ORIGINAL PAPER

CONSTRUCTION OF BINORMAL MOTION AND CHARACTERIZATION OF CURVES ON SURFACE BY SYSTEM OF DIFFERENTIAL EQUATIONS FOR POSITION VECTOR

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Abstract. *The main purpose of this paper is to investigate unit speed curve with constant geodesic, normal curvature and geodesic torsion of curve on a surface in the Euclidean 3-space. In accordance with this scope, the position vector of a curve is stated by a linear combination of its Darboux Frame with differentiable functions. Some special results have been obtained within the scope of this position curve and differentiable functions. As a physical application of obtained results, differential geometric properties of a surface with binormal motion of a given curve are given with the obtained characterization of the curve.*

Keywords: *Binormal motion; Darboux frame; Surface-twisted curve.*

1. INTRODUCTION

Surfaces are a part of our daily lives. We encounter surfaces almost every day such as cans, tubes or balls etc. In order to study the differential geometry of surfaces, we sometimes use curves on the surfaces. For this purpose, we get help from different frame fields of the curve to complete the investigation. Since the curve lies on a surface, then we need to construct a new frame, namely Darboux frame, which includes both terms of curve and surface.

In 3-dimensional Euclidean space some surfaces can be defined by integrable equations. An example of these types of surfaces is Hasimoto surfaces. These surfaces are known as the smoke ring equation or localized induction equation of a regular space curve. According to the following equation

$$r_t = r_s \times r_{ss}$$

is obtained by evolving over time. Since the equations are as following

$$r_s = T, r_{ss} = T' = \kappa N,$$

It can be also written as

$$r_t = \kappa B.$$

The solution of this equation is as follows

$$r(s, v) = \alpha(s) + v\,\kappa B(s).$$

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The $r(s,v)$ surface is the ruled surface formed by the line moving with the binormal direction and base curve. Because the binormal line moves on the α (s) curve, this surface is called the binormal motion of the curve [1].

In Euclidean 3-space, Darboux frame is constructed by unit tangent vector field of the curve, unit normal vector field of the surface and their vector product. Similar to Frenet-Serret formulas, the derivative of frame vector fields can be expressed in terms of vector fields. As a result, the normal curvature, geodesic curvature and geodesic torsion of Darboux frame were defined [2]. Using these curvature functions, characterizations of some special curves on a given surface, are investigated in [2-4].

It is preferable to use the position vector to study the character of a curve. If the position vector of a curve can be expressed as a linear combination of frame fields of the curve, the characterization of the curve becomes a problem of differential equations. Many studies dealing with this problem are also available in the literature. Such as in [5-10].

In all aforementioned studies, the position vector of the curve obtained with the help of the Serret- Frenet frame is examined. Since the main purpose of this study is to examine some special curves on the surface, Darboux frame fields will be preferred over the Serret-Frenet frame fields.

This study consists of three main parts. Firstly, a brief summary of Frenet and Darboux frames for unit speed curves in in Euclidean 3-space are stated. Later, the relations between Frenet and Darboux frames are stated. Moreover, curvature functions of (α, M) curve-surface couple are given in terms of curvature κ and torsion τ. The fundamental forms, the Gaussian and Mean curvature of a given surface are stated for necessary background. Secondly, the characterization of unit speed curves on a surface with constant curvatures according to Darboux Frame is investigated with the use of the position vector of the curve. Thirdly, differential geometric properties of a surface with binormal motion of a given curve are obtained with the obtained characterization of the curve.

2. PRELIMINARIES

In this section, the necessary information to understand the main subject of the study will be given.

Since the curve $\alpha(s)$ is also in space, there exists Frenet frame $\{t, n, b\}$ at each points of the curve where t is unit tangent vector, n is principal normal vector and b is binormal vector, respectively. The Frenet equations of the curve $\alpha(s)$ is given by

$$t' = \kappa n$$

$$n' = -\kappa t + \tau b$$

$$b' = -\tau n$$

The functions κ and τ are determined by the first and third formula and called the first and second curvatures of $\alpha(s)$, or the curvature and the torsion of $\alpha(s)$ respectively.

