ORIGINAL PAPER

ANALYSIS OF RECENT ANALYTICAL TECHNIQUES ON THE KdVB EQUATION

EHSAN UL HAQ[1], TASAWAR ABBAS[1], QAZI MAHMOOD UL HASSAN[1], BILAL AHMAD[1]

Manuscript received: 06.08.2021; Accepted paper: 10.11.2021; Published online: 30.12.2021.

Abstract. *The Korteweg-de Vries Burgers (KdVB) is significant in applied mathematics and physical sciences. Particularly, it is a fundamental equation in the study of shallow water waves. The traditional techniques which have been suggested to solve the Korteweg-de Vries Burgers (KdVB) are labor-intensive and time-consuming. The primary goal of this study is to introduce various analytical techniques i.e., Exp-Function Method, Modified Exp-Function Method, Variational Iteration Method, and the Decomposition Method to solve the Korteweg-de Vries Burgers (KdVB) equation. These methods are quickly implemented and give very accurate results of the KdVB equation. Among them, the Variational Iteration Method is particularly user-friendly and simple to implement for the aforementioned problem. The involvement of Lagrange Multiplier is a powerful tool to reduce the cumbersome integration. At the end, Maple18 is used to find the analytical and graphic outcomes. These results show that the proposed methods are effective and applicable to other nonlinear equations of physical interest as well.*

Keywords: *Variational Iteration Method; Lagrange Multiplier; Differential Equation, KdVB Equation; Exp-Function Method; Modified Exp-Function Method; Maple 18.*

1. INTRODUCTION

In recent years, applications of the nonlinear differential problems [1-21] have a very important role in different sciences. One of the most prominent topics of research in the domains of fluid mechanics and nonlinear evolution equations is the dynamics of shallow water waves. There are numerous equations in literature to govern this dynamics. In this since, several numerical and analytical techniques have been suggested to deal with this phenomena. For instance, Sulaiman et al.[1] illustrated the dynamical behaviour of Boussinesq equation in the shallow water waves for two-layered fluid flow. Morover, Bhatta, D. D., Bhatti, M. I. [2] presented the numerical solution of the Korteweg-de Vries (KdV) equation. Similarly, Ahmad et al. [3]. applied modified Riemann-Liouville derivative method on the fractional KdV equation. In this study the main purpose is to examine the KdV-Burgers (KdVB) equation which was initially introduced by Su and Gardner [5]. The KdV-Burgers (KdVB) equation is using to scruntinize the weak effects of dispersion, dissipation, and nonlinearity in waves travelling in a liquid-filled elastic tube. Further, Chukkol et al. [6] scan the KDV-Burgers equation to obtain the exact travelling wave solutions.

[1] University of Wah, Department of Mathematics, 47040 Punjab, Pakistan.
E-mails: ehsan.shah61@gmail.com; tasawar.abbas@uow.edu.pk; qazimahmood@uow.edu.pk; qazimahmood@yahoo.com; bilal.ahmadr@uow.edu.pk.

Analytic solutions of the problem play significant role in understanding colorful features of the phenomenon involved in nonlinear sciences. In the recent years, there are many methods has been developed to solve nonlinear differential equation, howevere, now a days many researcher are analysis the most advance techniques. In this research work, the solution of Korteweg-de Vries Burgers (KdVB) equations will be found out using the advanced analytical techniques such as Variational iterative method, Adomian decomposition method, Exp-Function Method and Modified Exp-Function Method [6-19]. The Variational iteration method is the most comprehensive, simple and user friendly technique to solve the differential equations as compared to a forehand mentioned techniques. Initially introduced by JI-Huan He [6]. He used this technique for approximate solutions for nonlinear differential equations It has been extensively used by many authors to solve problems with high nonlinearity [7,8]. The involvement of Lagrange Multipliers in VIM reduces successive integration and the cumbersome computational work while still maintaining a very high level of accuracy. Likewise, the Adomain decomposition method [9-12] is a powerful technique to solve a large amount of problems. It is very easy to implement and can effortlessly so;ve the wide range of nonlinear systems. Exp-Function Method [13-16] is straightforward and simple techniques particularly for finding solution solutions.

