numbers. Binet's formula of the binomial transform of generalized Padovan numbers can be given as

$$b_n = \frac{c_1\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{c_2\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{c_3\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)} \quad (17)$$

where

$$c_1 = b_2 - (\theta_2+\theta_3)b_1 + \theta_2\theta_3 b_0 = (V_0+2V_1+V_2) - (\theta_2+\theta_3)(V_0+V_1) + \theta_2\theta_3 V_0,$$
$$c_2 = b_2 - (\theta_1+\theta_3)b_1 + \theta_1\theta_3 b_0 = (V_0+2V_1+V_2) - (\theta_1+\theta_3)(V_0+V_1) + \theta_1\theta_3 V_0,$$
$$c_3 = b_2 - (\theta_1+\theta_2)b_1 + \theta_1\theta_2 b_0 = (V_0+2V_1+V_2) - (\theta_1+\theta_2)(V_0+V_1) + \theta_1\theta_2 V_0.$$

Here, θ_1, θ_2 and θ_3 are the roots of the cubic equation $x^3-3x^2+2x-1=0$. Moreover,

$$\theta_1 = 1 + \left(\frac{9+\sqrt{69}}{18}\right)^{1/3} + \left(\frac{9-\sqrt{69}}{18}\right)^{1/3},$$
$$\theta_2 = 1 + \omega\left(\frac{9+\sqrt{69}}{18}\right)^{1/3} + \omega^2\left(\frac{9-\sqrt{69}}{18}\right)^{1/3},$$
$$\theta_3 = 1 + \omega^2\left(\frac{9+\sqrt{69}}{18}\right)^{1/3} + \omega\left(\frac{9-\sqrt{69}}{18}\right)^{1/3},$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\theta_1+\theta_2+\theta_3 = 3,$$
$$\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3 = 2,$$
$$\theta_1\theta_2\theta_3 = 1.$$

For all integers n, (Binet's formulas of) binomial transforms of Padovan , Perrin, Padovan-Perrin, modified Padovan, adjusted Padovan numbers (using initial conditions in (17)) can be expressed using Binet's formulas as

$$\hat{P}_n = \frac{(2\theta_1^2-2\theta_1+1)\theta_1^{n-1}}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{(2\theta_2^2-2\theta_2+1)\theta_2^{n-1}}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{(2\theta_3^2-2\theta_3+1)\theta_3^{n-1}}{(\theta_3-\theta_1)(\theta_3-\theta_2)},$$
$$\hat{E}_n = \theta_1^n + \theta_2^n + \theta_3^n,$$
$$\hat{S}_n = \frac{\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)},$$
$$\hat{A}_n = \frac{(4\theta_1^2-4\theta_1+3)\theta_1^{n-1}}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{(4\theta_2^2-4\theta_2+3)\theta_2^{n-1}}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{(4\theta_3^2-4\theta_3+3)\theta_3^{n-1}}{(\theta_3-\theta_1)(\theta_3-\theta_2)},$$
$$\hat{U}_n = \frac{(-1+\theta_1)\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{(-1+\theta_2)\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{(-1+\theta_3)\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)}$$

respectively.

3. GENERATING FUNCTIONS AND OBTAINING BINET FORMULA OF BINOMIAL TRANSFORM FROM GENERATING FUNCTION

The generating function of the binomial transform of the generalized Padovan sequence V_n is a power series centered at the origin whose coefficients are the binomial transform of the generalized Padovan sequence. Next, we give the ordinary generating function $f_{b_n}(x)=\sum_{n=0}^{\infty} b_n x^n$ of the sequence b_n.

Lemma 4. Suppose that $f_{b_n}(x)=\sum_{n=0}^{\infty} b_n x^n$ is the ordinary generating function of the binomial transform of the Padovan sequence $\{V_n\}_{n\geq 0}$. Then, $f_{b_n}(x)$ is given by

$$f_{b_n}(x)=\frac{V_0+(V_1-2V_0)x+(V_2-V_1)x^2}{1-3x+2x^2-x^3}. \tag{18}$$

Proof: Using Lemma 1, we obtain

$$\begin{aligned} f_{b_n}(x) &= \frac{b_0+(b_1-r_1b_0)x+(b_2-r_1b_1-s_1b_0)x^2}{1-r_1x-s_1x^2-t_1x^3} \\ &= \frac{V_0+((V_0+V_1)-3V_0)x+((V_0+2V_1+V_2)-3(V_0+V_1)-(-2)V_0)x^2}{1-3x+2x^2-x^3} \\ &= \frac{V_0+(V_1-2V_0)x+(V_2-V_1)x^2}{1-3x+2x^2-x^3} \end{aligned}$$

where

$$\begin{aligned} b_0 &= V_0, \\ b_1 &= V_0+V_1, \\ b_2 &= V_0+2V_1+V_2. \end{aligned}$$

Note that P. Barry shows in [21] that if $A(x)$ is the generating function of the sequence $\{a_n\}$, then

$$S(x)=\frac{1}{1-x}A(\frac{x}{1-x})$$

is the generating function of the sequence $\{b_n\}$ with $b_n=\sum_{i=0}^{n}\binom{n}{i}a_i$. In our case, since

$$A(x)=\frac{V_0+V_1x+(V_2-V_0)x^2}{1-x^2-x^3}, \quad \text{see}(15),$$

we obtain

$$\begin{aligned} S(x) &= \frac{1}{1-x}\frac{V_0+V_1\left(\frac{x}{1-x}\right)+(V_2-V_0)\left(\frac{x}{1-x}\right)^2}{1-\left(\frac{x}{1-x}\right)^2-\left(\frac{x}{1-x}\right)^3} \\ &= \frac{V_0+(V_1-2V_0)x+(V_2-V_1)x^2}{1-3x+2x^2-x^3}. \end{aligned}$$

The previous lemma gives the following results as particular examples.

Corollary 2. Generating functions of the binomial transform of the Padovan, Perrin, Padovan-Perrin, modified Padovan, adjusted Padovan numbers are

$$\sum_{n=0}^{\infty} \widehat{P}_n x^n = \frac{1-x}{1-3x+2x^2-x^3},$$
$$\sum_{n=0}^{\infty} \widehat{E}_n x^n = \frac{3-6x+2x^2}{1-3x+2x^2-x^3},$$
$$\sum_{n=0}^{\infty} \widehat{S}_n x^n = \frac{x^2}{1-3x+2x^2-x^3},$$
$$\sum_{n=0}^{\infty} \widehat{A}_n x^n = \frac{3-5x+2x^2}{1-3x+2x^2-x^3},$$
$$\sum_{n=0}^{\infty} \widehat{U}_n x^n = \frac{x-x^2}{1-3x+2x^2-x^3},$$

respectively.

We next find Binet's formula of the Binomial transform of the generalized Padovan numbers $\{V_n\}$ by the use of generating function for b_n.

Theorem 3. (Binet's formula of the Binomial transform of the generalized Padovan numbers)

$$b_n = \frac{d_1\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{d_2\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{d_3\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)} \tag{19}$$

where

$$d_1 = V_0\theta_1^2 + (V_1-2V_0)\theta_1 + (V_2-V_1),$$
$$d_2 = V_0\theta_2^2 + (V_1-2V_0)\theta_2 + (V_2-V_1),$$
$$d_3 = V_0\theta_3^2 + (V_1-2V_0)\theta_3 + (V_2-V_1).$$

Proof: By using Lemma 4, the proof follows from Theorem 1.

Note that from (17) and (19), we have

$$b_2 - (\theta_2+\theta_3)b_1 + \theta_2\theta_3 b_0 = V_0\theta_1^2 + (V_1-2V_0)\theta_1 + (V_2-V_1),$$
$$b_2 - (\theta_1+\theta_3)b_1 + \theta_1\theta_3 b_0 = V_0\theta_2^2 + (V_1-2V_0)\theta_2 + (V_2-V_1),$$
$$b_2 - (\theta_1+\theta_2)b_1 + \theta_1\theta_2 b_0 = V_0\theta_3^2 + (V_1-2V_0)\theta_3 + (V_2-V_1),$$

or

$$(V_0+2V_1+V_2) - (\theta_2+\theta_3)(V_0+V_1) + \theta_2\theta_3V_0 = V_0\theta_1^2 + (V_1-2V_0)\theta_1 + (V_2-V_1),$$
$$(V_0+2V_1+V_2) - (\theta_1+\theta_3)(V_0+V_1) + \theta_1\theta_3V_0 = V_0\theta_2^2 + (V_1-2V_0)\theta_2 + (V_2-V_1),$$
$$(V_0+2V_1+V_2) - (\theta_1+\theta_2)(V_0+V_1) + \theta_1\theta_2V_0 = V_0\theta_3^2 + (V_1-2V_0)\theta_3 + (V_2-V_1).$$

Note that we can also write

$$(b_0+2b_1+b_2) - (\theta_2+\theta_3)(b_0+b_1) + \theta_2\theta_3b_0 = b_0\theta_1^2 + (b_1-2b_0)\theta_1 + (b_2-b_1),$$
$$(b_0+2b_1+b_2) - (\theta_1+\theta_3)(b_0+b_1) + \theta_1\theta_3b_0 = b_0\theta_2^2 + (b_1-2b_0)\theta_2 + (b_2-b_1),$$
$$(b_0+2b_1+b_2) - (\theta_1+\theta_2)(b_0+b_1) + \theta_1\theta_2b_0 = b_0\theta_3^2 + (b_1-2b_0)\theta_3 + (b_2-b_1).$$

Next, using Theorem 3, we present the Binet's formulas of binomial transform of Padovan, Perrin, Padovan-Perrin, modified Padovan, adjusted Padovan sequences.