The planes spanned by $\{t, n\}, \{t, b\}, \{n, b\}$ are called as the osculating plane, the rectifying plane and the normal plane, respectively. Since the curve $\alpha(s)$ lies on the surface M there exists another frame of the curve $\alpha(s)$ which is called Darboux frame and denoted by $\{t, y, N\}$. In this frame t is the unit tangent of the curve, N is the unit normal of the surface M

and $y = t \times N$ is a unit vector. So that the relations between these frames can be given as follows

$$\begin{bmatrix} t \\ y \\ N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

where θ is the angle between the vector fields N and b.

If the derivatives of Darboux frame are taken with respect to s, then obtained as follows

$$t' = k_g y + k_n N$$

$$y' = -k_g t + t_r N$$

$$N' = -k_n t - t_r y.$$

So above equations can be obtained in following form

$$\begin{bmatrix} t' \\ y' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & t_r \\ -k_n & -t_r & 0 \end{bmatrix} \begin{bmatrix} t \\ y \\ N \end{bmatrix}$$

where, k_g defined by equality

$$k_g(s) = \langle \alpha''(s), y(s) \rangle$$

is the geodesic curvature, k_n is the normal curvature defined by equality

$$k_n(s) = \langle \alpha''(s), N(s) \rangle$$

and t_r defined by equality

$$t_r(s) = -\langle N'(s), y(s) \rangle$$

is the geodesic torsion of $\alpha(s)$. Then, following relations are provided

$$k_g = \kappa cos\theta$$

$$k_n = \kappa sin\theta$$

$$t_r = \tau - \theta'.$$

For a curve $\alpha(s)$ lying on a surface, the following cases are satisfied:

i) $\alpha(s)$ is a geodesic curve if and only if $k_g(s) = 0$,

ii) $\alpha(s)$ is a asymptotic curve if and only if $k_n(s) = 0$,

iii) $\alpha(s)$ is a principal line curve if and only if $t_r(s) = 0$ [11-13].

The unit normal vector field N on a surface M can be defined by

$$N = \frac{\langle M_S, M_v \rangle}{\| M_S, M_v \|}.$$

The first fundamental form characterizes the interior geometry of the surface in a neighbourhood of a given point. This means that measurements on the surface can be carried out by means of it. The first fundamental form is given by

$$I = Eds^2 + 2Fdsdv + Gdv^2$$

such that

$$E = \langle M_s, M_s \rangle, F = \langle M_s, M_v \rangle, G = \langle M_v, M_v \rangle.$$

At the same time the second fundamental form

$$II = eds^2 + 2fdsdv + gdv^2$$

where

$$e = \langle M_{ss}, N \rangle, f = \langle M_{sv}, N \rangle, g = \langle M_{vv}, N \rangle.$$

The Gauss curvature is defined as Weingarten map

$$K = detS$$

and the mean curvature is related to the trace as follows

$$H = \frac{1}{2} trS.$$

The Gauss and mean curvatures of a parametrized surface can be computed as

$$K = \frac{eg - f^2}{EG - F^2}$$

$$H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)}$$

A surface is called minimal surface if its mean curvatures vanishes [11, 12].

3. THE CHARACTERIZATION OF CURVES WITH CONSTANT CURVATURES ACCORDING TO DARBOUX FRAME

In this section, the characterization of a curve $\alpha: I \to E^3$ given by arc length parameter is investigated in terms of its geodesic curvatures, normal curvatures and geodesic torsion functions.

Definition 3.1. Let $\alpha: I \to M$ be a regular curve given by arc length parameter. If the position vector of the curve α on the surface M can be written in the following form

$$\alpha(s) = p_0(s)t(s) + p_1(s)y(s) + p_2(s)N(s),$$

then the curve α is called a surface-twisted curve on M where $p_0, p_1, p_2: I \to \mathbb{R}$ are differentiable functions and $\{t, y, N\}$ is Darboux frame of (α, M).

