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DIRICHLET TYPE PROBLEM WITH P(.)-TRIHARMONIC OPERATOR

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Abstract. *In this study, we consider the existence of multiple weak solutions to a class of Dirichlet type problem involving $p(.)$-triharmonic*

$$\begin{cases} -\Delta^3_{p(.)}u = \lambda|u|^{q(.)-2}u \;\; in\; \Omega, \\ u = \Delta u = \Delta^2 u = 0 \quad on\; \partial\Omega, \end{cases}$$

under some suitable conditions.

Keywords: *p(.)-triharmonic operator; Variational methods; Fountain Theorem.*

1. INTRODUCTION

In this paper, we discuss the sixth-order nonlinear problem

$$\begin{cases} -\Delta^3_{p(.)}u = \lambda|u|^{q(.)-2}u \;\; in\; \Omega, \\ u = \Delta u = \Delta^2 u = 0 \quad on\; \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $p, q\epsilon C(\Omega)$ with $inf_{x\in\overline{\Omega}}p(x) > 1$, $\Delta^3_{p(.)} = div\big(\Delta(|\nabla\Delta u|)^{p(x)-2}\nabla\Delta u\big)$ is the $p(.)$-triharmonic operator of sixth order, λ >0 is a real number.

In recent years, variational mathematical problems with $p(.)$-growth have been studied in several topics, such as electrorheological fluids, image processing, elastic mechanics, fluid dynamics and calculus of variations [1-5]. Moreover, using compact embedding theorems and equivalent norms in variable exponent Sobolev spaces (weighted or unweighted) give good results to find weak solutions for elliptic and parabolic problems involving $p(.)$-Laplacian operator [3, 6-11].

In 2019, Rahal [12] investigate the existence of weak solutions to a class of nonlinear elliptic Navier boundary value problem involving the $p(.)$-Kirchhoff type triharmonic operator using Ekeland's variational principle and Mountain Pass Theorem. In addition, Shokooh [13] study infinitely many weak solutions for the nonlinear elliptic problem with $p(.)$-triharmonic operator. Since the problem (1) is the special case of the problem (1.1) in Rahal [12], we give only multiple solutions of (1) using Fountain Theorem.

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2. NOTATION AND PRELIMINARIES

To obtain the weak solutions of the p(x)-triharmonic problem (1), we give some basic properties of the variable exponent Lebesgue and Sobolev spaces $L^{p(.)}(\Omega)$ and $W^{k,p(.)}(\Omega)$.

Let

$$C_+(\overline{\Omega}) = \{p \epsilon C(\overline{\Omega}) : inf_{x \in \overline{\Omega}} p(x) > 1\},$$

and

$$p^- = essinf_{x\in\Omega} p(x) \text{ and } p^+ = esssup_{x\in\Omega} p(x)$$

for $p \epsilon C_+(\overline{\Omega})$. Define the space

$$L^{p(.)}(\Omega) = \left\{u | u: \Omega \to \mathbb{R} \text{ } is \text{ } measurable \text{ } and \text{ } \int_\Omega |u(x)|^{p(x)} dx < \infty\right\},$$

with the (Luxemburg) norm

$$\|u\|_{p(.)} = inf \left\{\beta > 0 \left| \rho_{p(.)}\left(\frac{u}{\beta}\right) \leq 1\right.\right\},$$

where

$$\rho_{p(.)}(u) = \int_\Omega |u(x)|^{p(x)} dx$$

for $p \epsilon C_+(\overline{\Omega})$ and $1 < p^- \leq p^+ < \infty$.

Let $k \epsilon \mathbb{Z}^+$. Then, the space $W^{k,p(.)}(\Omega)$ is defined by

$$W^{k,p(.)}(\Omega) = \{u \epsilon L^{p(.)}(\Omega) : D^\alpha u \epsilon L^{p(.)}(\Omega), 0 \leq \alpha \leq k\},$$

where $\alpha \epsilon \mathbb{N}_0^N$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N$ and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_N}^{\alpha_N}}$. Hence $W^{k,p(.)}(\Omega)$ is a separable and reflexive Banach space equipped with the norm

$$\|u\|_{k,p(.)} = \sum_{0\leq\alpha\leq k} \|D^\alpha u\|_{p(.)}.$$

The space $W_0^{1,p(.)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p(.)}(\Omega)$. We denote by

