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VIETA-PELL-LIKE POLYNOMIALS AND SOME IDENTITIES

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Abstract. *In this paper, we introduce some new generalizations of the Vieta-Pell polynomial, which is called the Vieta-Pell-Like polynomial. We also give the generating function, the Binet's formula, the sum formula, and some well-known identities for this Vieta polynomial. Furthermore, the relations between the Vieta-Pell-Like polynomial and the previously well-known identities are presented.*

Keywords: *Vieta-Pell polynomials; Vieta-Pell-Lucas polynomials; Vieta-Pell-Like polynomial.*

1. INTRODUCTION

The Vieta polynomials were first introduced in 1991 by Robbins [1]. After that, in 2002, Horadam [2] introduced and studied the Vieta-Fibonacci polynomial $V_n(x)$ and Vieta-Lucas polynomials $v_n(x)$. These polynomials are defined respectively by

$$V_0(x) = 0, V_1(x) = 1, V_n(x) = xV_{n-1}(x) - V_{n-2}(x), \text{ for } n \geq 2$$

and

$$v_0(x) = 2, v_1(x) = x, V_n(x) = xv_{n-1}(x) - v_{n-2}(x), for n \geq 2.$$

The Vieta-Pell polynomials $t_n(x)$ and Vieta-Pell-Lucas polynomials $s_n(x)$ were studied in 2013 by Tasci and Yalcin [3]. They defined these polynomials for $|x| > 1$ by

$$t_0(x) = 0, t_1(x) = 1, t_n(x) = 2xt_{n-1}(x) - t_{n-2}(x), \text{ for } n \geq 2$$

and

$$s_0(x) = 2, s_1(x) = 2x, s_n(x) = 2xs_{n-1}(x) - s_{n-2}(x), \text{ for } n \geq 2.$$

They obtained the Binet form and generating functions of Vieta-Pell and Vieta-Pell-Lucas polynomials. Also, they received some differentiation rules and the finite summation formulas. Moreover, they show that Vieta-Pell and Vieta-Pell-Lucas polynomials are closely related to the well-known Chebyshev polynomials of the first kinds $T_n(x)$ and the second kinds $U_n(x)$. The related features of Vieta-Pell, Vieta-Pell-Lucas polynomials, and Chebyshev polynomials are given as

$$s_n(x) = 2T_n(x),$$

and

$$t_{n+1}(x) = U_n(x).$$

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For more detail about Vieta-Pell and Vieta-Pell-Lucas polynomials, see [3].

Recently, Yalcin et al. [4] introduced the Vieta-Jacobsthal polynomials $G_n(x)$ and Vieta-Jacobsthal-Lucas polynomials $g_n(x)$ which defined respectively by

$$G_0(x) = 0, G_1(x) = 1, G_n(x) = G_{n-1}(x) - 2xG_{n-2}(x), \text{for } n \geq 2$$

and

$$g_0(x) = 2, g_1(x) = 1, g_n(x) = g_{n-1}(x) - 2xg_{n-2}(x), \text{for } n \geq 2.$$

Moreover, they introduced the generalization of the Vieta-Jacobsthal and Vieta-Jacobsthal-Lucas polynomials, and many identities for these polynomials are derived.

In this paper, we investigated the generalization of the Vieta-Pell polynomials. We give the generating function, the Binet formula, and some well-known identities for this polynomial. Also, the relations between this polynomial and the Vieta-Pell and Vieta-Pell-Lucas polynomials are presented.

2. MATERIALS AND METHODS

This section collects some basic definition and helpful lemmas that we will use in the main results.

Definition 2.1. [4] For $|x| > 1$, the Vieta-Pell polynomials sequence $\{t_n(x)\}_{n=0}^{\infty}$ and Vieta-Pell-Lucas polynomials sequence $\{s_n(x)\}_{n=0}^{\infty}$ are defined respectively by

$$t_n(x) = 2xt_{n-1}(x) - t_{n-2}(x), \text{for } n \geq 2, \tag{1}$$

$$s_n(x) = 2xs_{n-1}(x) - s_{n-2}(x), \text{for } n \geq 2, \tag{2}$$

with the initial conditions $t_0(x) = 0$, $t_1(x) = 1$, and $s_0(x) = 2$, $s_1(x) = 2x$.

