ORIGINAL PAPER

ON GAUSSIAN MERSENNE NUMBERS

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Abstract. *In this paper, we define Gaussian Mersenne numbers and we give some properties of them. Moreover we present some relations among Gaussian Mersenne numbers, Gaussian Jacobsthal numbers and Gaussian Jacobsthal-Lucas numbers. We also present some results with matrices involving Gaussian Mersenne numbers.*

Keywords: *Gaussian Mersenne number; Gaussian Jacobsthal number; Gaussian Jacobsthal-Lucas number.*

1. INTRODUCTION

In this study we consider Mersenne sequences, some studies about this sequence have been worked Koshy and Gaz in [1, 2] and Gay T. studied in [3]. In number theory, recall that a Mersenne number of order n, denoted by M_n is a number of form $2^{\mathrm{n}} - 1$, where n is a non negative number. This identity is called the Binet formula for the Mersenne sequence. Mersenne numbers can also be defined recursively by

$$M_{\mathrm{n}+1} = 2M_n + 1, \tag{1.1}$$

with initial conditions $M_0 = 0$ and $M_1 = 1$. Since this recurrence is inhomogeneous, substituting n by $n+1$, we obtain the new form

$$M_{\mathrm{n}+2} = 2M_{n+1} + 1. \tag{1.2}$$

Subtracting (1.1) and (1.2), we find

$$M_{\mathrm{n}+2} = 3M_{n+1} - 2M_n \tag{1.3}$$

the equality (1.3) states another form of the recurrence relation for the Mersenne sequence, with initial values $M_0 = 0$ and $M_1 = 1$. The roots of the respective characteristic equation $r^2 - 3r + 2 = 0$ are $r_1 = 1$ and $r_2 = 2$ and we easily get the Binet formula

$$M_n = 2^n - 1 \tag{1.4}$$

We note that some Mersenne numbers are prime and the search for Mersenne Primes is an active field in number theory, computer science and coding theory. We remark that if M_n is prime then n is prime and the converse is false.

Horadam in [4] defined the Jacobsthal and Jacobsthal- Lucas sequences J_n and j_n by the following recurrence relations.

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$$J_{n+2} = J_{n+1} + 2\,J_n,\ J_0 = 0,\ J_1 = 1 \tag{1.5}$$

and

$$\mathrm{j}_{n+2} = \mathrm{j}_{n+1} + 2\,\mathrm{j}_n,\ \mathrm{j}_0 = 2,\ \mathrm{j}_1 = 1\ , \tag{1.6}$$

respectively, where -1 and 2 are the roots of the characteristic equation associated with the above recurrence relations (1.5) and (1.6).

There are large number of sequences indexed in the Online Encyclopedia of Integer Sequences, being in this case

$\{J_n\}=\{0,1,1,3,5,11,21,43,85,171,...\}$: A001045,

$\{j_n\}=\{2,1,5,7,17,31,65,127,257,...\}$: A014551,

$\{M_n\}=\{0,1,3,7,15,31,63,127,255,...\}$: A000225.

The Gaussian Fibonacci sequence and the Gaussian Lucas sequence in [5] are studied by Jordan, J.H. The complex Fibonacci numbers, generalized Gaussian Fibonacci numbers and their interesting properties are studied in [6] by Berzsanyi, G. and in [7] by Pethe S. and by Horadam A.F. in [4]. Also by Tasci, D. In [8] Gaussian Padovan and Gaussian Pell-Padovan sequences and by Tasci, D., Yalcin, F. in [9] Complex Fibonacci p-Numbers are studied.

2. GAUSSIAN MERSENNE SEQUENCES

Firstly we give the definition of the Gaussian Mersenne sequence.

Definition 1. The Mersenne sequence is the sequence of complex numbers, GM_n, defined by the initial values $GM_0 = -i/2, GM_1 = 1$ and the recurrence relation

$$GM_n = 3GM_{n-1} - 2GM_{n-2},\quad \text{for } n \geq 2.$$

The first few values of GM_n are $-\frac{i}{2}$, 1, 3+i, 7+3i, 15+7i, 31+15i, $\cdots$.