$$X = W_0^{1,p(.)}(\mathit{\Omega}) \cap W^{3,p(.)}(\Omega),$$

and define a norm $\|.\|_X$ by

$$\|u\|_X = \|u\|_{1,p(.)} + \|u\|_{2,p(.)} + \|u\|_{3,p(.)}.$$

Moreover, the norms $\|u\|_X$ and $\|\nabla\Delta u\|_{p(.)}$ are equivalent on X (see [12, 13]). Let

$$\|u\| = inf\left\{\mu > 0 \left| \int_\Omega \left|\frac{\nabla\Delta u}{\mu}\right|^{p(x)} dx \leq 1\right.\right\}$$

for any $u \in X$. Hence, we see that $\|u\|$ is equivalent to the norms $\|u\|_X$ and $\|\nabla\Delta u\|_{p(.)}$ in X (see [12]). Throughout the paper, we take the norm $\|u\|$ on the space X.

Proposition 2.1. ([12], Proposition 2.3) Let $q \epsilon C_+(\overline{\Omega})$ satisfying $q(x) < p^*(x)$ on Ω. Then, there exists a compact embedding $X \hookrightarrow L^{q(.)}(\Omega)$, where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-3p(x)}, p(x) < \frac{N}{3} \\ \infty \qquad , p(x) \geq \frac{N}{3} \end{cases}.$$

3. MAIN RESULTS

We say that u∈X is a weak solution of the problem (1) if

$$\int_\Omega |\nabla\Delta u|^{p(x)-2}\nabla\Delta u\nabla\Delta v dx - \lambda\int_\Omega |u|^{q(x)-2}uv dx = 0$$

for all $v \in X$.

Let us introduce the energy functional $\phi_\lambda : X \to \mathbb{R}$ defined by

$$\phi_\lambda(u) = \int_\Omega \frac{1}{p(x)}|\nabla\Delta u|^{p(x)}dx - \lambda\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx$$

for any $\lambda > 0$. It is easy to see that ϕ_λ is sequentially weakly lower semicontinuous, $\phi_\lambda \in C^1(X, \mathbb{R})$, and its Gâteaux derivative ϕ'_λ at $u \in X$ is given by

$$< \phi'_\lambda(u), v >= \int_\Omega |\nabla\Delta u|^{p(x)-2}\nabla\Delta u\nabla\Delta v dx - \lambda\int_\Omega |u|^{q(x)-2}uv dx$$

for all $v \in X$.

Set

$$\Psi_{p(.)}(u) = \int_\Omega |\nabla\Delta u|^{p(x)}dx$$

for any $u \in X$. Then, we have

$$\|u\| \leq 1 \Rightarrow \|u\|^{p^+} \leq \Psi_{p(.)}(u) \leq \|u\|^{p^-}$$

and

$$\|u\| \geq 1 \Rightarrow \|u\|^{p^-} \leq \Psi_{p(.)}(u) \leq \|u\|^{p^+}$$

(see [9]).

For the multiple solutions of the problem (1), we need the following well-known Lemmas.

Lemma 3.1. (see [9]) Let X be a reflexive and separable Banach space. Then there exists $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{span\{e_j : j = 1,2,\cdots\}},\ X^* = \overline{span\{e_j^* : j = 1,2,\dots\}}$$

and

$$< e_j^*, e_j >= \begin{cases} 1, & i = j \\ 0, & i \neq j' \end{cases}$$

where $<.,.>$ denotes the duality product between X and X^*.

For convenience, we write $X_j = span\{e_j\}$, $Y_k = \oplus_{j=1}^{k} X_j$, $Z_k = \oplus_{j=k}^{\infty} X_j$.

Lemma 3.2. Let $q \epsilon C_+(\overline{\Omega})$ satisfying $q(x) < p^*(x)$ on Ω. If the set α_k is defined by

$$\alpha_k = sup\{\|u\|_{q(.)} : \|u\| = 1, u \in Z_k\},$$

then $\lim_{k\to\infty} \alpha_k = 0$.

Proof: Using the continuous embedding $X \hookrightarrow L^{q(.)}(\Omega)$ by Proposition 2.1 and the method in Lemma 4.9 in [9], then we have $\lim_{k\to\infty} \alpha_k = 0$.