Corollary 3. Binet's formulas of binomial transform of Padovan, Perrin, Padovan-Perrin, modified Padovan, adjusted Padovan sequences are

$$\hat{P}_n = \frac{(2\theta_1^2 - 2\theta_1 + 1)\theta_1^{n-1}}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{(2\theta_2^2 - 2\theta_2 + 1)\theta_2^{n-1}}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{(2\theta_3^2 - 2\theta_3 + 1)\theta_3^{n-1}}{(\theta_3-\theta_1)(\theta_3-\theta_2)},$$
$$\hat{E}_n = \theta_1^n + \theta_2^n + \theta_3^n,$$
$$\hat{S}_n = \frac{\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)},$$
$$\hat{A}_n = \frac{(4\theta_1^2 - 4\theta_1 + 3)\theta_1^{n-1}}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{(4\theta_2^2 - 4\theta_2 + 3)\theta_2^{n-1}}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{(4\theta_3^2 - 4\theta_3 + 3)\theta_3^{n-1}}{(\theta_3-\theta_1)(\theta_3-\theta_2)},$$
$$\hat{U}_n = \frac{(-1+\theta_1)\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{(-1+\theta_2)\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{(-1+\theta_3)\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)},$$

respectively.

4. SIMSON FORMULAS

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized Padovan sequence $\{W_n\}$.

Theorem 4 (Simson Formula of Generalized Tribonacci Numbers). For all integers n, we have

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = t^n \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{vmatrix}. \tag{20}$$

Proof: (20) is given in Soykan [22].

Taking $\{W_n\} = \{b_n\}$ in the above theorem and considering $b_{n+3} = 3b_{n+2} - 2b_{n+1} + b_n$, $r = 3, s = -2, t = 1$, we have the following proposition.

Proposition 1. For all integers n, Simson formula of binomial transforms of generalized Padovan numbers is given as

$$\begin{vmatrix} b_{n+2} & b_{n+1} & b_n \\ b_{n+1} & b_n & b_{n-1} \\ b_n & b_{n-1} & b_{n-2} \end{vmatrix} = \begin{vmatrix} b_2 & b_1 & b_0 \\ b_1 & b_0 & b_{-1} \\ b_0 & b_{-1} & b_{-2} \end{vmatrix}.$$

The previous proposition gives the following results as particular examples.

Corollary 4. For all integers n, Simson formula of binomial transforms of the Padovan, Perrin, Padovan-Perrin, modified Padovan, adjusted Padovan numbers are given as

$$\begin{vmatrix} \widehat{P}_{n+2} & \widehat{P}_{n+1} & \widehat{P}_{n} \\ \widehat{P}_{n+1} & \widehat{P}_{n} & \widehat{P}_{n-1} \\ \widehat{P}_{n} & \widehat{P}_{n-1} & \widehat{P}_{n-2} \end{vmatrix} = -1,$$
$$\begin{vmatrix} \widehat{E}_{n+2} & \widehat{E}_{n+1} & \widehat{E}_{n} \\ \widehat{E}_{n+1} & \widehat{E}_{n} & \widehat{E}_{n-1} \\ \widehat{E}_{n} & \widehat{E}_{n-1} & \widehat{E}_{n-2} \end{vmatrix} = -23,$$
$$\begin{vmatrix} \hat{S}_{n+2} & \hat{S}_{n+1} & \hat{S}_{n} \\ \hat{S}_{n+1} & \hat{S}_{n} & \hat{S}_{n-1} \\ \hat{S}_{n} & \hat{S}_{n-1} & \hat{S}_{n-2} \end{vmatrix} = -1,$$
$$\begin{vmatrix} \hat{A}_{n+2} & \hat{A}_{n+1} & \hat{A}_{n} \\ \hat{A}_{n+1} & \hat{A}_{n} & \hat{A}_{n-1} \\ \hat{A}_{n} & \hat{A}_{n-1} & \hat{A}_{n-2} \end{vmatrix} = -19,$$
$$\begin{vmatrix} \widehat{U}_{n+2} & \widehat{U}_{n+1} & \widehat{U}_{n} \\ \widehat{U}_{n+1} & \widehat{U}_{n} & \widehat{U}_{n-1} \\ \widehat{U}_{n} & \widehat{U}_{n-1} & \widehat{U}_{n-2} \end{vmatrix} = -1,$$

respectively.

5. SOME IDENTITIES

In this section, we obtain some identities of Padovan, Perrin, Padovan-Perrin, modified Padovan, adjusted Padovan numbers. First, we can give a few basic relations between $\{\widehat{P}_n\}$ and $\{\widehat{E}_n\}$.

Lemma 5. The following equalities are true:

$$\begin{aligned} 23\widehat{P}_n &= 5\widehat{E}_{n+4} - 6\widehat{E}_{n+3} - 10\widehat{E}_{n+2}, \\ 23\widehat{P}_n &= 9\widehat{E}_{n+3} - 20\widehat{E}_{n+2} + 5\widehat{E}_{n+1}, \\ 23\widehat{P}_n &= 7\widehat{E}_{n+2} - 13\widehat{E}_{n+1} + 9\widehat{E}_{n}, \\ 23\widehat{P}_n &= 8\widehat{E}_{n+1} - 5\widehat{E}_{n} + 7\widehat{E}_{n-1}, \\ 23\widehat{P}_n &= 19\widehat{E}_{n} - 9\widehat{E}_{n-1} + 8\widehat{E}_{n-2}, \end{aligned} \tag{21}$$

and

$$\begin{aligned} \widehat{E}_n &= 6\widehat{P}_{n+4} - 19\widehat{P}_{n+3} + 12\widehat{P}_{n+2}, \\ \widehat{E}_n &= -\widehat{P}_{n+3} + 6\widehat{P}_{n+1}, \\ \widehat{E}_n &= -3\widehat{P}_{n+2} + 8\widehat{P}_{n+1} - \widehat{P}_{n}, \\ \widehat{E}_n &= -\widehat{P}_{n+1} + 5\widehat{P}_{n} - 3\widehat{P}_{n-1}, \\ \widehat{E}_n &= 2\widehat{P}_{n} - \widehat{P}_{n-1} - \widehat{P}_{n-2}. \end{aligned}$$

Proof: Note that all the identities hold for all integers n. We prove (21). To show (21), writing

$$\hat{P}_n = a \times \hat{E}_{n+4} + b \times \hat{E}_{n+3} + c \times \hat{E}_{n+2}$$

and solving the system of equations

$$\begin{gathered}\hat{P}_0 = a \times \hat{E}_4 + b \times \hat{E}_3 + c \times \hat{E}_2 \\ \hat{P}_1 = a \times \hat{E}_5 + b \times \hat{E}_4 + c \times \hat{E}_3 \\ \hat{P}_2 = a \times \hat{E}_6 + b \times \hat{E}_5 + c \times \hat{E}_4\end{gathered}$$

we find that $a = \frac{5}{23}, b = -\frac{6}{23}, c = -\frac{10}{23}$. The other equalities can be proved similarly.

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{\hat{P}_n\}$ and $\{\hat{S}_n\}$.

Lemma 6. The following equalities are true:

$$\begin{gathered}\hat{P}_n = -\hat{S}_{n+4} + 3\hat{S}_{n+3} - \hat{S}_{n+2}, \\ \hat{P}_n = \hat{S}_{n+2} - \hat{S}_{n+1}, \\ \hat{P}_n = 2\hat{S}_{n+1} + 2\hat{S}_n + \hat{S}_{n-1}, \\ \hat{P}_n = 4\hat{S}_n - 3\hat{S}_{n-1} + 2\hat{S}_{n-2},\end{gathered}$$

and

$$\begin{gathered}\hat{S}_n = -3\hat{P}_{n+4} + 7\hat{P}_{n+3}, \\ \hat{S}_n = -2\hat{P}_{n+3} + 6\hat{P}_{n+2} - 3\hat{P}_{n+1}, \\ \hat{S}_n = \hat{P}_{n+1} - 2\hat{P}_n, \\ \hat{S}_n = \hat{P}_n - 2\hat{P}_{n-1} + \hat{P}_{n-2}.\end{gathered}$$

Now, we give a few basic relations between $\{\hat{P}_n\}$ and $\{\hat{A}_n\}$.