Theorem 3.1. Let $\alpha: I \to M$ be a surface-twisted curve given by arc length parameter. If $\alpha(s)$ is a curve with constant geodesic curvatures, normal curvatures and geodesic torsion functions, then the position vector $\alpha(s)$ is stated

$$\alpha(s) = p_0(s)t(s) + p_1(s)y(s) + p_2(s)N(s)$$

with the following differentiable functions

$$p_0(s) = c_0 t_r + c_1\left(ak_n sin(as) - k_g t_r cos(as)\right) - c_2\left(ak_n cos(as) + k_g t_r sin(as)\right) + \frac{1}{a^2} s t_r^2,$$

$$p_1(s) = -c_0 k_n + c_1\left(at_r sin(as) - k_g k_n cos(as)\right) - c_2\left(at_r cos(as) - k_g k_n sin(as)\right) - \frac{1}{a^2}\left(k_g + s k_n t_r\right),$$

$$p_2(s) = c_0 k_g + c_1(k_n^2 cos(as) + t_r^2 cos(as)) + c_2\left(k_n^2 sin(as) + t_r^2 sin(as)\right) - \frac{1}{a^2}\left(k_n + s k_g t_r\right)$$

where c_i are arbitrary constants for $0 < i < 2,\ a^2 = \mathrm{k}_\mathrm{g}^2 + \mathrm{k}_\mathrm{n}^2 + \mathrm{t}_\mathrm{r}^2$.

Proof: The position vector of the curve with Darboux Frame is stated as follows

$$\alpha(s) = p_0(s)t(s) + p_1(s)y(s) + p_2(s)N(s) \tag{3.1}$$

Differentiating equation 3.1 with respect to the arc length parameter s, then folowing equality is obtained

$$\alpha'(s) = (p_0{}'(s) - k_g p_1(s) - k_n p_2(s))t(s) + (p_0(s)k_g + p'_1(s) - p_2(s)t_r)\,y(s) + (p_0(s)k_n + p_1(s)t_r + p'_2(s))N(s).$$

It follows that

$$p_0{}'(s) = 1 + k_g p_1(s) + k_n p_2(s)$$

$$p'_1(s) = p_2(s)t_r - p_0(s)k_g$$

$$p'_2(s) = -p_0(s)k_n - p_1(s)t_r.$$

It is clear that the above equation can be written in the following form

$$\begin{bmatrix} p_0{}' \\ p'_1 \\ p'_2 \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & t_r \\ -k_\mathrm{n} & -t_r & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The eigenvalues of the matrix of the nonhomogeneous linear system is given by

$$\lambda_1 = \sqrt{-k_g^2 - k_n^2 - t_r^2}\,,$$
$$\lambda_2 = \sqrt{-k_g^2 - k_n^2 - t_r^2},$$
$$\lambda_3 = 0$$

and the eigenvectors corresponding to these eigenvalues are as follows

$$V_1 = \begin{bmatrix} -(k_g t_r + k_n\sqrt{-k_g^2 - k_n^2 - t_r^2}) \\ k_g k_n - t_r\sqrt{-k_g^2 - k_n^2 - t_r^2} \\ k_n^2 + t_r^2 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} -(k_g t_r - k_n\sqrt{-k_g^2 - k_n^2 - t_r^2}) \\ k_g k_n + t_r\sqrt{-k_g^2 - k_n^2 - t_r^2} \\ k_n^2 + t_r^2 \end{bmatrix},$$

$$V_3 = \begin{bmatrix} t_r \\ -k_n \\ k_g \end{bmatrix},$$

respectively. If substitute a^2 for $k_g^2 + k_n^2 + t_r^2$ in the eigenvalues and eigenvectors obtained above, the followings are obtained