We remark that the equality $GM_n = M_n + iM_{n-1}$ is valid.

Theorem 2. The generating function of the Gaussian Mersenne sequence is

$$f(x) = \frac{-\frac{i}{2} + \left(1 + \frac{3}{2}i\right)x}{1 - 3x + 2x^2}.$$

Proof: Let

$$f(x) = \sum_{n=0}^{\infty} GM_n x^n = GM_0 + GM_1 x + GM_2 x^2 + \cdots + GM_n x^n + \cdots$$

be the generating function of the Gaussian Mersenne sequence. On the other hand, since

$$-3xf(x) = -3\sum_{n=0}^{\infty} GM_n x^{n+1} = -3GM_0x - 3GM_1x^2 - 3GM^2x^3 - \cdots - 3GM_{n-1}x^{\mathrm{n}} - \cdots$$

and

$$2x^2f(x) = 2\sum_{n=0}^{\infty} GM_n x^{n+2} = 2GM_0x^2 + 2GM_1x^3 + \ 2GM_2\ x^4\ + \cdots + 2GM_{n-2}x^{\mathrm{n}} + \cdots$$

we have

$$(1-3x+2x^2)f(x) = GM_0 + (GM_1 - 3GM_0)x + (GM_2 - 3GM_1 + 2GM_0)x^2 + \cdots + (GM_n - 3GM_{n-1} + 2GM_{n-2})x^n + \cdots$$

Now we consider $GM_0 = -\frac{i}{2}, GM_1 = 1\ , GM_n = 3GM_{n-1} - 2GM_{n-2}$. Thus we obtain

$$f(x) = \frac{-\frac{i}{2} + \left(1 + \frac{3}{2}i\right)x}{1 - 3\mathrm{x} + 2\mathrm{x}^2}$$

Thus the proof is complete.

Theorem 3. The Binet formula for the Gaussian Mersenne sequence is

$$GM_n = 2^{n-1}(2+i) - (1+i) \tag{2.1}$$

Proof: Considering the Binet formula for Mersenne sequences $M_n = 2^n - 1$ and $GM_n = M_n + iM_{n-1}$, it is easily seen

$$GM_n = 2^n - 1 + i(2^{n-1} - 1)$$

or

$$GM_n = 2^{n-1}(2+i) - (1+i)$$

Theorem 4. Catalan's identity for Gaussian Mersenne sequence is

$$GM_{n+r}GM_{n-r} - {GM_n}^2 = -(1+3i)2^{n-r-1}(2^r-1)^2 \tag{2.2}$$

where $n\ \geq\ r$ and $n, r\ \in\ Z^{+}$.

Proof: Using Binet formula for Gaussian Mersenne sequence, we have

$$\begin{aligned} GM_{n+r}GM_{n-r} - {GM_n}^2 &= [2^{n+r-1}(2+i) - (1+i)][2^{n-r-1}(2+i) - (1+i)] \\ &\quad -[2^{n-1}(2+i) - (1+i)]^2 \\ &= -2^{n-1}(2+i)(1+i)(2^r + 2^{-r} - 2) \\ &= -2^{n-1}(1+3i)(1+i)\left(\frac{2^{2r}+1-2.2^r}{2^r}\right) \end{aligned}$$

$$= -2^{n-r-1}(1+3i)(2^r-1)^2$$

So the theorem is proved.

Corollary 5. (Cassini's Identity for the Gaussian Mersenne Sequence) For $n \geq 1$, we have

$$GM_{n+1}GM_{n-1} - {GM_n}^2 = -(1+3i)2^{n-2} \tag{2.3}$$

Proof: In fact, the equation (2.2), for $r = 1$ yields the equation (2.3).

