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QUANTUM SPLIT QUATERNIONS

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Abstract. *In this study we introduce q-deformed split quaternions, that is, this deformation reduces to classical split quaternions as $q \to 1$ where q is a real parameter. It is also shown that there is a quantum group associated with q-deformed split quaternions, which is isomorphic to $SU_q(1,1)$.*

Keywords: *Hopf algebra; split quaternions; quantum group.*

1. INTRODUCTION

It is well known that from a mathematical point of view, a *q*-deformation of Lie algebra is regarded as a noncommutative Hopf algebra, but it is said to be a quantum group in the context of quantum integrable models [1]. In fact, the theory of quantum groups was studied extensively by L.D. Faddeev [2], M. Jimbo [3] and V.G. Drinfeld [4]. According to the study of Y. Manin in 1988, the quantum group can be realized to be the group of linear transformations acting on the quantum plane, which is a special kind of quantum spaces [5].

In 1844, R. Hamilton constructed the algebra of real quaternions in order to gain more insight into the geometry of the 3-dimensional Euclidean real space (see [6]). In particular, in 1992, S.Marchiafava and J. Rembielinski [7] introduced the notion of quantum quaternions. In [7], the coproduct and counit for quantum quaternions were introduced, and the differential quaternionic calculus was studied by using the approach given in the paper [8]. In 2003, S. Celik studied the Hopf algebra structure of quantum quaternionic groups and the isomorphism between the group of unit quantum quaternions and the quantum group $SU_q(2)$ [9].

In 1849, J. Cockle [10] introduced the notion of split quaternions, also known as coquaternions or paraquaternions. Like the algebra of quaternions, the split quaternions have been considered by many people as an efficient tool for studying different branches of mathematics and physics [11-13] .

2. PRELIMINARIES

In the present paper, we first introduce *q*-deformed split quaternion which reduces to classical split quaternions as $q \to 1$. Then, we show that there is a quantum group associated with *q*-deformed split quaternions. Let's recall some basic definitions and structures which will be necessary to present our main results. The algebra of split quaternions is given by the following set:

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$$P = \{ae_0 + be_1 + ce_2 + de_3 \mid a,b,c,d \in \mathbb{R}\} \quad (1)$$

where the algebra generators e_i's are subject to the following noncommutative relations:

$$e_1e_2 = e_3 = -e_2e_1, e_2e_3 = -e_1 = -e_3e_2, e_3e_1 = e_2 = -e_1e_3, e_1^2 = -1, e_2^2 = 1, e_3^2 = 1. \quad (2)$$

Note that an element of P can be represented by a matrix as follows:

$$\begin{bmatrix} a+ib & c+id \\ c-id & a-ib \end{bmatrix} \in GL(2,C).$$

Furthermore, the conjugate of a split quaternion h is defined as

$$\bar{h} = ae_0 - be_1 - ce_2 - de_3 .$$

It is clear that the quadratic form is given by the determinant of the above matrix, that is,

$$N(h) = h\bar{h} = \bar{h}h = a^2 + b^2 - c^2 - d^2,$$

which implies

$$[h,\bar{h}] = h\bar{h} - \bar{h}h = 0.$$

Here $N(h)$ is at the center of P. Unlike quaternions, every nonzero element in P is not invertible. However, the collection of all split quaternions with nonzero quadratic form is a multiplicative group where the multiplicative inverse of h with $N(h) \neq 0$ is given by $h^{-1} = \bar{h}/N(h)$. Note that the set of all split quaternions with $N(h) = 1$ gives rise to a noncompact topological group, which is isomorphic to $SU(1,1)$. From now on, we denote the space of all split quaternions by P throughout the paper.