The first few terms of $\{t_n(x)\}_{n=0}^{\infty}$ and $\{s_n(x)\}_{n=0}^{\infty}$ are as follows:

$t_0(x) = 0,$	$s_0(x) = 2,$
$t_1(x) = 1,$	$s_1(x) = 2x,$
$t_2(x) = 2x,$	$s_2(x) = 4x^2 - 2,$
$t_3(x) = 4x^2 - 1,$	$s_3(x) = 8x^3 - 6x,$
$t_4(x) = 8x^3 - 4x,$	$s_4(x) = 16x^4 - 16x^2 + 2,$
$t_5(x) = 16x^4 - 12x^2 + 1,$	$s_5(x) = 32x^5 - 40x^3 + 10x,$
$\vdots$	$\vdots$

Terms of these sequences are called the Vieta-Pell polynomials and Vieta-Pell-Lucas polynomials, respectively. The Binet's formulas for Vieta-Pell and Vieta-Pell-Lucas polynomials are given as in the following Lemma.

Lemma 2.2. [4] (Binet's formula). *Let $\{t_n(x)\}_{n=0}^{\infty}$ and $\{s_n(x)\}_{n=0}^{\infty}$ be the sequences of Vieta-Pell and Vieta-Pell-Lucas polynomials, respectively. Then*

$$t_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)},$$

$$s_n(x) = \alpha^n(x) + \beta^n(x),$$

where, $\alpha(x) = x + \sqrt{x^2 - 1}$ *and* $\beta(x) = x - \sqrt{x^2 - 1}$ *are the roots of the characteristic equation* $r^2 - 2xr + 1 = 0$.

The following Lemma is helpful for proof our main result in section 3.2.

Lemma 2.3. [4] *Let* $\{t_n(x)\}_{n=0}^{\infty}$ *be the sequence of Vieta-Pell polynomials and let*

$$V = \begin{bmatrix} 2x & -1 \\ 1 & 0 \end{bmatrix}. \text{ Then } V^n = \begin{bmatrix} t_{n+1}(x) & -t_n(x) \\ t_n(x) & -t_{n-1}(x) \end{bmatrix}.$$

3. MAIN RESULTS

3.1. VIETA-PELL-LIKE POLYNOMIALS AND SOME IDENTITIES

In this section, we introduce the polynomial sequence with the same recurrence relation as the Vieta-Pell polynomials but has different initial conditions as the following definition.

Definition 3.1. For $|x| > 1$, the Vieta-Pell-Like polynomials sequence $\{\mathcal{R}_n(x)\}_{n=0}^{\infty}$ is defined by

$$\mathcal{R}_n(x) = 2x\mathcal{R}_{n-1}(x) - \mathcal{R}_{n-2}(x), \quad \text{for } n \geq 2, \tag{3}$$

with the initial conditions $\mathcal{R}_0(x) = 2$, $\mathcal{R}_1(x) = x$.

The first few terms of $\{\mathcal{R}_n(x)\}_{n=0}^{\infty}$ are as follows:

$\mathcal{R}_0(x) = 2,$
$\mathcal{R}_1(x) = x,$
$\mathcal{R}_2(x) = 2x^2 - 2,$
$\mathcal{R}_3(x) = 4x^3 - 5x,$
$\mathcal{R}_4(x) = 8x^4 - 12x^2 + 2,$
$\mathcal{R}_5(x) = 16x^5 - 28x^3 + 9x,$
$\vdots$

Terms of the Vieta-Pell-Like polynomial sequence are called Vieta-Pell-Like polynomial.

The characteristic equation of (3) is also $r^2 - 2xr + 1 = 0$ and the roots of this equation are $\alpha(x) = x + \sqrt{x^2 - 1}$ and $\beta(x) = x - \sqrt{x^2 - 1}$.

We note that $\alpha(x) + \beta(x) = 2x$, $\alpha(x)\beta(x) = 1$, and $\alpha(x) - \beta(x) = 2\sqrt{x^2 - 1}$.