Theorem 6. (d'Ocagne's Identity) For the sequence $\{GM\}_{n=0}^{\infty}$, if $m > n$, we have,

$$GM_mGM_{n+1} - GM_{m+1}GM_n = (1+3i)(2^{m-1} - 2^{n-1}).$$

Proof: Considering the Binet formula for the Gaussian Mersenne sequence, the proof is immediately seen.

Theorem 7. The explicit formula for the Gaussian Mersenne sequence is

$$GM_n = \left(\sum_{k=0}^{n}\binom{n}{k} - 1\right) + i\left(\sum_{k=0}^{n-1}\binom{n-1}{k} - 1\right).$$

Proof: Using the sum of binomial coefficients of Mersenne numbers

$$M_n = \sum_{k=0}^{n}\binom{n}{k} - 1$$

and considering the equality

$$GM_n = M_n + iM_{n-1}$$

the proof is easily seen.

Theorem 8. (Summation formula of Gaussian Mersenne numbers)

$$\sum_{k=0}^{n} GM_k = GM_{n+1} - (n+1)(1+i) + \frac{i}{2}.$$

Proof: Using the equality

$$GM_{n+1} = 2GM_n + 1 + i\,; \quad n \geq 0.$$

We write

$$\begin{gathered} GM_1 = 2GM_0 + 1 + i \\ GM_2 = 2GM_1 + 1 + i \\ GM_3 = 2GM_2 + 1 + i \\ \vdots \\ GM_{n+1} = 2GM_n + 1 + i. \end{gathered}$$

Thus from the above equations, we obtain

$$\sum_{k=1}^{n+1} GM_k = 2\sum_{k=0}^{n} GM_k + (n+1)(1+i)$$

which implies (using the fact that $GM_0 = -\frac{i}{2}$) that

$$2\sum_{k=0}^{n} GM_k = \sum_{k=1}^{n+1} GM_k - (n+1)(1+i)$$

$$= \sum_{k=0}^{n} GM_k - (n+1)(1+i) - GM_0 + GM_{n+1}$$

$$= \sum_{k=0}^{n} GM_k - (n+1)(1+i) + \frac{i}{2} + GM_{n+1}$$

Consequently we obtain

$$\sum_{k=0}^{n} GM_k = GM_{n+1} - (n+1)(1+i) + \frac{i}{2}.$$

3. SOME RELATIONS AMONG GAUSSIAN MERSENNE NUMBERS, GAUSSIAN JACOBSTHAL NUMBERS AND GAUSSIAN JACOBSTHAL- LUCAS NUMBERS

Definition 9. Gaussian Jacobsthal numbers are defined by

$$GJ_n = GJ_{n-1} + 2GJ_{n-2}, \; n \geq 2.$$

with initial values $GJ_0 = \frac{i}{2}, GJ_1 = 1$.

Definition 10. Gaussian Jacobsthal-Lucas numbers are defined by

$$Gj_n = Gj_{n-1} + 2Gj_{n-2}, \; n \geq 2.$$

with initial values $Gj_0 = 2\left(1 - \frac{i}{4}\right), Gj_1 = 1 + 2i$.

Binet formulas for these sequences are given by

$$GJ_n = \frac{1}{3}[2^n - (-1)^n + i(2^{n-1} - (-1)^{n-1})] \tag{3.1}$$

and

$$Gj_n = 2^n + (-1)^n + i[2^{n-1} + (-1)^{n-1}] \tag{3.2}$$

Theorem 11. If GM_n, GJ_n *and* Gj_n are the n th term of the Gaussian Mersenne sequence, Gaussian Jacobsthal sequence and Gaussian Jacobsthal- Lucas sequence, respectively, then we have

1. For an even subscript n

$$GM_n = 3GJ_n - 2i \tag{3.3}$$

$$GM_n = Gj_n - 2 \tag{3.4}$$

2. For an odd subscipt n

$$GM_n = 3GJ_n - 2 \tag{3.5}$$

$$GM_n = Gj_n - 2i \tag{3.6}$$

Proof: Using Binet formulas for Gaussian Mersenne numbers, Gaussian Jacobsthal numbers and Gaussian Jacobsthal- Lucas numbers, the proof is immediately seen.