The KdVB is a non-linear equation and it is a special form of equation which is generated by the combination of Korteweg-de Vries (KdV) equation [20] and Burgers equation [21]. The KdVB equation has equal importance in mathematics, fluid and viscous mechanics. Su and Gardner first derived KdVB equation in 1969 [5] which is given as:

$$u_t + \varepsilon u u_x - \nu u_{xx} + \mu u_{xxx} = 0; \tag{1}$$

where ε, ν and μ are +ve parameters.

If $\nu = 0$, (1) will take the form of KdVequation, which is as follows:

$$u_t + \varepsilon u u_x + \mu u_{xxx} = 0. \tag{2}$$

and if $\mu = 0$,then (1) will take the form of Burgers equation, which is given as:

$$u_t + \varepsilon u u_x - \nu u_{xx} = 0. \tag{3}$$

2. ANALYSIS OF THE METHOD

In this section, we will discuss the general analysis of the proposed analytical techniques.

2.1. EXP-FUNCTION METHOD

He and Wu [15] constructed this technique in 2006, which is very effective for finding the solutions of non-linear differential equations, such as solitary and periodic wave equations. Many researchers are still working on this method and trying to modify it to get the results more efficiently and quickly. This method is basically an efficient and easy to implement method.

Methodology

Consider the general non-linear PDEs as follows:

$$P\left(u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{tt}, u_{xy}, u_{xt}, u_{yt}, \ldots\right) = 0. \tag{4}$$

By applying the transformation

$$\eta = kx + ly + \omega t. \tag{5}$$

where k, l and ω are constant. The equation (4) becomes an ode of the form:

$$Q(u, u', u'', u''', \ldots) = 0. \tag{6}$$

The solution can be expressed in the following form according to the Exp-Function method:

$$u(\eta) = \frac{\sum_{n=-d}^{c} a_n \exp[n\eta]}{\sum_{m=-q}^{p} b_m \exp[m\eta]}, \tag{7}$$

The positive integers c, d, p and q are to be determined, a_n and b_m are unknowns. The equivalent from of trivial solution can be expressed in the form of:

$$u(\eta) = \frac{a_c \exp[c\eta] + \ldots + a_{-d} \exp[-d\eta]}{b_p \exp[p\eta] + \cdots + b_{-q} \exp[-q\eta]}, \tag{8}$$

Equation (8) is used to find the solution of the given problems. In equation (6), balance the highest order linear term with highest order non-linear term to evaluate c and p, in the same way, balance the lowest order linear term with lowest order non-linear term to obtain d and q of equation (6).

2.2. MODIFIED EXP-FUNCTION METHOD

Modified Exp-Function method [17] is very efficient and gives the result in very few steps. It reduces the step size and computational work as compared to Exp-Functional method. It is also very easy and elegant method to find the solution of non-linear partial differential equations like KdVB equation. Basically, homogeneous balancing is needed to be done in Exp-Function method to make it modifies form. Here is the methodology of this method described in detail:

Methodology

Consider the general non-linear pde as follows:

$$P\left(u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{tt}, u_{xy}, u_{xt}, u_{yt}, \ldots\right) = 0.$$

By applying the transformation

$$\eta = kx + ly + \omega t.$$

where k, l and ω are constants. The equation (4) becomes an ODE of the form as given:

$$Q(u, ku', lu', \omega u', \dots) = 0. \tag{9}$$

Integrate the equation (9) for simplicity and put constants of integration equals to zero. In the next step, suppose the solution given by modified Exp-function method as follows:

$$u(\eta) = \frac{\sum_{n=0}^{2M} a_n \exp[n\eta]}{\sum_{n=0}^{2M} b_n \exp[n\eta]}, \tag{10}$$

a_n and b_n are constants and M is an integer and they are unknowns to be determined.

Next, to find the value of M, balance the linear terms of highest order with the non-linear terms of highest order of equation (10).

Putting equation (10) in equation (9), we get an equation in the powers of$\exp[\eta]$. Then we can further solve the problem by any computer program. Maple 18 may use to obtain the results. Exact solution can be found by using values of all parameters in equation (10).