We first give the generating function for this Vieta-Pell-Like polynomials sequence.

Theorem 3.2. (The generating function). *Let* $g(x,t) = \sum_{n=0}^{\infty} \mathcal{R}_n(x)t^n$ *be the generating function of the Vieta-Pell-Like polynomials sequence. Then*

$$g(x,t) = \frac{2 - 3xt}{1 - t + 2xt^2}. \tag{4}$$

Proof: Consider,

$$g(x,t)=\sum_{n=0}^{\infty}\mathcal{R}_n(x)t^n=\mathcal{R}_0(x)+\mathcal{R}_1(x)t+\mathcal{R}_2(x)t^2+\cdots+\mathcal{R}_n(x)t^n+\cdots.$$

Then we get that

$$2xtg(x,t)=2x\mathcal{R}_0(x)t+2x\mathcal{R}_1(x)t^2+2x\mathcal{R}_2(x)t^3+\cdots+2x\mathcal{R}_{n-1}(x)t^n+\cdots$$
$$t^2g(x,t)=\mathcal{R}_0(x)t^2+\mathcal{R}_1(x)t^3+\mathcal{R}_2(x)t^4+\cdots+\mathcal{R}_{n-2}(x)t^n+\cdots.$$

Thus,

$$\begin{aligned}g(x,t)(1-2xt+t^2)&\\&=\mathcal{R}_0(x)+\big(\mathcal{R}_1(x)-2x\mathcal{R}_0(x)\big)t+\sum_{n=2}^{\infty}\big(\mathcal{R}_n(x)-2x\mathcal{R}_{n-1}+\mathcal{R}_{n-2}(x)\big)t^n\\&=\mathcal{R}_0(x)+\big(\mathcal{R}_1(x)-2x\mathcal{R}_0(x)\big)t\\&=2-3xt.\end{aligned}$$

It implies that

$$g(x,t)=\frac{2-3xt}{1-2xt+t^2}.$$ □

Next, we give Binet's formula for this Vieta-Pell-Like polynomials as follows.

Theorem 3.3. (Binet's formula). *Let* $\{\mathcal{R}_n(x)\}_{n=0}^{\infty}$ *be the sequence of Vieta-Pell-Like polynomials. Then*

$$\mathcal{R}_n(x)=A\alpha^n(x)+B\beta^n(x), \tag{5}$$

where , $A=\frac{\alpha-2\beta(x)}{\alpha(x)-\beta(x)}$, $B=\frac{2\alpha(x)-x}{\alpha(x)-\beta(x)}$, *and* $\alpha(x),\beta(x)$ *are the roots of the characteristics equation* $r^2-2xr+1=0$.

Proof: Since the roots of the characteristic equation $r^2-2xr+1=0$ are distinct, we get that

$$\mathcal{R}_n(x)=c\alpha^n(x)+d\beta^n(x),\quad \text{for all } n\geq 0,$$

for some real numbers c, and d. Taking $n=0$, $n=1$, and then solving the system of linear equations, we obtain

$$\mathcal{R}_n(x)=\frac{\alpha-2\beta(x)}{\alpha(x)-\beta(x)}\alpha^n(x)+\frac{2\alpha(x)-x}{\alpha(x)-\beta(x)}\beta^n(x)$$

Setting $A=\frac{\alpha-2\beta(x)}{\alpha(x)-\beta(x)}$ and $B=\frac{2\alpha(x)-x}{\alpha(x)-\beta(x)}$, then we get the result. □

We note that

$$A + B = 2,$$

$$A - B = \frac{-2x}{\alpha(x) - \beta(x)},$$

$$AB = \frac{3x^2 - 4}{\left(\alpha(x) - \beta(x)\right)^2},$$

$$A(2\alpha(x) - x) = \frac{3x^2 - 4}{\alpha(x) - \beta(x)} = B\left(x - 2\beta(x)\right),$$

and

$$A\beta(x) + B\alpha(x) = 3x.$$

Using Binet's formula, we obtained some well-known identities and the sum formula for the Vieta-Pell-Like polynomials, and we begin with the following Lemma.