The main goal of this study is to obtain a q-deformation of P by using the setting of quantum group $GL_q(2)$. In the sequel, we recall from [5, 13, 14] some basic definitions and notions about quantum group to present our results.ℂ

Let $\mathcal{A}$ be a vector space over $K = \mathbb{R}$ or $\mathbb{C}$ and $m: \mathcal{A} \otimes \mathcal{A} \to K$ (the multiplication), $\eta: K \to \mathcal{A}$ (the unit mapping) be two linear mappings. Then the triple $(\mathcal{A}, m, \eta)$ is called an algebra if the following conditions hold:

$$\begin{aligned} m \circ (m \otimes \mathrm{id}) &= m \circ (\mathrm{id} \otimes m) \\ m \circ (\eta \otimes \mathrm{id}) &= m \circ (\mathrm{id} \otimes \eta) \end{aligned} \quad (3)$$

where id is the identity mapping. A *coalgebra* $\mathcal{A}$ over K is a K vector space $\mathcal{A}$, together with two linear mappings $\Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and $\varepsilon: \mathcal{A} \to K$ satisfying the following rules:

$$\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \quad (4)$$

$$m \circ ((\varepsilon \otimes \mathrm{id}) \circ \Delta) = \mathrm{id} = m \circ ((\mathrm{id} \otimes \varepsilon) \circ \Delta). \quad (5)$$

A *bialgebra* is a unital associative algebra endowed with a coalgebra structure admitting the compatibility conditions, that is Δ and ε are both algebra homomorphisms with

$\Delta(1_{\mathcal{A}}) = 1_{\mathcal{A}} \otimes 1_{\mathcal{A}}$ and $\varepsilon(1_{\mathcal{A}}) = 1_K$. A *Hopf algebra* is a bialgebra $\mathcal{A}$ endowed with an algebra antihomomorphism from $\mathcal{A} \to \mathcal{A}$ satisfying the following condition:

$$m \circ ((S \otimes \mathrm{id}) \circ \Delta) = \eta \circ \varepsilon = m \circ ((\mathrm{id} \otimes S) \circ \Delta). \tag{6}$$

A *quantum plane* is given as an associative polynomial algebra generated by *x* and *y* satisfying *q*-commutation rule

$$xy = qyx \tag{7}$$

where *q* is a nonzero complex parameter.

Moreover, a linear transformation acting on this quantum plane is given in [5] as follows:

$$\begin{aligned} x' &= Ax + By \\ y' &= Cx + Dy \end{aligned} \tag{8}$$

where x' and y' are *q*-commutative in the sense of (7), and the matrix entries *A*, *B*, *C*, *D* are commutative with the generators *x*, *y*. Thus, an element of the quantum group acting on the quantum plane is defined as a matrix *M* of the following form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{9}$$

where *A*, *B*, *C*, *D* hold the following noncommutative relations:

$$\begin{aligned} &AB = qBA,\ AC = qCA,\ AD = DA + (q - q^{-1})BC, \\ &BC = CB,\ BD = qDB,\ CD = qDC, \end{aligned} \tag{10}$$

and the quantum determinant $det_q M = AD - qBC = DA - q^{-1}BC$ is nonzero.

Note that $det_q M$ is commutative with all elements in an associative algebra generated by *A*, *B*, *C*, *D* and the inverse matrix is described as follows:

$$M^{-1} = \frac{1}{det_q M} \begin{pmatrix} D & -q^{-1}B \\ -qC & A \end{pmatrix}. \tag{11}$$

It is also seen from [5] that the algebra with generators satisfying the relations in (10) can be endowed with a Hopf algebra structure due to the coproduct, counit and coinverse mappings acting on the generators as follows

$$\begin{aligned} \Delta(A) &= A \otimes A + B \otimes C \\ \Delta(B) &= A \otimes B + B \otimes D \\ \Delta(C) &= C \otimes A + D \otimes C \\ \Delta(D) &= C \otimes B + D \otimes D, \end{aligned} \tag{12}$$

$$\varepsilon(A) = 1,\ \varepsilon(B) = 0,\ \varepsilon(C) = 0,\ \varepsilon(D) = 1, \tag{13}$$

$$S(A)=\frac{1}{det_q M}D,\ S(B)=\frac{1}{det_q M}(-q^{-1}B)$$
$$S(C)=\frac{1}{det_q M}(-qC),\ S(D)=\frac{1}{det_q M}A. \tag{14}$$