2.3. VARIATIONAL ITERATION METHOD

The Variational iteration method [6, 7] is very useful and efficiently technique for both linear /non-linear ODEs and PDEs equations. Unlike ADM, this method deals with non-linear terms in the same manner as it deals with linear terms in the same steps. So, it is considered more capable method than ADM to handle non-linear problems.
Consider the differential equation of the form:

$$L(u) + N(u) = g(x), \tag{11}$$

In this equation, L is linear and N is a non-linear operator where $g(x)$ is a non-homogeneous function. The correction functional method of VIM can be written as:

$$u_{n+1}(x) = u_n(x) + \lambda \int_0^x \{Lu_n(t) + N\widetilde{u_n}(t) - g(t)\}\, dt, \tag{12}$$

where λ is a langrange multiplier and $\widetilde{u_n}$ is restricted variation such that $\delta(\widetilde{u_n}) = 0$.

In this method, u_0 can be selected from the given initial guess. The final solution is obtained by:

$$u(x) = \lim_{n\to\infty} u_n. \tag{13}$$

2.4. THE ADOMIAN DECOMPOSITION METHOD

The Adomian Decomposition Method [9] is applicable for both linear and non-linear problems but the Adomian polynomials are much important to be found before solving any non-linear problem. The method was given the name after the name of George Adomian, who first developed this method in 1970.

Consider a non-linear equation of the form:

$$L(u) + N(u) + R(u) = g(x), \tag{14}$$

where L is the highest order operator, N is the non-linear term and R is remaining linear terms. Apply the inverse linear operator L^{-1} on both sides of equation (14), after applying the initial conditions, this equation takes the form:

$$u(x) = f(x) - L^{-1}\{N(u) + R(u)\}, \tag{15}$$

where $f(x)$ is the function obtained after applying L^{-1} on $g(x)$. The non-linear terms are represented by an infinite number of Adomian polynomials: $A_0, A_1, A_2, \dots$.

The solution by ADM is obtained by:

$$u(x) = \sum_{n=0}^{\infty} u_n. \tag{16}$$

3. APPLICATION OF METHODS ON KdVB EQUATION

In this section, we presented the results of KdVB equation to show the efficiency and high accuracy of the proposed methods. Maple is use to give the numerical values in easy way.

3.1. ANALYSIS OF EXP-FUNCTION METHOD ON KDVB EQUATION

Consider the equation (1) and apply the transformation$\eta = kx + \omega t$, becomes:

$$\omega u' + \varepsilon k u u' - \nu k^2 u'' + \mu k^3 u''' = 0, \tag{17}$$

Equation (17) is now an ordinary differential equation. The trial solution of above equation can be written by using Exp-Function Method is discussed in equation (7) already. The equivalent solution for the given equation is written in equation (8) above. Now, we will balance the linear term of highest order with the non-linear term of highest order to find the values of c and p. The highest order linear term is u''' and highest order non-linear term is uu'. We get:

$$u''' = \frac{c_1 \exp[(c+3p)\eta] + \cdots}{c_2 \exp[4p\eta] + \cdots}, \tag{18}$$

and:

$$uu' = \frac{c_3 \exp[(2c+2p)\eta] + \cdots}{c_4 \exp[4p\eta] + \cdots}, \tag{19}$$

where $c_1, c_2, c_3$and$c_4$are the coefficients taken for simplicity.

Now balance (18) and (19), we get:

$$c + 3p = 2c + 2p, \tag{20}$$

From this, we get:

$$c = p, \tag{21}$$

Continuing in the same way as discussed in the methodology of this method, now, we balance the linear term of lowest order with the non-linear term of lowest order, which is given below:

$$u''' = \frac{\ldots + d_1 \exp[-(q+3d)\eta]}{\ldots + d_2 \exp[-4q\eta]}, \tag{22}$$

and

$$u''' = \frac{\ldots + d_3 \exp[-(2q+2d)\eta]}{\ldots + d_4 \exp[-4q\eta]}, \tag{23}$$

Now balancing above equation and we get:

$$-(2q+2d) = -(q+3d),$$

Hence, we get:

$$q = d. \tag{24}$$

We can solve it further by using Maple or any other computer aid. And the solution of this equation in the case when $p = c = 1$ *and* $q = d = 1$ is given as follows:

$$u(x,t) = \frac{\left[\frac{1}{25}\left(\frac{25\varepsilon a_{-1}\mu - 12\nu^2}{\varepsilon}\right) + a_{-1}\right]\cos(k(x+\alpha t)) \pm 2a_{-1}}{2\cos(k(x+\alpha t)) \pm 2} \tag{25}$$

Equation (25) is required solution. Now by using Maple 20, we plot the graph for the KdVB equation, which is given below.