$$\lambda_1 = ai \rightarrow V_1 = \begin{bmatrix} -k_g t_r - k_n ai \\ k_g k_n - t_r ai \\ k_n^2 + t_r^2 \end{bmatrix}$$

$$\lambda_2 = -ai \rightarrow V_2 = \begin{bmatrix} -k_g t_r + k_n ai \\ k_g k_n + t_r ai \\ k_n^2 + t_r^2 \end{bmatrix}$$

$$\lambda_3 = 0 \rightarrow V_3 = \begin{bmatrix} t_r \\ -k_n \\ k_g \end{bmatrix}.$$

Thus, homogeneous solution of the differential equation is obtained as follows

$$X_h = c_0 X_1 + d_1 X_2 + d_2 X_3$$

such that

$$X_1 = \begin{bmatrix} -\big(cos(as) - isin(as)\big)(k_g t_r - iak_n) \\ \big(cos(as) - isin(as)\big)(iat_r + k_g k_n) \\ \big(cos(as) - isin(as)\big)\,(k_n^2 + t_r^2) \end{bmatrix}$$

$$X_2 = \begin{bmatrix} -\left(cos(as) - isin(as)\right)(k_g t_r - iak_n) \\ \left(cos(as) - isin(as)\right)(iat_r + k_g k_n) \\ \left(cos(as) - isin(as)\right)(k_n^2 + t_r^2) \end{bmatrix}$$

$$X_3 = \begin{bmatrix} t_r \\ -k_n \\ k \end{bmatrix}$$

If the values of X_1, X_2, X_3 are replaced above matrices, the homogeneous solution of the differential equation is found as follows

$$X_h = c_0 \begin{bmatrix} t_r \\ -k_n \\ k_g \end{bmatrix} + (d_1 + d_2) \begin{bmatrix} -k_g t_r cos(as) + k_g t_r sin(as) \\ k_g k_n cos(as) + t_r a sin(as) \\ k_n^2 cos(as) + t_r^2 cos(as) \end{bmatrix}$$
$$+(-id_1 + id_2) \begin{bmatrix} k_n a cos(as) + k_g t_r sin(as) \\ -k_g k_n sin(as) + t_r a cos(as) \\ -k_n^2 sin(as) - t_r^2 sin(as) \end{bmatrix}$$

Substituting $d_1 + d_2 = c_1$, $id_1 - id_2 = c_2$, following equality is obtained

$$X_h = c_0 \begin{bmatrix} t_r \\ -k_n \\ k_g \end{bmatrix} + c_1 \begin{bmatrix} -k_g t_r cos(as) + k_g t_r sin(as) \\ k_g k_n cos(as) + t_r a sin(as) \\ k_n^2 cos(as) + t_r^2 cos(as) \end{bmatrix} + c_2 \begin{bmatrix} k_n a cos(as) + k_g t_r sin(as) \\ -k_g k_n sin(as) + t_r a cos(as) \\ -k_n^2 sin(as) - t_r^2 sin(as) \end{bmatrix}.$$

The fundamental matrix of the nonhomogeneous linear differential system of the equation can be written as

$$\varphi(s) = \begin{bmatrix} t_r & -k_g t_r cos(as) + k_g t_r sin(as) & k_n a cos(as) + k_g t_r sin(as) \\ -k_n & k_g k_n cos(as) + t_r a sin(as) & -k_g k_n sin(as) + t_r a cos(as) \\ k & k_n^2 cos(as) + t_r^2 cos(as) & -k_n^2 sin(as) - t_r^2 sin(as) \end{bmatrix}.$$