3. QUANTUM SPLIT QUATERNIONS

In order to obtain *q*-deformed split quaternions, we will try to combine the theory of the classical split quaternions with that of quantum group. For this, we first consider an associative unital algebra over $\mathbb{C}$ generated by a_0, a_1, a_2, a_3 that commute with unit split quaternions e_0, e_1, e_2, e_3. Then, we need to obtain commutation relations among the generators a_i's by using the relations in (10). To manage this, by setting

$$A=a_0+ia_1,\ B=a_2+ia_3,\ C=a_2-ia_3,\ D=a_0-ia_1, \tag{15}$$

we compute all commutation relations among a_i's as follows

$$\begin{aligned}
&[a_2,a_3]=0\\
&[a_0,a_1]=i\lambda_-(a_2^2+a_3^2)\\
&[a_0,a_2]_q=-i[a_1,a_2]_q\\
&[a_1,a_3]_q=i[a_0,a_3]_q\\
&[a_2,a_0]_q=i[a_2,a_1]_q\\
&[a_3,a_1]_q=-i[a_3,a_0]_q\\
&[a_0,a_2]_{q^{-1}}=i[a_1,a_2]_{q^{-1}}\\
&[a_1,a_3]_{q^{-1}}=-i[a_0,a_3]_{q^{-1}}\\
&[a_0,a_2]_{\lambda_+}=i\lambda_-a_2a_1\\
&[a_1,a_2]_{\lambda_+}=-i\lambda_-a_2a_0\\
&[a_1,a_3]_{\lambda_+}=-i\lambda_-a_3a_0\\
&[a_1,a_2]_{\lambda_+}=i\lambda_-a_3a_1
\end{aligned} \tag{16}$$

where

$$\lambda_+:=\frac{q+q^{-1}}{2},\ \lambda_-:=\frac{q-q^{-1}}{2} \tag{17}$$

for nonzero real number *q*, and note that $[A,B]_c=AB-cBA$ for complex number *c*. For instance, using the fact $AD-DA=(q-q^{-1})BC$, we have

$$\begin{aligned}
&(a_0+ia_1)(a_0-ia_1)-(a_0-ia_1)(a_0+ia_1)=(q-q^{-1})(a_2+ia_3)(a_3-ia_2)\\
&\Rightarrow (2i)(a_1a_0-a_0a_1)=(q-q^{-1})(a_2^2+a_3^2)\\
&\Rightarrow [a_0,a_1]=i\lambda_-(a_2^2+a_3^2).
\end{aligned}$$

In similar manner, other commutation relations can be obtained straightforwardly by using the corresponding relations in (10). From now on, the noncommutative algebra generated by a_i's where $i \in \{0,1,2,3\}$ is denoted by $\mathcal{A}$ throughout the paper. Note that this noncommutative algebra reduces to a commutative algebra as $q \to 1$.

Lemma. *Let* $*: \mathcal{A} \to \mathcal{A}$ *be an antilinear antimultiplicative mapping acting on the generators as follows:*

$$\begin{aligned} a_0^* &= a_0 \\ a_1^* &= a_1 \\ a_2^* &= \lambda_+ a_2 - i\lambda_- a_3 \\ a_3^* &= \lambda_+ a_3 + i\lambda_- a_2 \end{aligned} \tag{18}$$

Then $\mathcal{A}$ *is a *-algebra.*

Proof: By (18), one can see that $(a_l^*)^* = a_l$ for $0 \le l \le 3$. Now, to show $*^2 = id$ for all elements of $\mathcal{A}$, it is sufficient to see that * preserves the algebra relations given in (16). For example, in the sequel, we show that the relation $[a_0, a_1] = i\lambda_-(a_2^2 + a_3^2)$ is preserved under the action of * :