Lemma 3.4. *Let $\{\mathcal{R}_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta-Pell-Like polynomials. Then*

$$\frac{2\mathcal{R}_{n+1}(x) - x\mathcal{R}_n(x)}{3x^2 - 4} = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)},$$

where $\alpha(x)$ and $\beta(x)$ are the roots of the characteristic equation $r^2 - 2xr + 1 = 0$.

Proof: By using Binet's formula (5), we obtain

$$\begin{aligned}\frac{2\mathcal{R}_{n+1}(x) - x\mathcal{R}_n(x)}{3x^2 - 4} &= \frac{1}{3x^2 - 4}\left(2\left(A\alpha^{n+1}(x) + B\beta^{n+1}(x)\right) - x\left(A\alpha^n(x) + B\beta^n(x)\right)\right) \\ &= \frac{1}{3x^2 - 4}\left(\alpha^n(x)A(2\alpha(x) - x) - \beta^n(x)B\left(x - 2\beta(x)\right)\right) \\ &= \frac{1}{3x^2 - 4}\left(\frac{\alpha^n(x)(3x^2 - 4)}{\alpha(x) - \beta(x)} - \frac{\beta^n(x)(3x^2 - 4)}{\alpha(x) - \beta(x)}\right) \\ &= \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}.\end{aligned}$$

□

By using Binet's formula (5) and Lemma 3.4, we obtain the Catalan identity.

Theorem 3.5. (Catalan's identity). *Let $\{\mathcal{R}_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta-Pell-Like polynomials. Then*

$$\mathcal{R}_n^2(x) - \mathcal{R}_{n+r}(x)\mathcal{R}_{n-r}(x) = \frac{1}{4 - 3x^2}\left(2\mathcal{R}_{r+1}(x) - x\mathcal{R}_r(x)\right)^2, \quad \text{for } n \geq r \geq 1 \tag{6}$$

Proof: By using Binet's formula, we obtain

$$
\begin{aligned}
&\mathcal{R}_n^2(x) - \mathcal{R}_{n+r}(x)\mathcal{R}_{n-r}(x) \\
&\quad = \left(A\alpha^n(x) + B\beta^n(x)\right)^2 - \left(A\alpha^{n+r}(x) + B\beta^{n+r}(x)\right)\left(A\alpha^{n-r}(x) + B\beta^{n-r}(x)\right) \\
&\quad = -AB\left(\alpha(x)\beta(x)\right)^{n-r}\left(\alpha^r(x) - \beta^r(x)\right)^2 \\
&\quad = -\frac{3x^2-4}{\left(\alpha(x)-\beta(x)\right)^2}\left(\alpha^r(x) - \beta^r(x)\right)^2 \\
&\quad = -(3x^2-4)\left(\frac{\alpha^r(x) - \beta^r(x)}{\alpha(x)-\beta(x)}\right)^2 \\
&\quad = \frac{1}{4-3x^2}\left(2\mathcal{R}_{r+1}(x) - x\mathcal{R}_r(x)\right)^2
\end{aligned}
$$

This completes the proof. □

Take $r = 1$ in Catalan identity (6), we obtain Cassini's identity as the following Corollary.

Corollary 3.6. (Cassini's identity). *Let* $\{\mathcal{R}_n(x)\}_{n=0}^{\infty}$*be the sequence of Vieta-Pell-Like polynomials. Then*

$$\mathcal{R}_n^2(x) - \mathcal{R}_{n+r}(x)\mathcal{R}_{n-r}(x) = 4 - 3x^2, \qquad \text{for } n \geq 1$$

Proof: Take $r = 1$ in Catalan's identity (6), we obtain the result. □

Theorem 3.7. (d'Ocagne's identity). *Let* $\{\mathcal{R}_n(x)\}_{n=0}^{\infty}$ *be the sequence of Vieta-Pell-Like polynomials. Then*

$$\mathcal{R}_m(x)\mathcal{R}_{n+1}(x) - \mathcal{R}_{m+1}(x)\mathcal{R}_n(x) = -2\mathcal{R}_{m-n+1}(x) + x\mathcal{R}_{m-n}(x), \quad \text{for } m \geq n \geq 1.$$