By using the equality $X_p = \varphi(s)u(s)$, particular solution of differential equation and vector values function $u(s)$ can be found with following equality

$$\varphi(s)u'(s) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Using linear equation by Crammers method, the vector values function u(s) obtained as follows

$$u_1(s) = \frac{1}{a^3} s t_r$$

$$u_2(s) = -\frac{1}{a^3 (k_n^2 + t_r^2)} (ak_n cos(as) + k_g t_r sin(as))$$

$$u_3(s) = \frac{1}{a^3 (k_n^2 + t_r^2)} \left(ak_n sin(as) - k_g t_r sin(as)\right).$$

Then

$$u(s) = \begin{bmatrix} \frac{t_r}{\alpha^3} s \\ -\frac{1}{a^3(k_n^2 + t_r^2)}(ak_n cos(as) + \ k_g t_r sin(as)) \\ \frac{1}{a^3(k_n^2 + t_r^2)}(ak_n sin(as) + \ k_g t_r cos(as)) \end{bmatrix}$$

particular solution of the equation is obtained as follows

$$X_p(s) = \ \varphi(s)u(s) = \begin{bmatrix} \frac{1}{\alpha^2} s t_r^2 \\ -\frac{1}{\alpha^2}(k_g + \ sk_n t_r) \\ -\frac{1}{a^2}(k_n - sk_g t_r) \end{bmatrix}.$$

Since $X_g = X_h + \ X_p$, then it is seen that

$$p_0(s) = c_0 t_r + c_1\left(ak_n sin(as) - k_g t_r cos(as)\right) - c_2\left(ak_n cos(as) + k_g t_r sin(as)\right) + \frac{1}{a^2} s t_r^2$$

$$p_1(s) = - c_0 k_n + c_1\left(at_r sin(as) - k_g k_n cos(as)\right) - c_2\left(at_r cos(as) - k_g k_n sin(as)\right) - \frac{1}{a^2}\left(k_g + sk_n t_r\right)$$

$$p_2(s) = c_0 k_g + c_1(k_n^2 cos(as) + t_r^2 cos(as)\ ) + c_2\left(k_n^2 sin(as) + t_r^2 sin(as)\right) - \frac{1}{a^2}\left(k_n + sk_g t_r\right)$$

Theorem 3.2. Let $\alpha: I \to M$ be a surface-twisted curve given by arc length parameter. If α is an asymptotic curve, then the position vector of $\alpha(s)$ can be stated as follows

$$\alpha(s) = p_0(s)t(s) + p_1(s)y(s) + p_2(s)N(s)$$

such that

$$p_0 = c_0 t_r - c_1 k_g t_r cos(as) - c_2 k_g t_r sin(as) + \frac{1}{a^2} s t_r^2,$$

$$p_1 = c_1 a t_r sin(as) - c_2 a t_r cos(as) - \frac{1}{a^2} k_g,$$

$$p_2 = c_0 k_g + c_1 t_r^2 cos(as) + c_2 t_r^2 sin(as)$$

where c_i are arbitrary constants for $0 < i < 2,\ a^2 = \mathrm{k}_\mathrm{g}^2 + \mathrm{k}_\mathrm{n}^2 + \mathrm{t}_\mathrm{r}^2$.

Theorem 3.3. Let $\alpha: I \to M$ be a surface-twisted curve given by arc length parameter. If α is a geodesic curve, then the position vector of $\alpha(s)$ can be written as follows

$$\alpha(s) = p_0(s)t(s) + p_1(s)y(s) + p_2(s)N(s)$$

such that

$$p_0 = c_1\big(ak_n sin(as)\big) - c_2\big(ak_n cos(as)\big) + \ c_0 t_r + \frac{1}{a^2} st_r^2,$$

$$p_1 = c_1\big(at_r sin(as)\big) - \frac{1}{a^2}(sk_n t_r) - c_2\big(at_r cos(as)\big) - \ c_0 k_n$$

$$p_2 = -\frac{1}{a^2}(k_n) + c_2\big(k_n^2 sin(as) + t_r^2 sin(as)\big) + c_1(k_n^2 cos(as) + t_r^2 cos(as)\ )$$

where c_i are arbitrary constants for $0 < i < 2,\ a^2 = \mathrm{k_g^2} + \mathrm{k_n^2} + \mathrm{t_r^2}$.