Lemma 7. The following equalities are true:

$$\begin{gathered}19\hat{P}_n = 4\hat{A}_{n+4} - 3\hat{A}_{n+3} - 13\hat{A}_{n+2}, \\ 19\hat{P}_n = 9\hat{A}_{n+3} - 21\hat{A}_{n+2} + 4\hat{A}_{n+1}, \\ 19\hat{P}_n = 6\hat{A}_{n+2} - 14\hat{A}_{n+1} + 9\hat{A}_n, \\ 19\hat{P}_n = 4\hat{A}_{n+1} - 3\hat{A}_n + 6\hat{A}_{n-1}, \\ 19\hat{P}_n = 9\hat{A}_n - 2\hat{A}_{n-1} + 4\hat{A}_{n-2},\end{gathered}$$

and

$$\begin{gathered}\hat{A}_n = 4\hat{P}_{n+4} - 13\hat{P}_{n+3} + 9\hat{P}_{n+2}, \\ \hat{A}_n = -\hat{P}_{n+3} + \hat{P}_{n+2} + 4\hat{P}_{n+1}, \\ \hat{A}_n = -2\hat{P}_{n+2} + 6\hat{P}_{n+1} - \hat{P}_n, \\ \hat{A}_n = 3\hat{P}_n - 2\hat{P}_{n-1}.\end{gathered}$$

Next, we present a few basic relations between $\{\hat{P}_n\}$ and $\{\hat{U}_n\}$.

Lemma 8. The following equalities are true:

$$\begin{gathered}\hat{P}_n = \hat{U}_{n+4} - 3\hat{U}_{n+3} + 2\hat{U}_{n+2}, \\ \hat{P}_n = \hat{U}_{n+1}, \\ \hat{P}_n = 3\hat{U}_n - 2\hat{U}_{n-1} + \hat{U}_{n-2},\end{gathered}$$

and

$$\begin{gathered}\hat{U}_n = \hat{P}_{n+4} - \hat{P}_{n+3} - 3\hat{P}_{n+2}, \\ \hat{U}_n = 2\hat{P}_{n+3} - 5\hat{P}_{n+2} + \hat{P}_{n+1},\end{gathered}$$

$$\widehat{U}_n = \widehat{P}_{n+2} - 3\widehat{P}_{n+1} + 2\widehat{P}_n,$$
$$\widehat{U}_n = \widehat{P}_{n-1}.$$

Now, we give a few basic relations between $\{\widehat{E}_n\}$ and $\{\hat{S}_n\}$.

Lemma 9. The following equalities are true:

$$\begin{aligned}
\widehat{E}_n &= -2\hat{S}_{n+4} + 8\hat{S}_{n+3} - 7\hat{S}_{n+2},\\
\widehat{E}_n &= 2\hat{S}_{n+3} - 3\hat{S}_{n+2} - 2\hat{S}_{n+1},\\
\widehat{E}_n &= 3\hat{S}_{n+2} - 6\hat{S}_{n+1} + 2\hat{S}_n,\\
\widehat{E}_n &= 3\hat{S}_{n+1} - 4\hat{S}_n + 3\hat{S}_{n-1},\\
\widehat{E}_n &= 5\hat{S}_n - 3\hat{S}_{n-1} + 3\hat{S}_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
23\hat{S}_n &= -\widehat{E}_{n+4} - 8\widehat{E}_{n+3} + 25\widehat{E}_{n+2},\\
23\hat{S}_n &= -11\widehat{E}_{n+3} + 27\widehat{E}_{n+2} - \widehat{E}_{n+1},\\
23\hat{S}_n &= -6\widehat{E}_{n+2} + 21\widehat{E}_{n+1} - 11\widehat{E}_n,\\
23\hat{S}_n &= 3\widehat{E}_{n+1} + \widehat{E}_n - 6\widehat{E}_{n-1},\\
23\hat{S}_n &= 10\widehat{E}_n - 12\widehat{E}_{n-1} + 3\widehat{E}_{n-2}.
\end{aligned}$$

Next, we present a few basic relations between $\{\hat{A}_n\}$ and $\{\widehat{E}_n\}$.

Lemma 10. The following equalities are true:

$$\begin{aligned}
19\widehat{E}_n &= 50\hat{A}_{n+4} - 123\hat{A}_{n+3} + 18\hat{A}_{n+2},\\
19\widehat{E}_n &= 27\hat{A}_{n+3} - 82\hat{A}_{n+2} + 50\hat{A}_{n+1},\\
19\widehat{E}_n &= -\hat{A}_{n+2} - 4\hat{A}_{n+1} + 27\hat{A}_n,\\
19\widehat{E}_n &= -7\hat{A}_{n+1} + 29\hat{A}_n - \hat{A}_{n-1},\\
19\widehat{E}_n &= 8\hat{A}_n + 13\hat{A}_{n-1} - 7\hat{A}_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
23\hat{A}_n &= 35\widehat{E}_{n+4} - 88\widehat{E}_{n+3} + 22\widehat{E}_{n+2},\\
23\hat{A}_n &= 17\widehat{E}_{n+3} - 48\widehat{E}_{n+2} + 35\widehat{E}_{n+1},\\
23\hat{A}_n &= 3\widehat{E}_{n+2} + \widehat{E}_{n+1} + 17\widehat{E}_n,\\
23\hat{A}_n &= 10\widehat{E}_{n+1} + 11\widehat{E}_n + 3\widehat{E}_{n-1},\\
23\hat{A}_n &= 41\widehat{E}_n - 17\widehat{E}_{n-1} + 10\widehat{E}_{n-2}.
\end{aligned}$$

Now, we give a few basic relations between $\{\widehat{E}_n\}$ and $\{\widehat{U}_n\}$.

Lemma 11. The following equalities are true:

$$\begin{aligned}
\widehat{E}_n &= -\widehat{U}_{n+4} + 6\widehat{U}_{n+2},\\
\widehat{E}_n &= -3\widehat{U}_{n+3} + 8\widehat{U}_{n+2} - \widehat{U}_{n+1},\\
\widehat{E}_n &= -\widehat{U}_{n+2} + 5\widehat{U}_{n+1} - 3\widehat{U}_n,\\
\widehat{E}_n &= 2\widehat{U}_{n+1} - \widehat{U}_n - \widehat{U}_{n-1},\\
\widehat{E}_n &= 5\widehat{U}_n - 5\widehat{U}_{n-1} + 2\widehat{U}_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
23\widehat{U}_n &= -10\widehat{E}_{n+4} + 35\widehat{E}_{n+3} - 26\widehat{E}_{n+2},\\
23\widehat{U}_n &= 5\widehat{E}_{n+3} - 6\widehat{E}_{n+2} - 10\widehat{E}_{n+1},\\
23\widehat{U}_n &= 9\widehat{E}_{n+2} - 20\widehat{E}_{n+1} + 5\widehat{E}_n,\\
23\widehat{U}_n &= 7\widehat{E}_{n+1} - 13\widehat{E}_n + 9\widehat{E}_{n-1},
\end{aligned}$$

$$23\widehat{U}_n = 8\widehat{E}_n - 5\widehat{E}_{n-1} + 7\widehat{E}_{n-2}.$$

Next, we present a few basic relations between $\{\hat{S}_n\}$ and $\{\hat{A}_n\}$.

Lemma 12. The following equalities are true:

$$\begin{gathered}
19\hat{S}_n = \hat{A}_{n+4} - 15\hat{A}_{n+3} + 30\hat{A}_{n+2}, \\
19\hat{S}_n = -12\hat{A}_{n+3} + 28\hat{A}_{n+2} + \hat{A}_{n+1}, \\
19\hat{S}_n = -8\hat{A}_{n+2} + 25\hat{A}_{n+1} - 12\hat{A}_n, \\
19\hat{S}_n = \hat{A}_{n+1} + 4\hat{A}_n - 8\hat{A}_{n-1}, \\
19\hat{S}_n = 7\hat{A}_n - 10\hat{A}_{n-1} + \hat{A}_{n-2},
\end{gathered}$$

and

$$\begin{gathered}
\hat{A}_n = -\hat{S}_{n+4} + 5\hat{S}_{n+3} - 5\hat{S}_{n+2}, \\
\hat{A}_n = 2\hat{S}_{n+3} - 3\hat{S}_{n+2} - \hat{S}_{n+1}, \\
\hat{A}_n = 3\hat{S}_{n+2} - 5\hat{S}_{n+1} + 2\hat{S}_n, \\
\hat{A}_n = 4\hat{S}_{n+1} - 4\hat{S}_n + 3\hat{S}_{n-1}, \\
\hat{A}_n = 8\hat{S}_n - 5\hat{S}_{n-1} + 4\hat{S}_{n-2}.
\end{gathered}$$

Now, we give a few basic relations between $\{\hat{S}_n\}$ and $\{\widehat{U}_n\}$.