$$\begin{aligned} &([a_0, a_1])^* = (i\lambda_-(a_2^2 + a_3^2))^* \\ &\Rightarrow a_1^* a_0^* - a_0^* a_1^* = [-i\lambda_-(a_2^*)^2 + (a_3^*)^2] \\ &\Rightarrow a_1 a_0 - a_0 a_1 = -i\lambda_-[(\lambda_+ a_2 - i\lambda_- a_3)^2 + [(\lambda_+ a_3 + i\lambda_- a_2)^2] \\ &\Rightarrow a_1 a_0 - a_0 a_1 = -i\lambda_-[\lambda_+^2(a_2^2 + a_3^2) - \lambda_-^2(a_2^2 + a_3^2)] \\ &\Rightarrow a_1 a_0 - a_0 a_1 = -i\lambda_-(a_2^2 + a_3^2) \end{aligned}$$

which implies that * leaves the relation invariant. Similarly, it can be shown that * preserves other relations.

Now we set up an $\mathcal{A}$-module generated by the split quaternionic units given in (2), that is,

$$P_q = \{a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \mid a_0, a_1, a_2, a_3 \in \mathcal{A}\}$$

together with the requirement that the all elements of $\mathcal{A}$ commute with the split quaternionic units.

Definition. *We define the algebra of quantum split quaternions as a pair* $(\mathcal{A}, P_q)$ *where the anti-involution on* $\mathcal{A}$ *is readily extended to* P_q *by*

$$h^* = a_0^* e_0 - a_1^* e_1 - a_2^* e_2 - a_3^* e_3, \tag{19}$$

for $h = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \in P_q$.

From now on, throughout the paper, we suppose that P_q stands for the algebra of quantum split quaternions. Note that, a q-split quaternion h and its inverse h^{-1} (if $det_q h \neq 0$) can be represented by the following matrices, respectively

$$h = \begin{pmatrix} a_0 + ia_1 & a_2 + ia_3 \\ a_2 - ia_3 & a_0 - ia_1 \end{pmatrix} \tag{20}$$

$$h^{-1} = \frac{1}{det_q h} \begin{pmatrix} a_0 - ia_1 & -q^{-1}(a_2 + ia_3) \\ -q(a_2 - ia_3) & a_0 + ia_1 \end{pmatrix} \tag{21}$$

where

$$det_q h = a_0^2 + a_1^2 - \lambda_+ (a_2^2 + a_3^2) = N_q(h) = hh^* = h^* h. \tag{22}$$

Furthermore, $det_q h$ is the central element of P_q. As seen from the constructions given above, P_q can be regarded as a q deformation of the algebra of the usual split quaternions. Now, in the sequel, we show that there exists a quantum group (Hopf algebra) associated with P_q.

Using (15), (12), and (13) one can straightforwardly see that the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and $\varepsilon : \mathcal{A} \to \mathbb{C}$ act on the generators of $\mathcal{A}$, respectively, as follows

$$\begin{aligned} \Delta(a_0) &= a_0 \otimes a_0 - a_1 \otimes a_1 + a_2 \otimes a_2 + a_3 \otimes a_3 \\ \Delta(a_1) &= a_1 \otimes a_0 + a_0 \otimes a_1 - a_2 \otimes a_3 + a_3 \otimes a_2 \\ \Delta(a_2) &= a_0 \otimes a_2 + a_2 \otimes a_0 - a_1 \otimes a_3 + a_3 \otimes a_1 \\ \Delta(a_3) &= a_1 \otimes a_2 - a_2 \otimes a_1 + a_0 \otimes a_3 + a_3 \otimes a_0 \end{aligned}$$

, and

$$\varepsilon(a_0) = 1, \varepsilon(a_1) = 0, \varepsilon(a_2) = 0, \varepsilon(a_3) = 0.$$

Moreover, it is clear that Δ and ε hold coassociative and counit conditions given by (4) and (5).

That is, $\mathcal{A}$ is coalgebra together with Δ and ε.