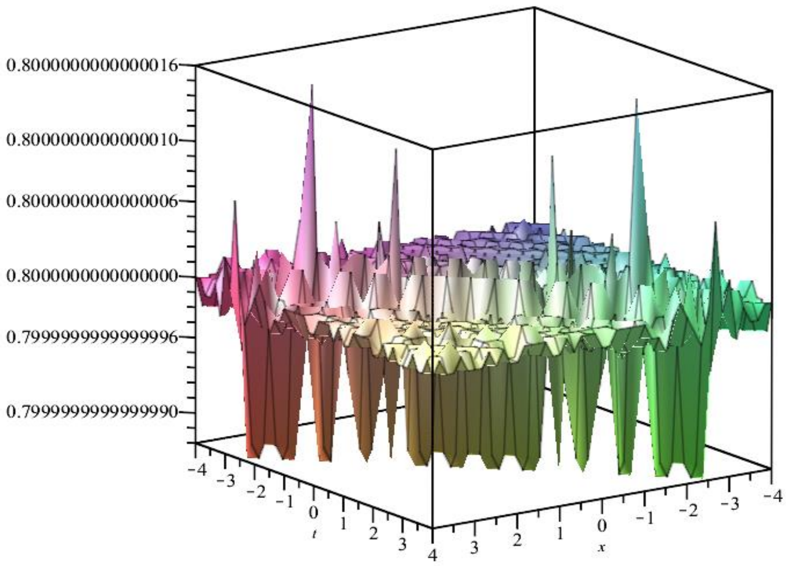

Figure 3.1. Graphical Representation for solution by Exp-Function Method.

3.2. ANALYSIS OF MODIFIED EXP-FUNCTION METHOD ON KDVB EQUATION

Consider the KdVB equation given in (1), now we will apply Modified Exp-Function Method on (1). Apply the transformation as we have already done in the above method and we get the equation (17) as above. Now integrate equation (17) once, we get:

$$\omega u + \varepsilon k u^2 - \nu k^2 u' + \mu k^3 u'' = 0, \tag{26}$$

Now balance the linear term of highest order u'' with the non-linear term of highest order u^2, such that:

$$2M = M + 2, \tag{27}$$

$$M = 2. \tag{28}$$

Then the trial solution (equation (10)) will become:

$$u(\eta) = \frac{a_0 + a_2 \exp[2\eta] + a_4 \exp[4\eta]}{b_0 + b_2 \exp[2\eta] + b_4 \exp[4\eta]}, \tag{29}$$

Putting (29) in (26), we will get an ODE which can be solved by any computer aid like Maple. By using Maple 20, the plot KdVB equation is as follows:

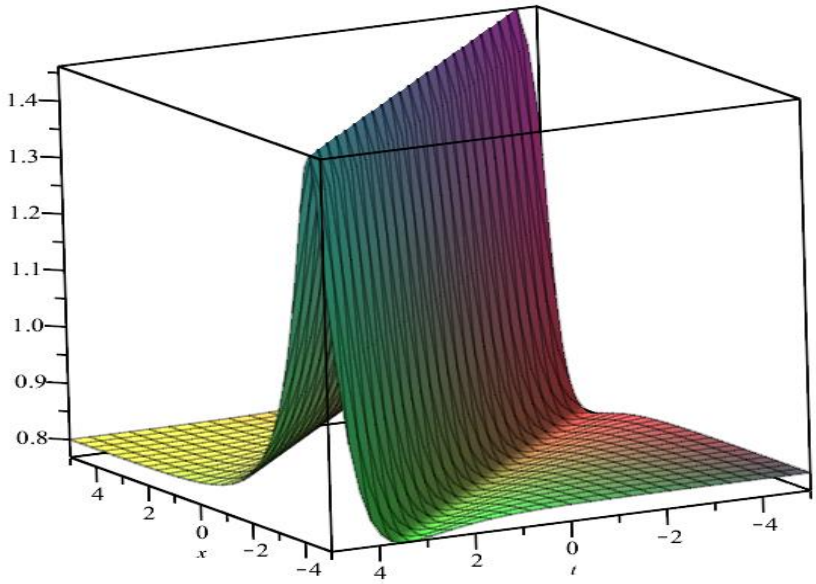

Figure 3.2. Graphical Representation for solution by Modified Exp-Function Method.