Lemma 13. The following equalities are true:

$$\begin{gathered}
\hat{S}_n = -2\widehat{U}_{n+4} + 6\widehat{U}_{n+3} - 3\widehat{U}_{n+2}, \\
\hat{S}_n = \widehat{U}_{n+2} - 2\widehat{U}_{n+1}, \\
\hat{S}_n = \widehat{U}_{n+1} - 2\widehat{U}_n + \widehat{U}_{n-1}, \\
\hat{S}_n = \widehat{U}_n - \widehat{U}_{n-1} + \widehat{U}_{n-2},
\end{gathered}$$

and

$$\begin{gathered}
\widehat{U}_n = -\hat{S}_{n+4} + 2\hat{S}_{n+3} + \hat{S}_{n+2}, \\
\widehat{U}_n = -\hat{S}_{n+3} + 3\hat{S}_{n+2} - \hat{S}_{n+1}, \\
\widehat{U}_n = \hat{S}_{n+1} - \hat{S}_n, \\
\widehat{U}_n = 2\hat{S}_n - 2\hat{S}_{n-1} + \hat{S}_{n-2}.
\end{gathered}$$

Next, we present a few basic relations between $\{\hat{A}_n\}$ and $\{\widehat{U}_n\}$.

Lemma 14. The following equalities are true:

$$\begin{gathered}
\hat{A}_n = -\widehat{U}_{n+4} + \widehat{U}_{n+3} + 4\widehat{U}_{n+2}, \\
\hat{A}_n = -2\widehat{U}_{n+3} + 6\widehat{U}_{n+2} - \widehat{U}_{n+1}, \\
\hat{A}_n = 3\widehat{U}_{n+1} - 2\widehat{U}_n, \\
\hat{A}_n = 7\widehat{U}_n - 6\widehat{U}_{n-1} + 3\widehat{U}_{n-2},
\end{gathered}$$

and

$$\begin{gathered}
19\widehat{U}_n = -13\hat{A}_{n+4} + 43\hat{A}_{n+3} - 29\hat{A}_{n+2}, \\
19\widehat{U}_n = 4\hat{A}_{n+3} - 3\hat{A}_{n+2} - 13\hat{A}_{n+1}, \\
19\widehat{U}_n = 9\hat{A}_{n+2} - 21\hat{A}_{n+1} + 4\hat{A}_n, \\
19\widehat{U}_n = 6\hat{A}_{n+1} - 14\hat{A}_n + 9\hat{A}_{n-1}, \\
19\widehat{U}_n = 4\hat{A}_n - 3\hat{A}_{n-1} + 6\hat{A}_{n-2}.
\end{gathered}$$

6. SUM FORMULAS

6.1. SUMS OF TERMS WITH POSITIVE SUBSCRIPTS

The following proposition presents some formulas of binomial transform of binomial transform of generalized Padovan numbers with positive subscripts.

Proposition 2. For $n \geq 0$, we have the following formulas:

a) $\sum_{k=0}^{n} b_k = b_{n+3} - 2b_{n+2} - b_2 + 2b_1$.
b) $\sum_{k=0}^{n} b_{2k} = \frac{1}{7}(3b_{2n+2} - 5b_{2n+1} + 4b_{2n} - 3b_2 + 5b_1 + 3b_0)$.
c) $\sum_{k=0}^{n} b_{2k+1} = \frac{1}{7}(4b_{2n+2} - 2b_{2n+1} + 3b_{2n} - 4b_2 + 9b_1 - 3b_0)$.

Proof: Take $r = 3, s = -2, t = 1$ in Theorem 2.1 in [23] (or take $x = 1, r = 3, s = -2, t = 1$ in Theorem 2.1 in [24]).

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of binomial transform of Padovan numbers (take $b_n = \widehat{P}_n$ with $\widehat{P}_0 = 1, \widehat{P}_1 = 2, \widehat{P}_2 = 4$).

Corollary 5. For $n \geq 0$, we have the following formulas:

a) $\sum_{k=0}^{n} \widehat{P}_k = \widehat{P}_{n+3} - 2\widehat{P}_{n+2}$.
b) $\sum_{k=0}^{n} \widehat{P}_{2k} = \frac{1}{7}(3\widehat{P}_{2n+2} - 5\widehat{P}_{2n+1} + 4\widehat{P}_{2n} + 1)$.
c) $\sum_{k=0}^{n} \widehat{P}_{2k+1} = \frac{1}{7}(4\widehat{P}_{2n+2} - 2\widehat{P}_{2n+1} + 3\widehat{P}_{2n} - 1)$.

Taking $b_n = \widehat{E}_n$ with $\widehat{E}_0 = 3, \widehat{E}_1 = 3, \widehat{E}_2 = 5$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of binomial transform of Perrin numbers.

Corollary 6. For $n \geq 0$, we have the following formulas:

a) $\sum_{k=0}^{n} \widehat{E}_k = \widehat{E}_{n+3} - 2\widehat{E}_{n+2} + 1$.
b) $\sum_{k=0}^{n} \widehat{E}_{2k} = \frac{1}{7}(3\widehat{E}_{2n+2} - 5\widehat{E}_{2n+1} + 4\widehat{E}_{2n} + 9)$.
c) $\sum_{k=0}^{n} \widehat{E}_{2k+1} = \frac{1}{7}(4\widehat{E}_{2n+2} - 2\widehat{E}_{2n+1} + 3\widehat{E}_{2n} - 2)$.

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of binomial transform of Padovan-Perrin numbers (take $b_n = \hat{S}_n$ with $\hat{S}_0 = 0, \hat{S}_1 = 0, \hat{S}_2 = 1$).

Corollary 7. For $n \geq 0$, we have the following formulas:

a) $\sum_{k=0}^{n} \hat{S}_k = \hat{S}_{n+3} - 2\hat{S}_{n+2} - 1$.
b) $\sum_{k=0}^{n} \hat{S}_{2k} = \frac{1}{7}(3\hat{S}_{2n+2} - 5\hat{S}_{2n+1} + 4\hat{S}_{2n} - 3)$.
c) $\sum_{k=0}^{n} \hat{S}_{2k+1} = \frac{1}{7}(4\hat{S}_{2n+2} - 2\hat{S}_{2n+1} + 3\hat{S}_{2n} - 4)$.

Taking $b_n = \hat{A}_n$ with $\hat{A}_0 = 3, \hat{A}_1 = 4, \hat{A}_2 = 8$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of binomial transform of modified Padovan numbers.

Corollary 8. For $n \geq 0$, we have the following formulas:

a) $\sum_{k=0}^{n} \hat{A}_k = \hat{A}_{n+3} - 2\hat{A}_{n+2}$.
b) $\sum_{k=0}^{n} \hat{A}_{2k} = \frac{1}{7}(3\hat{A}_{2n+2} - 5\hat{A}_{2n+1} + 4\hat{A}_{2n} + 5)$.
c) $\sum_{k=0}^{n} \hat{A}_{2k+1} = \frac{1}{7}(4\hat{A}_{2n+2} - 2\hat{A}_{2n+1} + 3\hat{A}_{2n} - 5)$.

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of binomial transform of adjusted Padovan numbers (take $b_n = \widehat{U}_n$ with $\widehat{U}_0 = 0, \widehat{U}_1 = 1, \widehat{U}_2 = 2$).

Corollary 9. For $n \geq 0$, we have the following properties:

a) $\sum_{k=0}^{n} \widehat{U}_k = \widehat{U}_{n+3} - 2\widehat{U}_{n+2}$.
b) $\sum_{k=0}^{n} \widehat{U}_{2k} = \frac{1}{7}(3\widehat{U}_{2n+2} - 5\widehat{U}_{2n+1} + 4\widehat{U}_{2n} - 1)$.
c) $\sum_{k=0}^{n} \widehat{U}_{2k+1} = \frac{1}{7}(4\widehat{U}_{2n+2} - 2\widehat{U}_{2n+1} + 3\widehat{U}_{2n} + 1)$.

6.2. SUMS OF TERMS WITH NEGATIVE SUBSCRIPTS

The following proposition presents some formulas of binomial transform of binomial transform of generalized Padovan numbers with negative subscripts.

Proposition 3. For $n \geq 1$, we have the following formulas:

a) $\sum_{k=1}^{n} b_{-k} = -2b_{-n-1} + b_{-n-2} - b_{-n-3} + b_2 - 2b_1$.
b) $\sum_{k=1}^{n} b_{-2k} = \frac{1}{7}(-4b_{-2n+1} + 9b_{-2n} - 3b_{-2n-1} + 3b_2 - 5b_1 - 3b_0)$.
c) $\sum_{k=1}^{n} b_{-2k+1} = \frac{1}{7}(-3b_{-2n+1} + 5b_{-2n} - 4b_{-2n-1} + 4b_2 - 9b_1 + 3b_0)$.