Suppose that Δ and ε are both algebra homomorphisms and $\Delta(1) = 1 \otimes 1$ and $\varepsilon(1) = 1$. Then one can reasonably check that Δ and ε preserve the commutation relations (16). For instance, for $[a_2, a_3] = 0$,

$$\begin{aligned} &\Delta([a_2, a_3]) = \Delta(a_2 a_3 - a_3 a_2) \\ &\Rightarrow \Delta([a_2, a_3]) = \Delta(a_2 a_3) - \Delta(a_3 a_2) \\ &\Rightarrow \Delta([a_2, a_3]) = \Delta(a_2)\Delta(a_3) - \Delta(a_3)\Delta(a_2) \\ &\Rightarrow \Delta([a_2, a_3]) = [a_0 a_1 \otimes a_2^2 - a_2^2 \otimes a_0 a_1 - a_1 a_0 \otimes a_3^2 + a_3^2 \otimes a_1 a_0] \\ &\qquad - [a_1 a_0 \otimes a_2^2 - a_2^2 \otimes a_1 a_0 - a_0 a_1 \otimes a_3^2 + a_3^2 \otimes a_0 a_1] \\ &\Rightarrow \Delta([a_2, a_3]) = a_1 a_0 \otimes a_2^2 + i\lambda_- (a_2^2 + a_3^2) \otimes a_2^2 - a_2^2 \otimes a_1 a_0 - i\lambda_- a_2^2 \otimes (a_2^2 + a_3^2) \\ &\qquad - a_1 a_0 \otimes a_3^2 + a_3^2 \otimes a_1 a_0 - a_1 a_0 \otimes a_2^2 + a_2^2 \otimes a_1 a_0 + a_1 a_0 \otimes a_3^2 \\ &\qquad + i\lambda_- (a_2^2 + a_3^2) \otimes a_3^2 - a_3^2 \otimes a_0 a_1 - i\lambda_- a_3^2 \otimes (a_2^2 + a_3^2) \\ &\Rightarrow \Delta([a_2, a_3]) = 0 \end{aligned}$$

Finally, we see that $\mathcal{A}$ is a bialgebra. Now we suppose $S : \mathcal{A} \to \mathcal{A}$ is an anti-linear and anti-homomorphism. Then, $N_q(h) \neq 0$ and the actions of S on generators of $\mathcal{A}$ are obtained from $S(h) := h^{-1}$ as follows

$$S(a_0) := \frac{1}{N_q(h)} a_0$$

$$S(a_1) := -\frac{1}{N_q(h)} a_1$$

$$S(a_2) := -\frac{1}{N_q(h)} [\lambda_+ a_2 - i\lambda_- a_3]$$

$$S(a_3) := -\frac{1}{N_q(h)} [i\lambda_- a_2 + \lambda_+ a_3].$$

One can obviously show that the mapping S holds the condition (14). Note that S preserves all commutation relations in (16). For example, for $[\mathrm{a}_2, \mathrm{a}_3] = 0$, we can show that $S([\mathrm{a}_2, \mathrm{a}_3]) = [S(\mathrm{a}_2), S(\mathrm{a}_3)] = 0$:

$$\begin{aligned}
& S([\mathrm{a}_2, \mathrm{a}_3]) = [S(\mathrm{a}_2), S(\mathrm{a}_3)] \\
& \Rightarrow S([a_2, a_3]) = S(a_2 a_3 - a_3 a_2) = S(a_2 a_3) - S(a_3 a_2) \\
& \Rightarrow S([a_2, a_3]) = S(a_3) S(a_2) - S(a_2) S(a_3) \\
& \Rightarrow S([a_2, a_3]) = (-\frac{1}{N_q(h)}[i\lambda_-(a_2) + \lambda_+ a_3])(-\frac{1}{N_q(h)}[\lambda_+ a_2 - i\lambda_- a_3]) \\
& \qquad - (-\frac{1}{N_q(h)}[\lambda_+ a_2 - i\lambda_- a_3])(-\frac{1}{N_q(h)}[i\lambda_-(a_2) + \lambda_+ a_3]) \\
& \Rightarrow S([a_2, a_3]) = (\frac{1}{N_q(h)})^2 (i\lambda_- \lambda_+ a_2^2 + \lambda_-^2 a_2 a_3 + \lambda_+^2 a_3 a_2 - i\lambda_+ \lambda_- a_3^2 \\
& \qquad - i\lambda_+ \lambda_- a_2^2 - \lambda_+^2 a_2 a_3 - \lambda_-^2 a_3 a_2 + i\lambda_- \lambda_+ a_3^2) \\
& \Rightarrow S([a_2, a_3]) = 0.
\end{aligned}$$

Thus, we see that there is a Hopf algebra structure associated with $\mathcal{A}$ together with the mappings Δ, ε and S mentioned above.