Theorem 3. 4. Let $\alpha: I \to M$ be a surface-twisted curve given by arc length parameter. If α is a asymptotic line, then the position vector of $\alpha(s)$ can be written as follows

$$\alpha(s) = p_0(s)t(s) + p_1(s)y(s) + p_2(s)N(s)$$

such that

$$p_0 = c_1\big(ak_n sin(as)\big) - c_2\big(ak_n cos(as)\big),$$

$$p_1 = c_1\left(-k_g k_n cos(as)\right) - \tfrac{1}{a^2}\left(k_g\right) - c_2\left(-k_g k_n sin(as)\right) - \ c_0 k_n,$$

$$p_2 = c_0 k_g - \frac{1}{a^2}(k_n) + \left(\big(sin(as)\big)k_n^2\right)c_2 + \left(\big(cos(as)\big)k_n^2\ \right)c_1$$

where c_i are arbitrary constants for $0 < i < 2,\ a^2 = \mathrm{k_g^2} + \mathrm{k_n^2} + \mathrm{t_r^2}$.

Example 3.1. Let us consider the cylinder $M = \varphi(\mathbb{R}^2)$ where

$$\varphi(u, v) = (\cos u\,, \sin u\,, v).$$

The unit speed curve $\alpha: \mathbb{R} \to M$ is given with the following parametrization

$$\alpha(s) = \varphi(\frac{s}{2}, \frac{\sqrt{3}s}{2}) = (\cos\frac{s}{2}, \sin\frac{s}{2}, \frac{\sqrt{3}s}{2}).$$

Then unit tangent vector field of the curve α obtained as follows

$$t(s) = \left(-\frac{1}{2}\sin\frac{s}{2}, \frac{1}{2}\cos\frac{s}{2}, \frac{\sqrt{3}}{2}\right).$$

Since

$$\frac{\partial}{\partial u}\varphi(u, v) = \varphi_u(u, v) = (-\sin u\,, \cos u\,, 0),$$

$$\tfrac{\partial}{\partial v}\varphi(u, v) = \varphi_v(u, v) = (0,0,1).$$

Then, the unit normal vector field of the surface is given as

$$N(u,v)=\frac{\varphi_u(u,v)\times\varphi_u(u,v)}{\|\varphi_u(u,v)\times\varphi_u(u,v)\|}=(\cos u\,,\sin u\,,0).$$

Therefore, unit normal vector field of the surface is obtained as

$$N(s)=N(\alpha(s))=(\cos\tfrac{s}{2},\sin\tfrac{s}{2},0).$$

Finally, $y(s)$is found as

$$y(s)=N(s)\times t(s)=\left(\frac{\sqrt{3}}{2}\sin\frac{s}{2},-\frac{\sqrt{3}}{2}\cos\frac{s}{2},\frac{1}{2}\right).$$

Thus, the normal curvature, geodesic curvature and geodesic torsion functions of the curve α are found as

$$k_n(s)=-\frac{1}{4},k_g(s)=0,t_r(s)=\frac{\sqrt{3}}{4},$$

respectively.

Since these functions are all constant, then Theorem 3.1 is used to obtain the position vector of the curve α as a linear combination of Darboux frame fields. Following equality is given

$$a^2=k_n^2+k_g^2+t_r^2=\frac{1}{4}.$$

By substituting the values of the constant curvatures and $a=\frac{1}{2}$ into the equations in Theorem 3.1, the differentiable functions are found as

$$p_0(s)=\frac{3}{4}s,$$

$$p_1(s)=\frac{\sqrt{3}}{4}s,$$

$$p_2(s)=1.$$

Therefore, the position vector of the curve α is expressed as follows:

$$\alpha(s)=\frac{3}{4}st(s)+\frac{\sqrt{3}}{4}sN(s)+y(s).$$

Example 3. 2 Let us consider the cylinder $M=\varphi(\mathbb{R}^2)$ where

$$\varphi(u,v)=(\cos u\,,\sin u\,,v).$$

The unit speed curve $\alpha\colon\mathbb{R}\to M$ is given with the following parametrization

$$\alpha(s)=\varphi(0,s)=(1,0,s).$$

Then, unit tangent vector field of the curve α obtained as follows

$$t(s) = (0,0,1).$$

Since

$$\frac{\partial}{\partial u}\varphi(u,v) = \varphi_u(u,v) = (-\sin u\,, \cos u\,, 0),$$

$$\frac{\partial}{\partial v}\varphi(u,v) = \varphi_v(u,v) = (0,0,1),$$

Then, the unit normal vector field of the surface obtained as

$$N(u,v) = \frac{\varphi_u(u,v) \times \varphi_u(u,v)}{\|\varphi_u(u,v) \times \varphi_u(u,v)\|} = (\cos u\,, \sin u\,, 0).$$

Therefore,

$$N(s) = N(\alpha(s)) = (1,0,0).$$

Finally, $y(s)$ is found as

$$y(s) = N(s) \times t(s) = (0,-1,0).$$

Thus,

$$k_n(s) = k_g(s) = t_r(s) = 0,$$

Since these functions are all zero, then it can bu used to obtain the position vector of the curve α as a linear combination of Darboux frame fields. The differentiable functions are given as

$$p_0(s) = s,$$

$$p_1(s) = 0,$$

$$p_2(s) = 1.$$

Therefore, the position vector of the curve α obtained as follows:

$$\alpha(s) = st(s) + y(s).$$

4. SURFACE WITH BINORMAL MOTION OF CURVES

In this section, firstly surface with binormal motion of curves is defined. Then, some properties of this surface are given and also get some special results depending on these properties. The surface with binormal motion of curves is parametrized by

$$M(s,v) = \alpha(s) + v\,\kappa\,b$$

such that

$$\alpha(s) = p_0(s)t(s) + p_1(s)y(s) + p_2(s)N(s).$$

Namely the surface is parametrized by

$$M(s,v) = \left(\frac{1}{\alpha^2} s t_r^2\right) t(\mathrm{s}) + \left(-\frac{1}{\alpha^2}\left(\left(k_g + s k_n t_r\right) + v\kappa sin\theta\right)\right) y(s) \\ + \left(-\frac{1}{\alpha^2}\left(k_n - s k_g t_r\right) + v\kappa cos\theta\right) N(s).$$

Theorem 4.1. The Gauss curvature K of $M(s,v)$ are obtained as follows

$$K = -\frac{\left(t_r\, \kappa^2 (1 - 2Acos\theta^2)\right)^2}{(v^2\, \kappa^4 {t_r}^2 + \kappa^2 A^2)^2}$$

such that

$$A = 1 - 2v k_g k_n.$$

Proof: The normal of ruled surface $M(s,v)$ is given by

$$N = \frac{1}{\sqrt{v^2\, \kappa^4 {t_r}^2 + \kappa^2 A^2}} (-v\, \kappa^2 t_r, -A\kappa cos\theta, A\kappa sin\theta)$$

The coefficients E, F and G of the first fundamental form of surface $M(s,v)$ are given as follows

$$E = v^2\, \kappa^2 {t_r}^2 + A^2$$

$$F = 0$$

$$G = \kappa^2.$$

After simple computations, one can easily obtain coefficients e, f and g of the second fundamental form of $M(s,v)$ obtained as follows

$$e = (sin\theta^2 - cos\theta^2)\sqrt{v^2\, \kappa^4 {t_r}^2 + \kappa^2 A^2}$$

$$f = \frac{\kappa^2 t_r (1 - 2Acos\theta^2)}{\sqrt{v^2\, \kappa^4 {t_r}^2 + \kappa^2 A^2}}$$

$$g = 0.$$

The Gauss curvature of a parametrized surface can be computed as follows

$$K = \frac{eg - f^2}{EG - F^2}.$$

So, values of coefficients of first and second fundamental form of $M(s,v)$ are substituted in above equation, Gauss curvature of $M(s,v)$ can be found as follows

$$K = -\frac{\left(t_r\, \kappa^2 (1 - 2Acos\theta^2)\right)^2}{\left(v^2\, \kappa^4 {t_r}^2 + \kappa^2 A^2\right)^2}.$$