Proof: Take $r = 3, s = -2, t = 1$ in Theorem 3.1 in [23] or (or take $x = 1, r = 3, s = -2, t = 1$ in Theorem 3.1 in [24]).

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of binomial transform of Padovan numbers (take $b_n = \hat{P}_n$ with $\hat{P}_0 = 1, \hat{P}_1 = 2, \hat{P}_2 = 4$).

Corollary 10. For $n \geq 1$, binomial transform of Padovan numbers have the following properties.

a) $\sum_{k=1}^{n} \hat{P}_{-k} = -2\hat{P}_{-n-1} + \hat{P}_{-n-2} - \hat{P}_{-n-3}$.
b) $\sum_{k=1}^{n} \hat{P}_{-2k} = \frac{1}{7}(-4\hat{P}_{-2n+1} + 9\hat{P}_{-2n} - 3\hat{P}_{-2n-1} - 1)$.
c) $\sum_{k=1}^{n} \hat{P}_{-2k+1} = \frac{1}{7}(-3\hat{P}_{-2n+1} + 5\hat{P}_{-2n} - 4\hat{P}_{-2n-1} + 1)$.

Taking $b_n = \hat{E}_n$ with $\hat{E}_0 = 3, \hat{E}_1 = 3, \hat{E}_2 = 5$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of binomial transform of Perrin numbers.

Corollary 11. For $n \geq 1$, binomial transform of Perrin numbers have the following properties.

a) $\sum_{k=1}^{n} \hat{E}_{-k} = -2\hat{E}_{-n-1} + \hat{E}_{-n-2} - \hat{E}_{-n-3} - 1.$
b) $\sum_{k=1}^{n} \hat{E}_{-2k} = \frac{1}{7}(-4\hat{E}_{-2n+1} + 9\hat{E}_{-2n} - 3\hat{E}_{-2n-1} - 9).$
c) $\sum_{k=1}^{n} \hat{E}_{-2k+1} = \frac{1}{7}(-3\hat{E}_{-2n+1} + 5\hat{E}_{-2n} - 4\hat{E}_{-2n-1} + 2).$

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of binomial transform of Padovan-Perrin numbers (take $b_n = \hat{S}_n$ with $\hat{S}_0 = 0, \hat{S}_1 = 0, \hat{S}_2 = 1$).

Corollary 12. For $n \geq 1$, binomial transform of Padovan-Perrin numbers have the following properties.

a) $\sum_{k=1}^{n} \hat{S}_{-k} = -2\hat{S}_{-n-1} + \hat{S}_{-n-2} - \hat{S}_{-n-3} + 1.$
b) $\sum_{k=1}^{n} \hat{S}_{-2k} = \frac{1}{7}(-4\hat{S}_{-2n+1} + 9\hat{S}_{-2n} - 3\hat{S}_{-2n-1} + 3).$
c) $\sum_{k=1}^{n} \hat{S}_{-2k+1} = \frac{1}{7}(-3\hat{S}_{-2n+1} + 5\hat{S}_{-2n} - 4\hat{S}_{-2n-1} + 4).$

Taking $b_n = \hat{A}_n$ with $\hat{A}_0 = 3, \hat{A}_1 = 4, \hat{A}_2 = 8$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of binomial transform of modified Padovan numbers.

Corollary 13. For $n \geq 1$, binomial transform of modified Padovan numbers have the following properties.

a) $\sum_{k=1}^{n} \hat{A}_{-k} = -2\hat{A}_{-n-1} + \hat{A}_{-n-2} - \hat{A}_{-n-3}.$
b) $\sum_{k=1}^{n} \hat{A}_{-2k} = \frac{1}{7}(-4\hat{A}_{-2n+1} + 9\hat{A}_{-2n} - 3\hat{A}_{-2n-1} - 5).$
c) $\sum_{k=1}^{n} \hat{A}_{-2k+1} = \frac{1}{7}(-3\hat{A}_{-2n+1} + 5\hat{A}_{-2n} - 4\hat{A}_{-2n-1} + 5).$

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of binomial transform of adjusted Padovan numbers (take $b_n = \widehat{U}_n$ with $\widehat{U}_0 = 0, \widehat{U}_1 = 1, \widehat{U}_2 = 2$).

Corollary 14. For $n \geq 1$, binomial transform of adjusted Padovan numbers have the following properties:

a) $\sum_{k=1}^{n} \widehat{U}_{-k} = -2\widehat{U}_{-n-1} + \widehat{U}_{-n-2} - U_{-n-3}.$
b) $\sum_{k=1}^{n} \widehat{U}_{-2k} = \frac{1}{7}(-4\widehat{U}_{-2n+1} + 9\widehat{U}_{-2n} - 3\widehat{U}_{-2n-1} + 1).$
c) $\sum_{k=1}^{n} \widehat{U}_{-2k+1} = \frac{1}{7}(-3\widehat{U}_{-2n+1} + 5\widehat{U}_{-2n} - 4\widehat{U}_{-2n-1} - 1).$

6.3. SUMS OF SQUARES OF TERMS WITH POSITIVE SUBSCRIPTS

The following proposition presents some formulas of binomial transform of generalized Padovan numbers with positive subscripts.

Proposition 4. For $n \geq 0$, we have the following formulas:

a) $\sum_{k=0}^{n} b_k^2 = \frac{1}{7}(-b_{n+3}^2 - 16b_{n+2}^2 - 8b_{n+1}^2 + 8b_{n+3}b_{n+2} + 2b_{n+3}b_{n+1} + 6b_{n+2}b_{n+1} + b_2^2 + 16b_1^2 + 8b_0^2 - 8b_2b_1 - 2b_2b_0 - 6b_1b_0)$.
b) $\sum_{k=0}^{n} b_{k+1}b_k = \frac{1}{7}(b_{n+3}^2 - 5b_{n+2}^2 + b_{n+1}^2 - b_{n+3}b_{n+2} + 5b_{n+3}b_{n+1} - 6b_{n+2}b_{n+1} - b_2^2 + 5b_1^2 - b_0^2 + b_2b_1 - 5b_2b_0 + 6b_1b_0)$.
c) $\sum_{k=0}^{n} b_{k+2}b_k = \frac{1}{7}(6b_{n+3}^2 + 12b_{n+2}^2 + 6b_{n+1}^2 - 20b_{n+3}b_{n+2} + 9b_{n+3}b_{n+1} - 15b_{n+2}b_{n+1} - 6b_2^2 - 12b_1^2 - 6b_0^2 + 20b_2b_1 - 9b_2b_0 + 15b_1b_0)$.

Proof: Take $x = 1, r = 3, s = -2, t = 1$ in Theorem 4.1 in [25], see also [26].

From the last proposition, we have the following Corollary which gives sum formulas of binomial transform of Padovan numbers (take $b_n = \hat{P}_n$ with $\hat{P}_0 = 1, \hat{P}_1 = 2, \hat{P}_2 = 4$).

Corollary 15. For $n \geq 0$, binomial transform of Padovan numbers have the following properties:

a) $\sum_{k=0}^{n} \hat{P}_k^2 = \frac{1}{7}(-\hat{P}_{n+3}^2 - 16\hat{P}_{n+2}^2 - 8\hat{P}_{n+1}^2 + 8\hat{P}_{n+3}\hat{P}_{n+2} + 2\hat{P}_{n+3}\hat{P}_{n+1} + 6\hat{P}_{n+2}\hat{P}_{n+1} + 4)$.
b) $\sum_{k=0}^{n} \hat{P}_{k+1}\hat{P}_k = \frac{1}{7}(\hat{P}_{n+3}^2 - 5\hat{P}_{n+2}^2 + \hat{P}_{n+1}^2 - \hat{P}_{n+3}\hat{P}_{n+2} + 5\hat{P}_{n+3}\hat{P}_{n+1} - 6\hat{P}_{n+2}\hat{P}_{n+1} + 3)$.
c) $\sum_{k=0}^{n} \hat{P}_{k+2}\hat{P}_k = \frac{1}{7}(6\hat{P}_{n+3}^2 + 12\hat{P}_{n+2}^2 + 6\hat{P}_{n+1}^2 - 20\hat{P}_{n+3}\hat{P}_{n+2} + 9\hat{P}_{n+3}\hat{P}_{n+1} - 15\hat{P}_{n+2}\hat{P}_{n+1} + 4)$.

Taking $b_n = \hat{E}_n$ with $\hat{E}_0 = 3, \hat{E}_1 = 3, \hat{E}_2 = 5$ in the last Proposition, we have the following Corollary which presents sum formulas of binomial transform of Perrin numbers.