3.3. ANALYSIS OF VARIATIONAL ITERATION METHOD KDVB EQUATION

Consider the KdVB equation already discussed above. Here, we will apply VIM to finding the solution KdVB equation. First, we consider the following correction functional for equation (1):

$$u_{n+1}(x,t) = u_n(x,t) + \lambda \int_0^x \{ u_{nt} + \varepsilon u_n \widetilde{u_{nx}} - \nu u_{nxx} + \mu u_{nxxx} \} dt, \tag{30}$$

and $\delta(u_n \widetilde{u_{nx}})$ is a restricted variation. The Lagrange multiplier can be taken as $\lambda = -1$, we have:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^x \{ u_{nt} + \varepsilon u_n \widetilde{u_{nx}} - \nu u_{nxx} + \mu u_{nxxx} \} dt, \tag{31}$$

This method can further be solvable by using any computational aid like Maple.
For $\varepsilon = 1$, we the obtained the required solution with the help of Maple, As shown.

$$u(x,t) = -\frac{6\nu^2}{25\mu}\left[1 + tanh\left(\frac{\nu}{10\mu}\left(x + \frac{6\nu^2}{25\mu}t\right)\right) - \frac{1}{2}sech^2\left(\frac{\nu}{10\mu}\left(x + \frac{6\nu^2}{25\mu}t\right)\right)\right] \quad (32)$$

Equation (32) is required solution. By using Maple 20, the plot of KdVB equation is as follows:

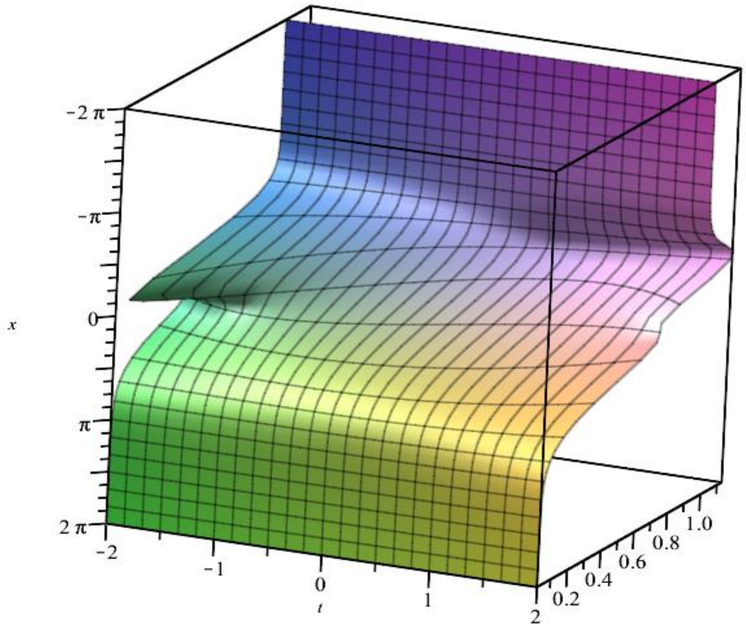

Figure 3.3. Graphical representation for solution by Variational Iteration Method.

3.4. ANALYSIS OF ADOMIAN DECOMPOSITION METHOD ON KDVB EQUATION

Now, we will apply ADM [11-12] on KdVB equation. The linear operators for the equation (1) are:

$$L_t = \frac{\partial}{\partial t}, L_x = \frac{\partial}{\partial x}, L_{xx} = \frac{\partial^2}{\partial x^2}, L_{xxx} = \frac{\partial^3}{\partial x^3}.$$

and the inverse operator is L_t^{-1}. The general solution of equation (1) can be written by using ADM is as follows:

$$u(x,t) = u(x,0) - L_t^{-1}\{\varepsilon\phi(u) - \nu L_{xx}(u) + \mu L_{xxx}(u)\}, \quad (33)$$

where $u(x,0) = f(x)$ is assumed initial condition, which is important to find the above solution.