Proof: By using Binet's formula and Lemma 3.4, we obtain

$$
\begin{aligned}
&\mathcal{R}_m(x)\mathcal{R}_{n+1}(x) - \mathcal{R}_{m+1}(x)\mathcal{R}_n(x) \\
&\quad = \left(A\alpha^m(x) + B\beta^m(x)\right)\left(A\alpha^{n+1}(x) + B\beta^{n+1}(x)\right) \\
&\qquad -\left(A\alpha^{m+1}(x) + B\beta^{m+1}(x)\right)\left(A\alpha^n(x) + B\beta^n(x)\right) \\
&\quad = -AB\left(\alpha(x)\beta(x)\right)^n\left(\alpha(x) - \beta(x)\right)\left(\alpha^{m-n}(x) - \beta^{m-n}(x)\right) \\
&\quad = -(3x^2-4)\left(\frac{\alpha^{m-n}(x) - \beta^{m-n}(x)}{\alpha(x)-\beta(x)}\right) \\
&\quad = -(3x^2-4)\left(\frac{2\mathcal{R}_{m-n+1}(x) - x\mathcal{R}_{m-n}(x)}{3x^2-4}\right) \\
&\quad = -2\mathcal{R}_{m-n+1}(x) + x\mathcal{R}_{m-n}(x)
\end{aligned}
$$

This completes the proof. □

Next, we give the finite sum formula for the Vieta-Pell-Like polynomials sequence.

Theorem 3.8. (The Sum formula). *Let* $\{\mathcal{R}_n(x)\}_{n=0}^{\infty}$ *be the sequence of Vieta-Pell-Like polynomials. Then*

$$\sum_{k=0}^{n-1}\mathcal{R}_k(x) = \frac{2-3x - \mathcal{R}_n(x) + \mathcal{R}_{n-1}(x)}{2(1-x)}.$$

Proof: By using Binet's formula, we obtain

$$\sum_{k=0}^{n-1}\mathcal{R}_k(x)=\sum_{k=0}^{n-1}\left(A\alpha^k(x)+B\beta^k(x)\right)$$

$$=A\frac{1-\alpha^n(x)}{1-\alpha(x)}+B\frac{1-\beta^n(x)}{1-\beta(x)}$$

$$=\frac{A+B-\left(A\beta(x)+B\alpha(x)\right)-\left(A\alpha^n(x)+B\beta^n(x)\right)+\left(A\alpha^{n-1}(x)+B\beta^{n-1}(x)\right)}{1-\left(\alpha(x)+\beta(x)\right)+\alpha(x)\beta(x)}$$

$$=\frac{2-3x-\mathcal{R}_n(x)+\mathcal{R}_{n-1}(x)}{2(1-x)}$$

This completes the proof. □

Again, by using Binet's formula, we derive the relation between the Vieta-Pell-Like polynomials, Vieta-Pell polynomials, and Vieta-Pell-Lucas polynomials.

Theorem 3.9. *Let* $\{\mathcal{R}_n(x)\}_{n=0}^{\infty}, \{t_n(x)\}_{n=0}^{\infty}$, *and* $\{s_n(x)\}_{n=0}^{\infty}$ *be the sequences of Vieta-Pell-Like, Vieta-Pell, and Vieta-Pell-Lucas polynomials, respectively. Then*