Now, it remains to show that one can extend this Hopf algebra structure to Hopf star algebra thanks to the involution * defined in Lemma. That is, we need to show that the following identities hold for all $a \in \mathcal{A}$:

$$((* \otimes *) \circ \Delta)(a) = (\Delta \circ *)(a) \tag{23}$$

$$\varepsilon(*(a)) = \overline{\varepsilon(a)} \tag{24}$$

$$(* \circ S \circ * \circ S)(a) = (S \circ * \circ S \circ *)(a). \tag{25}$$

To manage this, it is sufficient to show that the above conditions hold for the generators of $\mathcal{A}$. For instance, take $a_0 \in \mathcal{A}$, then we see that

$$\begin{aligned}
((* \otimes *) \circ \Delta)(a_0) &= (* \otimes *)(a_0 \otimes a_0 - a_1 \otimes a_1 + a_2 \otimes a_2 + a_3 \otimes a_3) \\
&= a_0^* \otimes a_0^* - a_1^* \otimes a_1^* + a_2^* \otimes a_2^* + a_3^* \otimes a_3^* \\
&= a_0 \otimes a_0 - a_1 \otimes a_1 + (\lambda_+ a_2 - i\lambda_- a_3) \otimes (\lambda_+ a_2 - i\lambda_- a_3) + (\lambda_+ a_3 + i\lambda_- a_2) \otimes (\lambda_+ a_3 + i\lambda_- a_2) \\
&= a_0 \otimes a_0 - a_1 \otimes a_1 + (\lambda_+^2 - \lambda_-^2)(a_2 \otimes a_2) + (\lambda_+^2 - \lambda_-^2)(a_3 \otimes a_3).
\end{aligned}$$

Thus, taking into account $\lambda_+^2 - \lambda_-^2 = 1$, we have

$$((*\otimes *)\circ \Delta)(a_0) = \Delta(a_0) = \Delta(a_0^*) = (\Delta \circ *)(a_0).$$

Furthermore, for $a_2 \in \mathcal{A}$, we observe that

$$(S\circ *)(a_2) = S(a_2^*) = S(\lambda_+ a_2 - i\lambda_- a_3) = \lambda_+ S(a_2) + i\lambda_- S(a_3) = \frac{-1}{N_q(h)} a_2$$

which implies

$$(S\circ * \circ S \circ *)(a_2) = \frac{1}{N_q(h)^2} a_2.$$

On the other hand

$$(*\circ S)(a_2) = *(S(a_2)) = \frac{-1}{N_q(h)}(\lambda_+ a_2^* - i\lambda_- a_3^*) = \frac{-1}{N_q(h)} a_2$$

thus $(*\circ S \circ * \circ S)(a_2) = \frac{1}{N_q(h)^2} a_2$. That is, the equation (25) holds for $a_2 \in \mathcal{A}$. It can be similarly shown that (23) and (25) are satisfied for other generators of $\mathcal{A}$. Finally, it is obvious that $\varepsilon(*(a_i)) = \overline{\varepsilon(a_i)}$ holds for $i \in \{0,1,2,3\}$. Consequently, $\mathcal{A}$ is a Hopf star algebra together with comappings given above.

4. CONCLUSION

Using the setting of the quantum group $Gl_q(2)$, we presented *q*-deformed version of the usual split quaternions. Furthermore, we show that there is a Hopf star algebra associated with the *q*-split quaternions with nonzero quadratic form. Therefore, one can easily conclude that the group of *q*-split quternions with $N_q(h) = 1$ is isomorphic to $SU_q(1,1)$.

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