Corollary 16. For $n \geq 0$, binomial transform of Perrin numbers have the following properties:

a) $\sum_{k=0}^{n} \hat{E}_k^2 = \frac{1}{7}(-\hat{E}_{n+3}^2 - 16\hat{E}_{n+2}^2 - 8\hat{E}_{n+1}^2 + 8\hat{E}_{n+3}\hat{E}_{n+2} + 2\hat{E}_{n+3}\hat{E}_{n+1} + 6\hat{E}_{n+2}\hat{E}_{n+1} + 37)$.
b) $\sum_{k=0}^{n} \hat{E}_{k+1}\hat{E}_k = \frac{1}{7}(\hat{E}_{n+3}^2 - 5\hat{E}_{n+2}^2 + \hat{E}_{n+1}^2 - \hat{E}_{n+3}\hat{E}_{n+2} + 5\hat{E}_{n+3}\hat{E}_{n+1} - 6\hat{E}_{n+2}\hat{E}_{n+1} + 5)$.
c) $\sum_{k=0}^{n} \hat{E}_{k+2}\hat{E}_k = \frac{1}{7}(6\hat{E}_{n+3}^2 + 12\hat{E}_{n+2}^2 + 6\hat{E}_{n+1}^2 - 20\hat{E}_{n+3}\hat{E}_{n+2} + 9\hat{E}_{n+3}\hat{E}_{n+1} - 15\hat{E}_{n+2}\hat{E}_{n+1} - 12)$.

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of Padovan-Perrin numbers (take $b_n = \hat{S}_n$ with $\hat{S}_0 = 0, \hat{S}_1 = 0, \hat{S}_2 = 1$).

Corollary 17. For $n \geq 0$, binomial transform of Padovan-Perrin numbers have the following properties:

a) $\sum_{k=0}^{n} \hat{S}_k^2 = \frac{1}{7}(-\hat{S}_{n+3}^2 - 16\hat{S}_{n+2}^2 - 8\hat{S}_{n+1}^2 + 8\hat{S}_{n+3}\hat{S}_{n+2} + 2\hat{S}_{n+3}\hat{S}_{n+1} + 6\hat{S}_{n+2}\hat{S}_{n+1} + 1)$.
b) $\sum_{k=0}^{n} \hat{S}_{k+1}\hat{S}_k = \frac{1}{7}(\hat{S}_{n+3}^2 - 5\hat{S}_{n+2}^2 + \hat{S}_{n+1}^2 - \hat{S}_{n+3}\hat{S}_{n+2} + 5\hat{S}_{n+3}\hat{S}_{n+1} - 6\hat{S}_{n+2}\hat{S}_{n+1} - 1)$.
c) $\sum_{k=0}^{n} \hat{S}_{k+2}\hat{S}_k =$
$\frac{1}{7}(6\hat{S}_{n+3}^2 + 12\hat{S}_{n+2}^2 + 6\hat{S}_{n+1}^2 - 20\hat{S}_{n+3}\hat{S}_{n+2} + 9\hat{S}_{n+3}\hat{S}_{n+1} - 15\hat{S}_{n+2}\hat{S}_{n+1} - 6)$.

Taking $b_n = \hat{A}_n$ with $\hat{A}_0 = 3, \hat{A}_1 = 4, \hat{A}_2 = 8$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of modified Padovan numbers.

Corollary 18. For $n \geq 0$, binomial transform of modified Padovan numbers have the following properties:

a) $\sum_{k=0}^{n} \hat{A}_k^2 = \frac{1}{7}(-\hat{A}_{n+3}^2 - 16\hat{A}_{n+2}^2 - 8\hat{A}_{n+1}^2 + 8\hat{A}_{n+3}\hat{A}_{n+2} + 2\hat{A}_{n+3}\hat{A}_{n+1} + 6\hat{A}_{n+2}\hat{A}_{n+1} + 16)$.
b) $\sum_{k=0}^{n} \hat{A}_{k+1}\hat{A}_k = \frac{1}{7}(\hat{A}_{n+3}^2 - 5\hat{A}_{n+2}^2 + \hat{A}_{n+1}^2 - \hat{A}_{n+3}\hat{A}_{n+2} + 5\hat{A}_{n+3}\hat{A}_{n+1} - 6\hat{A}_{n+2}\hat{A}_{n+1} - 9)$.
c) $\sum_{k=0}^{n} \hat{A}_{k+2}\hat{A}_k = \frac{1}{7}(6\hat{A}_{n+3}^2 + 12\hat{A}_{n+2}^2 + 6\hat{A}_{n+1}^2 - 20\hat{A}_{n+3}\hat{A}_{n+2} + 9\hat{A}_{n+3}\hat{A}_{n+1} - 15\hat{A}_{n+2}\hat{A}_{n+1} - 26)$.

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of adjusted Padovan numbers (take $b_n = \widehat{U}_n$ with $\widehat{U}_0 = 0, \widehat{U}_1 = 1, \widehat{U}_2 = 2$).

Corollary 19. For $n \geq 0$, binomial transform of adjusted Padovan numbers have the following properties:

a) $\sum_{k=0}^{n} \widehat{U}_k^2 = \frac{1}{7}(-\widehat{U}_{n+3}^2 - 16\widehat{U}_{n+2}^2 - 8\widehat{U}_{n+1}^2 + 8\widehat{U}_{n+3}\widehat{U}_{n+2} + 2\widehat{U}_{n+3}\widehat{U}_{n+1} + 6\widehat{U}_{n+2}\widehat{U}_{n+1} + 4)$.
b) $\sum_{k=0}^{n} \widehat{U}_{k+1}\widehat{U}_k = \frac{1}{7}(\widehat{U}_{n+3}^2 - 5\widehat{U}_{n+2}^2 + \widehat{U}_{n+1}^2 - \widehat{U}_{n+3}\widehat{U}_{n+2} + 5\widehat{U}_{n+3}\widehat{U}_{n+1} - 6\widehat{U}_{n+2}\widehat{U}_{n+1} + 3)$.
c) $\sum_{k=0}^{n} \widehat{U}_{k+2}\widehat{U}_k = \frac{1}{7}(6\widehat{U}_{n+3}^2 + 12\widehat{U}_{n+2}^2 + 6\widehat{U}_{n+1}^2 - 20\widehat{U}_{n+3}\widehat{U}_{n+2} + 9\widehat{U}_{n+3}\widehat{U}_{n+1} - 15\widehat{U}_{n+2}\widehat{U}_{n+1} + 4)$.

7. MATRICES RELATED WITH BINOMIAL TRANSFORM OF GENERALIZED PADOVAN NUMBERS

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{22}$$

For matrix formulation (22), see [27]. In fact, Kalman gave the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ r & s & t \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \end{pmatrix}.$$

We define the square matrix A of order 3 as:

$$A = \begin{pmatrix} 3 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 1$. From (16) we have

$$\begin{pmatrix} b_{n+2} \\ b_{n+1} \\ b_n \end{pmatrix} = \begin{pmatrix} 3 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} b_{n+1} \\ b_n \\ b_{n-1} \end{pmatrix} \tag{23}$$

and from (22) (or using (23) and induction) we have

$$\begin{pmatrix} b_{n+2} \\ b_{n+1} \\ b_n \end{pmatrix} = \begin{pmatrix} 3 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} b_2 \\ b_1 \\ b_0 \end{pmatrix}.$$

If we take $b_n = \hat{P}_n$ in (23) we have

$$\begin{pmatrix} \hat{P}_{n+2} \\ \hat{P}_{n+1} \\ \hat{P}_n \end{pmatrix} = \begin{pmatrix} 3 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{P}_{n+1} \\ \hat{P}_n \\ \hat{P}_{n-1} \end{pmatrix}. \tag{24}$$

For $n \geq 0$, we define

$$B_n = \begin{pmatrix} \sum_{k=0}^{n+1} \hat{P}_k & -2\sum_{k=0}^{n} \hat{P}_k + \sum_{k=0}^{n-1} \hat{P}_k & \sum_{k=0}^{n} \hat{P}_k \\ \sum_{k=0}^{n} \hat{P}_k & -2\sum_{k=0}^{n-1} \hat{P}_k + \sum_{k=0}^{n-2} \hat{P}_k & \sum_{k=0}^{n-1} \hat{P}_k \\ \sum_{k=0}^{n-1} \hat{P}_k & -2\sum_{k=0}^{n-2} \hat{P}_k + \sum_{k=0}^{n-3} \hat{P}_k & \sum_{k=0}^{n-2} \hat{P}_k \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} b_{n+1} & -2b_n + b_{n-1} & b_n \\ b_n & -2b_{n-1} + b_{n-2} & b_{n-1} \\ b_{n-1} & -2b_{n-2} + b_{n-3} & b_{n-2} \end{pmatrix}.$$

By convention, we assume that

$$\sum_{k=0}^{-1} \hat{P}_k = 0, \quad \sum_{k=0}^{-2} \hat{P}_k = 1, \qquad \sum_{k=0}^{-3} \hat{P}_k = 2.$$

Theorem 5. For all integers $m,n \geq 0$, we have

a) $B_n = A^n$.
b) $C_1A^n = A^nC_1$.
c) $C_{n+m} = C_nB_m = B_mC_n$.