Theorem 3.3. Let $p^+ < q^-$. There are infinite many pairs of solutions of the problem (1), i.e., the functional ϕ_λ has a sequence of critical points $\{u_n\}$ such that $\phi_\lambda(u_n) \to \infty$.

Proof: The functional ϕ_λ is an even functional and fulfills the (PS) condition (see [12]). We show that

$$(A_1)\quad b_k = inf\{\phi_\lambda(u) : u \in Z_k, \|u\| = \gamma_k\} \to \infty \text{ as } k \to \infty$$

and

$$(A_2)\quad a_k = max\{\phi_\lambda(u) : u \in Y_k, \|u\| = \eta_k\} \leq 0$$

for the reel numbers γ_k and η_k such that $\eta_k > \gamma_k > 0$ when k is large enough.

(A_1) For any $u \in Z_k$ such that $\|u\| = \gamma_k > 1$, we have

$$\phi_\lambda(u) \geq \frac{1}{p^+}\Psi_{p(.)}(u) - \frac{\lambda}{q^-}\int_\Omega |u|^{q(x)}dx$$

$$\geq \frac{1}{p^+}\Psi_{p(.)}(u) - \frac{\lambda}{q^-} max\left\{\|u\|_{q(.)}^{q^-}, \|u\|_{q(.)}^{q^+}\right\}$$

$$\geq \frac{1}{p^+}\|u\|^{p^-} - \frac{\lambda}{q^-} max\left\{\|u\|_{q(.)}^{q^-}, \|u\|_{q(.)}^{q^+}\right\}$$

$$\geq \begin{cases} \frac{1}{p^+}\|u\|^{p^-} - \lambda, \|u\|_{q(.)} \leq 1 \\ \frac{1}{p^+}\|u\|^{p^-} - \lambda\alpha_k^{q^+}\|u\|^{q^+}, \|u\|_{q(.)} > 1 \end{cases}$$

$$\geq \frac{1}{p^+}\|u\|^{p^-} - \lambda\alpha_k^{q^+}\|u\|^{q^+}$$

$$\geq \frac{1}{p^+}\gamma_k^{p^-} - \lambda\alpha_k^{q^+}\gamma_k^{q^+}.$$

If we take $\gamma_k = \left(\lambda q^+\alpha_k^{q^+}\right)^{\frac{1}{p^- - q^+}}$, then we obtain

$$\phi_\lambda(u) \geq \frac{1}{p^+}\left(\lambda q^+\alpha_k^{q^+}\right)^{\frac{p^-}{p^- - q^+}} - \lambda\alpha_k^{q^+}\left(\lambda q^+\alpha_k^{q^+}\right)^{\frac{q^+}{p^- - q^+}}$$

$$= \left(\frac{1}{p^+} - \frac{1}{q^+}\right)\left(\lambda q^+\alpha_k^{q^+}\right)^{\frac{p^-}{p^- - q^+}} \to \infty$$

as $k \to \infty$ because $p^+ < q^+$ and $\alpha_k \to 0$.

(A_2) Let $u \in Y_k$ be such that $\|u\| = \eta_k > \gamma_k > 1$. Then, we get

$$\phi_\lambda(u) \leq \frac{1}{p^-}\|u\|^{p^+} - \frac{\lambda}{q^+}\int_\Omega |u|^{q(x)}dx$$

$$\leq \frac{1}{p^-}\|u\|^{p^+} - \frac{\lambda}{q^+}min\left\{\|u\|_{q(.)}^{q^-}, \|u\|_{q(.)}^{q^+}\right\}.$$

Since the space Y_k has finite dimension, the norms $\|u\|$ and $\|u\|_{q(.)}$ are equivalent. Finally,

$$\phi_\lambda(u) \to -\infty \text{ as } \|u\| \to +\infty, u \in Y_k$$

due to $p^+ < q^-$ by the Fountain Theorem ([14], Theorem 3.6).

4. CONCLUSION

In this paper, we discuss the existence of multiple weak solutions to a class of Dirichlet type problem (1) involving p(.)-triharmonic. Using compact embeddings of the space X, variational methods and Fountain Theorem, we get infinite many pairs of solutions of the problem (1).

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