(1) $s_n(x)-xt_n(x)=\mathcal{R}_n(x)$, for $n\geq 0$,

(2) $xt_n(x)-2t_{n-1}(x)=\mathcal{R}_n(x)$, for $n\geq 1$,

(3) $2t_{n+1}(x)-3xt_n(x)=\mathcal{R}_n(x)$, for $n\geq 0$,

(4) $t_{n+1}(x)+\mathcal{R}_n(x)=\frac{3}{2}s_n(x)$, for ≥ 0,

(5) $\mathcal{R}_{4n}(x)-xt_{4n}(x)-2=4(x^2-1)t_{2n}^2(x)$, for $n\geq 0$,

(6) $2\mathcal{R}_{n+1}(x)-x\mathcal{R}_n(x)=(3x^2-4)t_n(x)$, for $n\geq 0$,

(7) $\mathcal{R}_n(x)s_n(x)-2=\mathcal{R}_{2n}(x)$ for $n\geq 0$,

(8) $\mathcal{R}_n(x)s_n(x)+xt_{2n}(x)-2=s_{2n}(x)$, for $n\geq 0$,

(9) $\mathcal{R}_n(x)s_n(x)+2t_{2n-1}(x)-2=xt_{2n}(x)$, for $n\geq 1$,

(10) $\mathcal{R}_m(x)s_n(x)-s_m(x)\mathcal{R}_n(x)=-2xt_{m-n}(x)$, for $m\geq n\geq 0$,

(11) $\mathcal{R}_m(x)t_n(x)-t_m(x)\mathcal{R}_n(x)=-2t_{m-n}(x)$, for $m\geq n\geq 0$,

(12) $t_n(x)\mathcal{R}_n(x)+xt_n^2(x)=t_{2n}(x)$, for $n\geq 0$,

(13) $\mathcal{R}_{n+1}(x)s_n(x)-s_{n+1}(x)\mathcal{R}_n(x)=-2x$, for $n\geq 0$,

(14) $\mathcal{R}_{n+1}(x)t_n(x)-t_{n+1}(x)\mathcal{R}_n(x)=-2$, for $n\geq 0$.

Proof: The results (1)-(14) are easily obtained by using Binet's formula (5). □

3.2. SOME IDENTITIES OF THE VIETA-PELL-LIKE POLYNOMIALS BY MATRIX METHODS

In this section, we establish some identities of the Vieta Pell-Like and Vieta-Pell polynomials by using elementary matrix methods.

Let $Q_{\mathcal{R}}$ be 2 x 2 matrix defined by

$$Q_{\mathcal{R}} = \begin{bmatrix} x & -2 \\ 2 & -3 \end{bmatrix} \tag{7}$$

Then by using this matrix and matrix V in Lemma 2.3, we can deduce some identities of Vieta-Pell-Like and Vieta Pell polynomials.

Theorem 3.10. *Let $\{\mathcal{R}_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta-Pell-Like polynomials, let $Q_{\mathcal{R}}$ be 2×2 matrix defined by (7), and let V be 2×2 matrix as in Lemma 2.3, then*

$$Q_{\mathcal{R}}V^n = \begin{bmatrix} \mathcal{R}_{n+1}(x) & -\mathcal{R}_n(x) \\ \mathcal{R}_n(x) & -\mathcal{R}_{n-1}(x) \end{bmatrix}, for\ all\ n \geq 1$$

Proof: From Lemma 2.3, we get

$$V^n = \begin{bmatrix} t_{n+1}(x) & -t_n(x) \\ t_n(x) & -t_{n-1}(x) \end{bmatrix}.$$

Thus,

$$Q_{\mathcal{R}}V^n = \begin{bmatrix} x & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} t_{n+1}(x) & -t_n(x) \\ t_n(x) & -t_{n-1}(x) \end{bmatrix}$$

$$= \begin{bmatrix} xt_{n+1}(x) - 2t_n(x) & -xt_n(x) + 2t_{n-1}(x) \\ 2xt_{n+1}(x) - 3t_n(x) & -2t_n(x) + 3t_{n-1}(x) \end{bmatrix}$$

By Theorem 3.9 (2) and (3), we obtain

$$Q_{\mathcal{R}}V^n = \begin{bmatrix} \mathcal{R}_{n+1}(x) & -\mathcal{R}_n(x) \\ \mathcal{R}_n(x) & -\mathcal{R}_{n-1}(x) \end{bmatrix}.$$

This completes the proof. □

From Theorem 3.10, Lemma 2.3, and the properties of the power matrix, we obtain many identities of the Vieta Pell-Like and Vieta-Pell polynomials.