Proof: a) Proof can be done by mathematical induction on n.
b) After matrix multiplication, (b) follows.
c) We have

$$\begin{aligned} AC_{n-1} &= \begin{pmatrix} 3 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} b_n & -2b_{n-1}+b_{n-2} & b_{n-1} \\ b_{n-1} & -2b_{n-2}+b_{n-3} & b_{n-2} \\ b_{n-2} & -2b_{n-3}+b_{n-4} & b_{n-3} \end{pmatrix} \\ &= \begin{pmatrix} b_{n+1} & -2b_n+b_{n-1} & b_n \\ b_n & -2b_{n-1}+b_{n-2} & b_{n-1} \\ b_{n-1} & -2b_{n-2}+b_{n-3} & b_{n-2} \end{pmatrix} = C_n. \end{aligned}$$

i.e. $C_n = AC_{n-1}$. From the last equation, using induction, we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^mC_1 = A^{n-1}C_1A^m = C_nB_m$$

and similarly

$$C_{n+m} = B_mC_n.$$

Some properties of matrix A^n can be given as

$$A^n = 3A^{n-1} - 2A^{n-2} + A^{n-3} = 2A^{n+1} - 3A^{n+2} + A^{n+3}$$

and

$$A^{n+m} = A^nA^m = A^mA^n$$

and

$$\det(A^n) = 1$$

for all integers $m,n \geq 0$.

Theorem 6. For $m,n \geq 0$, we have

$$\begin{aligned} b_{n+m} &= b_n \sum_{k=0}^{m+1} \widehat{P}_k + b_{n-1}\left(-2\sum_{k=0}^{m} \widehat{P}_k + \sum_{k=0}^{m-1} \widehat{P}_k\right) + b_{n-2}\sum_{k=0}^{m} \widehat{P}_k \\ &= b_n \sum_{k=0}^{m+1} \widehat{P}_k + (-2b_{n-1}+b_{n-2})\sum_{k=0}^{m} \widehat{P}_k + b_{n-1}\sum_{k=0}^{m-1} \widehat{P}_k. \end{aligned}$$

Proof: From the equation $C_{n+m} = C_nB_m = B_mC_n$, we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation, we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and C_nB_m. This completes the Proof:

Corollary 20. For $m, n \geq 0$, we have

$$\widehat{P}_{n+m} = \widehat{P}_n \sum_{k=0}^{m+1} \widehat{P}_k + \widehat{P}_{n-1}\left(-2\sum_{k=0}^{m} \widehat{P}_k + \sum_{k=0}^{m-1} \widehat{P}_k\right) + \widehat{P}_{n-2}\sum_{k=0}^{m} \widehat{P}_k,$$
$$\widehat{E}_{n+m} = \widehat{E}_n \sum_{k=0}^{m+1} \widehat{P}_k + \widehat{E}_{n-1}\left(-2\sum_{k=0}^{m} \widehat{P}_k + \sum_{k=0}^{m-1} \widehat{P}_k\right) + \widehat{E}_{n-2}\sum_{k=0}^{m} \widehat{P}_k,$$
$$\widehat{S}_{n+m} = \widehat{S}_n \sum_{k=0}^{m+1} \widehat{P}_k + \widehat{S}_{n-1}\left(-2\sum_{k=0}^{m} \widehat{P}_k + \sum_{k=0}^{m-1} \widehat{P}_k\right) + \widehat{S}_{n-2}\sum_{k=0}^{m} \widehat{P}_k,$$
$$\widehat{A}_{n+m} = \widehat{A}_n \sum_{k=0}^{m+1} \widehat{P}_k + \widehat{A}_{n-1}\left(-2\sum_{k=0}^{m} \widehat{P}_k + \sum_{k=0}^{m-1} \widehat{P}_k\right) + \widehat{A}_{n-2}\sum_{k=0}^{m} \widehat{P}_k,$$
$$\widehat{U}_{n+m} = \widehat{U}_n \sum_{k=0}^{m+1} \widehat{P}_k + \widehat{U}_{n-1}\left(-2\sum_{k=0}^{m} \widehat{P}_k + \sum_{k=0}^{m-1} \widehat{P}_k\right) + \widehat{U}_{n-2}\sum_{k=0}^{m} \widehat{P}_k.$$

From Corollary 5, we know that for $n \geq 0$,

$$\sum_{k=0}^{n} \widehat{P}_k = \widehat{P}_{n+3} - 2\widehat{P}_{n+2}.$$

So, Theorem 6 and Corollary 20 can be written in the following forms:

Theorem 7. For $m, n \geq 0$, we have

$$b_{n+m} = (\widehat{P}_{m+4} - 2\widehat{P}_{m+3})b_n + (-2P_{m+3} + 5P_{m+2} - 2P_{m+1})b_{n-1} + (\widehat{P}_{m+3} - 2\widehat{P}_{m+2})b_{n-2}. \tag{25}$$

Remark 2. By induction, it can be proved that for all integers $m, n \leq 0$, (25) holds. So, for all integers m, n, (25) is true.

Corollary 21. For all integers m, n, we have

$$\widehat{P}_{n+m} = (\widehat{P}_{m+4} - 2\widehat{P}_{m+3})\widehat{P}_n + (-2P_{m+3} + 5P_{m+2} - 2P_{m+1})\widehat{P}_{n-1} + (\widehat{P}_{m+3} - 2\widehat{P}_{m+2})\widehat{P}_{n-2},$$
$$\widehat{E}_{n+m} = (\widehat{P}_{m+4} - 2\widehat{P}_{m+3})\widehat{E}_n + (-2P_{m+3} + 5P_{m+2} - 2P_{m+1})\widehat{E}_{n-1} + (\widehat{P}_{m+3} - 2\widehat{P}_{m+2})\widehat{E}_{n-2},$$
$$\widehat{S}_{n+m} = (\widehat{P}_{m+4} - 2\widehat{P}_{m+3})\widehat{S}_n + (-2P_{m+3} + 5P_{m+2} - 2P_{m+1})\widehat{S}_{n-1} + (\widehat{P}_{m+3} - 2\widehat{P}_{m+2})\widehat{S}_{n-2},$$
$$\widehat{A}_{n+m} = (\widehat{P}_{m+4} - 2\widehat{P}_{m+3})\widehat{A}_n + (-2P_{m+3} + 5P_{m+2} - 2P_{m+1})\widehat{A}_{n-1} + (\widehat{P}_{m+3} - 2\widehat{P}_{m+2})\widehat{A}_{n-2},$$
$$\widehat{U}_{n+m} = (\widehat{P}_{m+4} - 2\widehat{P}_{m+3})\widehat{U}_n + (-2P_{m+3} + 5P_{m+2} - 2P_{m+1})\widehat{U}_{n-1} + (\widehat{P}_{m+3} - 2\widehat{P}_{m+2})\widehat{U}_{n-2}.$$

Now, we consider non-positive subscript cases. For $n \geq 0$, we define

$$B_{-n} = \begin{pmatrix} -\sum_{k=0}^{n-2} \widehat{P}_{-k} & 2\sum_{k=0}^{n-1} \widehat{P}_{-k} - \sum_{k=0}^{n} \widehat{P}_{-k} & -\sum_{k=0}^{n-1} \widehat{P}_{-k} \\ -\sum_{k=0}^{n-1} \widehat{P}_{-k} & 2\sum_{k=0}^{n} \widehat{P}_{-k} - \sum_{k=0}^{n+1} \widehat{P}_{-k} & -\sum_{k=0}^{n} \widehat{P}_{-k} \\ -\sum_{k=0}^{n} \widehat{P}_{-k} & 2\sum_{k=0}^{n+1} \widehat{P}_{-k} - \sum_{k=0}^{n+2} \widehat{P}_{-k} & -\sum_{k=0}^{n+1} \widehat{P}_{-k} \end{pmatrix}$$

and

$$C_{-n}=\begin{pmatrix} b_{-n+1} & -2b_{-n}+b_{-n-1} & b_{-n} \\ b_{-n} & -2b_{-n-1}+b_{-n-2} & b_{-n-1} \\ b_{-n-1} & -2b_{-n-2}+b_{-n-3} & b_{-n-2} \end{pmatrix}.$$

By convention, we assume that

$$\sum_{k=0}^{-1}\widehat{P}_{-k}=0, \qquad \sum_{k=0}^{-2}\widehat{P}_{-k}=-1.$$

Theorem 8. For all integers $m,n\geq 0$, we have

a) $B_{-n}=A^{-n}$.
b) $C_{-1}A^{-n}=A^{-n}C_{-1}$.
c) $C_{-n-m}=C_{-n}B_{-m}=B_{-m}C_{-n}$.