4. MATRICES WITH GAUSSIAN MERSENNE NUMBERS

Theorem12. Let Q, N and K_n denote the 2×2 matrices defined as

$$Q = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}, \qquad N = \begin{bmatrix} 3+i & 1 \\ 1 & -\frac{i}{2} \end{bmatrix} \; and \; K_n = \begin{bmatrix} GM_{n+2} & GM_{n+1} \\ GM_{n+1} & GM_n \end{bmatrix}.$$

Then for all $n \in Z^{+}$,

$$Q^{\mathrm{n}} N = K_n.$$

Proof: We can prove the theorem by induction on n. For $n = 1$, we have

$$\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3+i & 1 \\ 1 & -\frac{i}{2} \end{bmatrix} = \begin{bmatrix} 7+3i & 3+i \\ 3+i & 1 \end{bmatrix} = \begin{bmatrix} GM_3 & GM_2 \\ GM_2 & GM_1 \end{bmatrix}.$$

Assume that the theorem holds for $n = k$, that is

$$\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 3+i & 1 \\ 1 & -\frac{i}{2} \end{bmatrix} = \begin{bmatrix} GM_{k+2} & GM_{k+1} \\ GM_{k+1} & GM_k \end{bmatrix}.$$

Then for $n = k+1$, we have

$$\begin{aligned} \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}^{k+1} \begin{bmatrix} 3+i & 1 \\ 1 & -\frac{i}{2} \end{bmatrix} &= \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 3+i & 1 \\ 1 & -\frac{i}{2} \end{bmatrix} \\ &= \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} GM_{k+2} & GM_{k+1} \\ GM_{k+1} & GM_k \end{bmatrix} \\ &= \begin{bmatrix} GM_{k+3} & GM_{k+2} \\ GM_{k+2} & GM_{k+1} \end{bmatrix}. \end{aligned}$$

Thus the proof is complete.

Theorem 13. For all $n \in Z^{+}$, we have

$$\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}^{n} \begin{bmatrix} 1 \\ -\frac{i}{2} \end{bmatrix} = \begin{bmatrix} GM_{n+1} \\ GM_{n} \end{bmatrix}.$$

Proof: Using the induction method on n, the proof is easily seen.

Theorem 14. Let A_n denote the $n \times n$ tridiagonal matrix defined as

$$A_n = \begin{bmatrix} 1 & i & 0 & 0 & & 0 & 0 \\ -1 & 3 & -2 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 3 & -2 & & 0 & 0 \\ & \vdots & & & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 3 & -2 \\ 0 & 0 & 0 & 0 & & -1 & 3 \end{bmatrix}$$

and let $A_0 = -\frac{i}{2}$. Then

$$\det A_n = GM_n, \; n \geq 1$$

Proof: By the induction on *n*, we can prove the theorem. Since

$$det\, A_1 = 1 = GM_1$$

$$det\, A_2 = 3 + i = GM_2$$

It is true for $n = 1$ and $n = 2$. Assume that the theorem is true for $n-1$ and $n-2$, that is

$$det\, A_{n-1} = GM_{n-1}$$

and

$$det\, A_{n-2} = GM_{n-2}.$$

Then, we have

$$\begin{aligned} det\; A_n &= 3\, det\, A_{n-1} - 2\, det\, A_{n-2} \\ &= 3GM_{n-1} - 2GM_{n-2} \\ &= GM_n. \end{aligned}$$

So the theorem is proved.

4. CONCLUSIONS

In this study we defined a new sequence named the Gaussian Mersenne sequence. Furher we obtained some properties of this sequence. Moreover we found some relations among Gaussian Mersenne Numbers, Gaussian Jacobsthal Numbers and Gaussian Joacobsthal-Lucas numbers.

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