Corollary 3.11. *Let $\{\mathcal{R}_n(x)\}_{n=1}^{\infty}$ and $\{t_n(x)\}_{n=0}^{\infty}$ be the sequences of Vieta-Pell-Like and Vieta-Pell polynomials, respectively. Then for all integers $n > m \geq 1$, the following statements hold:*

(1) $\mathcal{R}_{n+1}(x) = \mathcal{R}_{(n-m)+1}(x)t_{m+1}(x) - \mathcal{R}_{n-m}(x)t_m(x)$,

(2) $\mathcal{R}_n(x) = \mathcal{R}_{(n-m)+1}(x)t_m(x) - \mathcal{R}_{n-m}(x)t_{m-1}(x)$,

(3) $\mathcal{R}_n(x) = \mathcal{R}_{n-m}(x)t_{m+1}(x) - \mathcal{R}_{(n-m)-1}(x)t_m(x)$,

(4) $\mathcal{R}_{n-1}(x) = \mathcal{R}_{n-m}(x)t_m(x) - \mathcal{R}_{(n-m)-1}(x)t_{m-1}(x)$.

Proof: By Theorem 3.10, Lemma 2.3 and the property of the power matrix $Q_{\mathcal{R}}V^n = Q_{\mathcal{R}}V^{n-m}V^m$, we obtained the results. □

Corollary 3.12. *Let* $\{\mathcal{R}_n(x)\}_{n=1}^{\infty}$ *and* $\{t_n(x)\}_{n=0}^{\infty}$ *be the sequences of Vieta-Pell-Like and Vieta-Pell polynomials, respectively. Then for all integers* $n > m \geq 1$, *the following statements hold:*

(1) $\mathcal{R}_{(m+n)+1}(x) = \mathcal{R}_{m+1}(x)t_{n+1}(x) - \mathcal{R}_m(x)t_n(x)$,

(2) $\mathcal{R}_{m+n}(x) = \mathcal{R}_{m+1}(x)t_n(x) - \mathcal{R}_m(x)t_{n-1}(x)$,

(3) $\mathcal{R}_{m+n}(x) = \mathcal{R}_m(x)t_{n+1}(x) - \mathcal{R}_{m-1}(x)t_n(x)$,

(4) $\mathcal{R}_{(m+n)-1}(x) = \mathcal{R}_m(x)t_n(x) - \mathcal{R}_{m-1}(x)t_{n-1}(x)$.

Proof: By Theorem 3.10, Lemma 2.3 and the property of the power matrix $Q_{\mathcal{R}}V^{m+n} = Q_{\mathcal{R}}V^mV^n$, we obtained the results. □

Corollary 3.13. *Let* $\{\mathcal{R}_n(x)\}_{n=1}^{\infty}$ *and* $\{t_n(x)\}_{n=0}^{\infty}$ *be the sequences of Vieta-Pell-Like and Vieta-Pell polynomials, respectively. Then for all integers* $n > m \geq 1$, *the following statements hold:*

(1) $\mathcal{R}_{(m-n)+1}(x) = -\mathcal{R}_{m+1}(x)t_{n-1}(x) - \mathcal{R}_m(x)t_n(x)$,

(2) $\mathcal{R}_{m-n}(x) = -\mathcal{R}_{m+1}(x)t_n(x) + \mathcal{R}_m(x)t_{n+1}(x)$,

(3) $\mathcal{R}_{m-n}(x) = -\mathcal{R}_m(x)t_{n-1}(x) - \mathcal{R}_{m-1}(x)t_n(x)$,

(4) $\mathcal{R}_{(m-n)-1}(x) = -\mathcal{R}_m(x)t_n(x) - \mathcal{R}_{m-1}(x)t_{n+1}(x)$.

Proof: By Theorem 3.10, Lemma 2.3 and the property of the power matrix $Q_{\boldsymbol{\mathcal{R}}}V^{m-n} = Q_{\boldsymbol{\mathcal{R}}}V^mV^{-n}$, we obtained the results. □

4. CONCLUSION

In this paper, the Vieta-Pell-Like polynomial is introduced, and the generating function, Binet's formula, some well-known identities, and the sum formula for this polynomial are established. Moreover, the relations between the Vieta-Pell-Like, Vieta-Pell, and Vieta-Pell-Lucas polynomials are presented in this study.

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