The solution of KdVB equation by ADM is obtained by decomposition series of the form:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \quad (34)$$

Now, using the initial condition in equation (33), we obtain the zero-th component u_0, following in this way, we can find the successive terms by the recursive formula given below:

$$u(x,t) = -L_t^{-1}\{\varepsilon A_n - \nu L_{xx}(u) + \mu L_{xxx}(u)\}, n \geq 0 \quad (35)$$

where $\phi(u) = uu_x = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \ldots, u_n)$ is the adomian polynomial. The general formula to calculate these polynomials is:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\phi\left(\sum_{k=0}^{\infty}\lambda^k u_k\right)\right]_{\lambda=0}, n \geq 0 \tag{36}$$

This formula can be used in computer aids like Maple, to find as many as adomian polynomials as possible. Some of the adomian polynomials for uu_x can easily be calculated and are mentioned below:

$$A_0 = u_0 u_{0_x},$$

$$A_1 = u_1\, u_{0_x} + u_0\, u_{1_x},$$

$$A_1 = u_2\, u_{0_x} + u_1\, u_{1_x} + u_0\, u_{2_x},$$

and so on.

The final solution can be obtained by using Maple as,

$$\begin{aligned} u(x,t) = 0.5 + 0.24 sech^2 x - 0.48\, tanh\, x + \frac{0.24t(sinh\, x + cosh\, x)}{cosh^3\, x} \\ + \frac{0.06t^2(2\, cosh\, x\, sinh\, x + 2cosh^2\, x - 3)}{cosh^4\, x} \\ + \frac{0.02t^3(-6\, sinh\, x + 2\, sinh\, x\, cosh^2\, x - 3\, cosh\, x + 2cosh^3\, x)}{cosh^5\, x} \end{aligned} \tag{37}$$

Equation (37) is required solution. Now, we plot the graph of solution by using Maple 20 and the result is as follows:

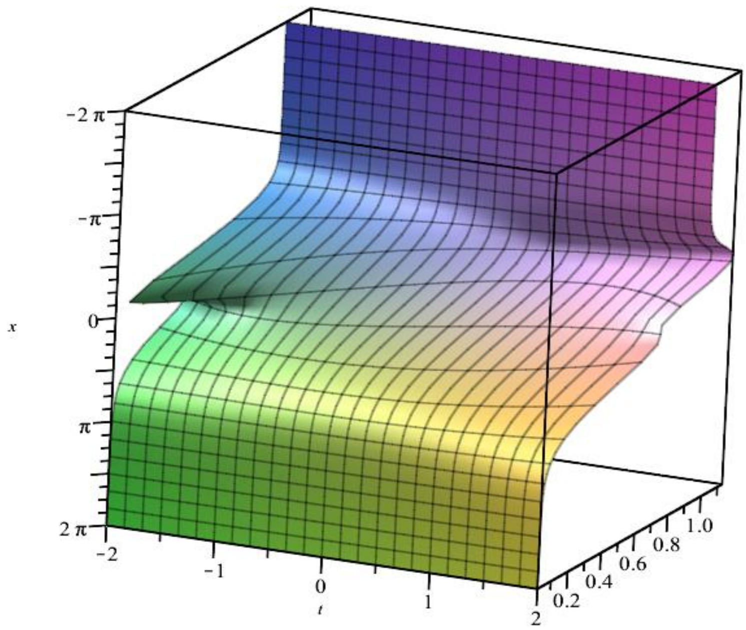

Figure 3.4. Graphical representation for solution by Adomian Decomposition Method.

4. CONCLUSION

In this work, we observe the solution KdVB equation by using different analytical techniques. We obtained these results by using the Maple. It can be seen that these techniques are very efficient and having powerful tools to solve the KdVB equation. We can observe that these methods can be effectively and elegantly applicable on other non-linear PDEs like solitary wave and periodic wave equations.

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