Corollary 4.1. It is known that the necessary condition for surface to be developable is $K = 0$. If $A = 0$ or

$$\theta = \frac{(2k-1)}{2}\pi, k \in Z^{+}$$

then the surface $M(s, v)$ is developable.

Corollary 4.2. If the curve $\alpha(s)$ is an asymptotic line, then the surface $M(s, v)$ is developable.

Theorem 4. 2. The mean curvature H of $M(s, v)$ are given as follows

$$H = \frac{(sin\theta^2 - cos\theta^2)\,\kappa^2}{2\sqrt{v^2\,\kappa^4 {t_r}^2 + \kappa^2 A^2}}$$

such that

$$A = 1 - 2vk_g k_n.$$

Proof: If the values of the components of the first and second fundamental forms obtained in the proof of theorem 3.1 are written in place of the mean curvature equation and the necessary operations are performed, then the poof is completed.

Corollary 4.3. It is known that minimal surface is surface satisfying $H = 0$. So, surface $M(s, v)$ is minimal if and only if $\theta = \frac{\pi}{4}$.

Example 4.1. In this example, the curve $\alpha(s)$ in example 3.1 will be used as the base curve of the surface. Consider the surface formed by binormal direction

$$M(s, v) = \alpha(s) + v\,\kappa\, b(s)$$

with

$$\alpha(s) = (\cos\frac{s}{2}, \sin\frac{s}{2}, \frac{\sqrt{3}s}{2})$$

for $s \in (-\pi, \pi)$ is shown in following figure:

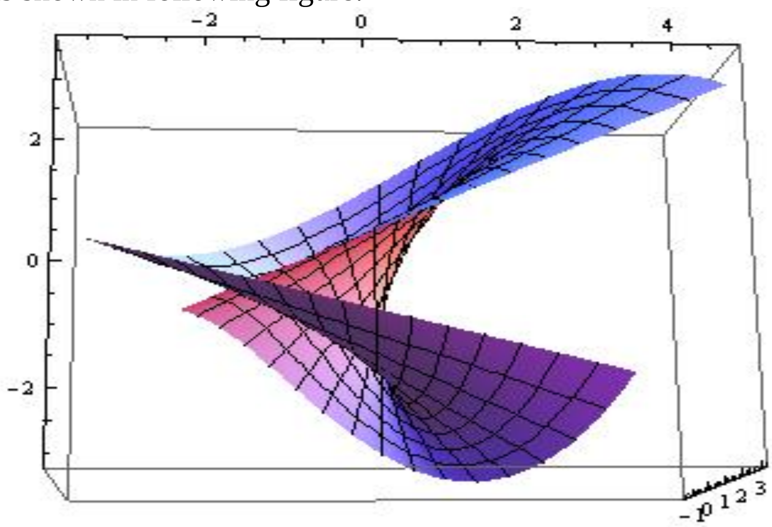

Figure 1. The ruled surface formed by binormal directions.

CONCLUSION

The paper is organized as follows. In section 2, the necessary informations to understand the main subject of the study is given. In section 3, characterization of a curve $\alpha: I \to E^3$ given by arc length parameter is obtained in terms of its geodesic curvatures, normal curvatures and geodesic torsion functions. In section 4, surface with binormal motion of curves is defined. Then, some properties of this surface is investigated and also some special results depending on these properties are given.

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