Proof: a) Proof can be done by mathematical induction on n.
b) After matrix multiplication, (b) follows.
c) We have

$$\begin{aligned} A^{-1}C_{-n-1} &= \begin{pmatrix} 3 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} b_{-n} & -2b_{-n-1}+b_{-n-2} & b_{-n-1} \\ b_{-n-1} & -2b_{-n-2}+b_{-n-3} & b_{-n-2} \\ b_{-n-2} & -2b_{-n-3}+b_{-n-4} & b_{-n-3} \end{pmatrix} \\ &= \begin{pmatrix} b_{-n+1} & -2b_{-n}+b_{-n-1} & b_{-n} \\ b_{-n} & -2b_{-n-1}+b_{-n-2} & b_{-n-1} \\ b_{-n-1} & -2b_{-n-2}+b_{-n-3} & b_{-n-2} \end{pmatrix}=C_{-n}, \end{aligned}$$

i.e. $C_{-n}=A^{-1}C_{-n-1}$. From the last equation, using induction, we obtain $C_{-n}=A^{-n-1}C_{-1}$. Now,

$$C_{-n-m}=A^{-n-m-1}C_{-1}=A^{-n-1}A^{-m}C_{-1}=A^{-n-1}C_{-1}A^{-m}=C_{-n}B_{-m}$$

and similarly,

$$C_{-n-m}=B_{-m}C_{-n}.$$

Some properties of matrix A^{-n} can be given as

$$A^{-n}=3A^{-n-1}-2A^{-n-2}+A^{-n-3}=2A^{-n+1}-3A^{-n+2}+A^{-n+3}$$

and

$$A^{-n-m}=A^{-n}A^{-m}=A^{-m}A^{-n}$$

and

$$\det(A^{-n})=1$$

for all integers $m,n\geq 0$.

Theorem 9. For $m,n\geq 0$, we have

$$b_{-n-m}=-b_{-n}\sum_{k=0}^{m-2}\widehat{P}_{-k}-b_{-n-1}\left(-2\sum_{k=0}^{m-1}\widehat{P}_{-k}+\sum_{k=0}^{m}\widehat{P}_{-k}\right)-b_{-n-2}\sum_{k=0}^{m-1}\widehat{P}_{-k}$$

$$= -b_{-n}\sum_{k=0}^{m-2}\widehat{P}_{-k} - (-2b_{-n-1}+b_{-n-2})\sum_{k=0}^{m-1}\widehat{P}_{-k} - b_{-n-1}\sum_{k=0}^{m}\widehat{P}_{-k}.$$

Proof: From the equation $C_{-n-m} = C_{-n}B_{-m} = B_{-m}C_{-n}$, we see that an element of C_{-n-m} is the product of row C_{-n} and a column B_{-m}. From the last equation, we say that an element of C_{-n-m} is the product of a row C_{-n} and column B_{-m}. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{-n-m} and $C_{-n}B_{-m}$. This completes the Proof:

Corollary 22. For $m, n \geq 0$, we have

$$\widehat{P}_{-n-m} = -\widehat{P}_{-n}\sum_{k=0}^{m-2}\widehat{P}_{-k} - \widehat{P}_{-n-1}\left(-2\sum_{k=0}^{m-1}\widehat{P}_{-k} + \sum_{k=0}^{m}\widehat{P}_{-k}\right) - \widehat{P}_{-n-2}\sum_{k=0}^{m-1}\widehat{P}_{-k},$$

$$\widehat{E}_{-n-m} = -\widehat{E}_{-n}\sum_{k=0}^{m-2}\widehat{P}_{-k} - \widehat{E}_{-n-1}\left(-2\sum_{k=0}^{m-1}\widehat{P}_{-k} + \sum_{k=0}^{m}\widehat{P}_{-k}\right) - \widehat{E}_{-n-2}\sum_{k=0}^{m-1}\widehat{P}_{-k},$$

$$\widehat{S}_{-n-m} = -\widehat{S}_{-n}\sum_{k=0}^{m-2}\widehat{P}_{-k} - \widehat{S}_{-n-1}\left(-2\sum_{k=0}^{m-1}\widehat{P}_{-k} + \sum_{k=0}^{m}\widehat{P}_{-k}\right) - \widehat{S}_{-n-2}\sum_{k=0}^{m-1}\widehat{P}_{-k},$$

$$\widehat{A}_{-n-m} = -\widehat{A}_{-n}\sum_{k=0}^{m-2}\widehat{P}_{-k} - \widehat{A}_{-n-1}\left(-2\sum_{k=0}^{m-1}\widehat{P}_{-k} + \sum_{k=0}^{m}\widehat{P}_{-k}\right) - \widehat{A}_{-n-2}\sum_{k=0}^{m-1}\widehat{P}_{-k},$$

$$\widehat{U}_{-n-m} = -\widehat{U}_{-n}\sum_{k=0}^{m-2}\widehat{P}_{-k} - \widehat{U}_{-n-1}\left(-2\sum_{k=0}^{m-1}\widehat{P}_{-k} + \sum_{k=0}^{m}\widehat{P}_{-k}\right) - \widehat{U}_{-n-2}\sum_{k=0}^{m-1}\widehat{P}_{-k}.$$

From Corollary 10, we know that for $n \geq 1$,

$$\sum_{k=1}^{n}\widehat{P}_{-k} = -2\widehat{P}_{-n-1} + \widehat{P}_{-n-2} - \widehat{P}_{-n-3}.$$

Since $\widehat{P}_0 = 0$, it follows that

$$\sum_{k=0}^{n}\widehat{P}_{-k} = -2\widehat{P}_{-n-1} + \widehat{P}_{-n-2} - \widehat{P}_{-n-3}.$$

So, Theorem 9 and Corollary 22 can be written in the following forms.

Theorem 10. For $m, n \geq 0$, we have

$$\begin{aligned} b_{-n-m} = {} & (2\widehat{P}_{-m+1} - \widehat{P}_{-m} + \widehat{P}_{-m-1})b_{-n} + (-4\widehat{P}_{-m} + 4P_{-m-1} - 3\widehat{P}_{-m-2} \\ & + \widehat{P}_{-m-3})b_{-n-1} + (2\widehat{P}_{-m} - \widehat{P}_{-m-1} + \widehat{P}_{-m-2})b_{-n-2}. \end{aligned} \tag{26}$$

Remark 3. By induction, it can be proved that for all integers $m, n \leq 0$, (26) holds. So, for all integers m, n, (26) is true.

Corollary 23. For all integers m, n, we have

$$\begin{aligned}
\hat{P}_{-n-m} &= (2\hat{P}_{-m+1} - \hat{P}_{-m} + \hat{P}_{-m-1})\hat{P}_{-n} + (-4\hat{P}_{-m} + 4P_{-m-1} - 3\hat{P}_{-m-2} + \hat{P}_{-m-3})\hat{P}_{-n-1} \\
&\quad + (2\hat{P}_{-m} - \hat{P}_{-m-1} + \hat{P}_{-m-2})\hat{P}_{-n-2}, \\
\hat{E}_{-n-m} &= (2\hat{P}_{-m+1} - \hat{P}_{-m} + \hat{P}_{-m-1})\hat{E}_{-n} + (-4\hat{P}_{-m} + 4P_{-m-1} - 3\hat{P}_{-m-2} + \hat{P}_{-m-3})\hat{E}_{-n-1} \\
&\quad + (2\hat{P}_{-m} - \hat{P}_{-m-1} + \hat{P}_{-m-2})\hat{E}_{-n-2}, \\
\hat{S}_{-n-m} &= (2\hat{P}_{-m+1} - \hat{P}_{-m} + \hat{P}_{-m-1})\hat{S}_{-n} + (-4\hat{P}_{-m} + 4P_{-m-1} - 3\hat{P}_{-m-2} + \hat{P}_{-m-3})\hat{S}_{-n-1} \\
&\quad + (2\hat{P}_{-m} - \hat{P}_{-m-1} + \hat{P}_{-m-2})\hat{S}_{-n-2}, \\
\hat{A}_{-n-m} &= (2\hat{P}_{-m+1} - \hat{P}_{-m} + \hat{P}_{-m-1})\hat{A}_{-n} + (-4\hat{P}_{-m} + 4P_{-m-1} - 3\hat{P}_{-m-2} + \hat{P}_{-m-3})\hat{A}_{-n-1} \\
&\quad + (2\hat{P}_{-m} - \hat{P}_{-m-1} + \hat{P}_{-m-2})\hat{A}_{-n-2}, \\
\hat{U}_{-n-m} &= (2\hat{P}_{-m+1} - \hat{P}_{-m} + \hat{P}_{-m-1})\hat{U}_{-n} + (-4\hat{P}_{-m} + 4P_{-m-1} - 3\hat{P}_{-m-2} + \hat{P}_{-m-3})\hat{U}_{-n-1} \\
&\quad + (2\hat{P}_{-m} - \hat{P}_{-m-1} + \hat{P}_{-m-2})\hat{U}_{-n-2}.
\end{aligned}$$

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