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# EXACT AND APPROXIMATE SOLUTIONS OF BOUSSINESQ EQUATION: A COMPARISON STUDY

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***Abstract.*** *In this paper, a technique called Tanh method is applied to obtain some traveling wave solutions for Boussinesq's equation, and by using new transform iterative method (NTIM) which is a combination of the new iterative method (NIM) and Laplace transform, we obtain an approximate solution to Boussinesq's equation. A comparison between the traveling wave solution (exact solution) and the approximate one of equation under study, indicate that new transform iterative method (NTIM) is highly accurate and can be considered a very useful and valuable method.*

***Keywords:*** *Boussinesq's equation; Laplace transform; new transforms iterative method; Tanh method.*

## 1. INTRODUCTION

Boussinesq's equations in fluid mechanics designate a system of wave equations obtained by approximation of Euler's equations for irrational incompressible flows at the free surface. They make it possible to predict gravity waves such as cnoidal waves, Stokes waves, swells, tsunamis, solitons, etc. These equations were introduced by Joseph Boussinesq in 1872 and are an example of dispersive partial differential equations.

In 1885, Boussinesq published equations to determine the state of stress in a subgrade material. He investigated the stresses in an elastic, semi-infinite, homogeneous, and isotropic soil solid medium. The assumed solid medium is loaded normally on its upper plane surface by a concentrated point load. The material is also considered weightless and unstressed.

In this article, we will look for the exact solutions and the approximate solutions to the following Boussinesq equation [1-3]:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^4 u(x,t)}{\partial x^4} - \frac{\partial^2 u(x,t)}{\partial x^2} + 3\frac{\partial^2 u^2(x,t)}{\partial x^2} = 0. \quad (1)$$

To get the exact solutions to (1) we propose a analytical method called Tanh method (or hyperbolic tangent method) was first introduced by Huibin and Kelin [4], because it is a powerful technique to search for traveling waves coming out from one-dimensional nonlinear wave and evolution equations. This method is used to solve ordinary and partial nonlinear differential equations (ODE and PDE). Many researchers have used this method to solve various equations [5-7].

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To get the approximate solutions to (1), we propose a semi-analytical method called new transform iterative method (NTIM), it is a combination of the new iterative method (NIM) and Laplace transform. The (NIM) was introduced by Daftardar-Gejji and Jafari [8] to solve linear and nonlinear ordinary and partial differential equations of fractional order.

Many researcher use new iterative method (NIM) for different type of linear and nonlinear differential equations [9-15].

## 2. PRELIMINARIES

Before the beginning of this research, we are trying in a hurry to get to know the supporting materials to accomplish this work.

**Definition 2.1.** [16] Let f(t) be a function of t specified for $t > 0$. Then the Laplace transform of $f(t)$, denoted by $\mathcal{L}\{f(t)\}$, is defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt, \tag{2}$$

where we assume at present that the parameter $s$ is strictly positive real number.

**Theorem 2.2.** [16] If $c_1$ and $c_2$ are any constants while $f_1(t)$ and $f_2(t)$ are functions with Laplace transforms $F_1(s)$ and $F_2(s)$ respectively, then

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\} = c_1 F_1(s) + c_2 F_2(s). \tag{3}$$

**Theorem 2.3.** [17] If

$$f(t) = \sum_{n=0}^{+\infty} a_n t^n \ , \tag{4}$$

converges for $t \geq 0$, with

$$|a_n| \leq \frac{K\alpha^n}{n!}, \tag{5}$$

for all $n$ sufficiently large and $\alpha > 0$, $K > 0$, then

$$\mathcal{L}\{f(t)\} = \sum_{n=0}^{+\infty} a_n \mathcal{L}\{t^n\} = \sum_{n=0}^{+\infty} \frac{a_n n!}{s^{n+1}} a_n \mathcal{L}\{t^n\} (\mathcal{R}e(s) > \alpha). \tag{6}$$

**Theorem 2.4.** [16] If $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\{F^n(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - sF^{(n-2)}(0) - F^{(n-1)}(0), \tag{7}$$

if $F(t), F'(t), \dots, F^{(n-1)}(t)$ are continuous for $0 \leq t \leq N$ and of exponential order for $t > N$ wile $F^{(n)}(t)$is sectionally continuous for $0 \leq t \leq N$.

## 3. OUTLINE OF THE TANH METHOD

The tanh method will be introduced as presented by Malfliet [18] and by Wazwaz [19, 20]. The tanh method is based on a priori assumption that the traveling wave solutions can be expressed in terms of the tanh function to solve the coupled KdVequations.

The tanh method is developed by Malfliet [18]. The method is applied to find out an exact solution of a nonlinear ordinary differential equation. The nonlinear wave and evolution equations we want to investigate (for simplicity, in one dimension) are commonly written as

$$u_t = \left[U, U_{x,} U_{xx}, \dots\right] \; or \; u_{tt} = \left[U, U_{x,} U_{xx}, \dots\right]. \quad (8)$$

We like to know if these equations admit exact traveling wave solutions and how to compute them. The first step is to combine the independent variables, $x$ and $t$, into a new variable, ξ =c(x − vt), which defines the traveling frame of reference. Here $c$ and $v$ represent the wave number and velocity of the traveling wave. Both are undetermined parameters but we assume that $(c > 0)$. Accordingly, thedependent variable $u(x,t)$is replaced by $U(\xi)$. Equations like (8) are then transformed into

$$-cv\frac{dU}{d\xi} = \left[U, c\frac{dU}{d\xi}, c^2\, d^2U\frac{d^2U}{d\xi^2}, \dots\right] \; or \; c^2\, v^2\frac{d^2U}{d\xi^2} = \left[U, c\frac{dU}{d\xi}, c^2\, d^2U\frac{d^2U}{d\xi^2}, \dots\right]. \quad (9)$$

Hence, in what follows we deal with ODEs rather than with PDEs. Our main goal is to find exact solutions for those ODEs in tanh form. If that is impossible, one can find approximate solutions [21]. So, we introduce a new independent variable $Y = tanh\,\xi$ into the ODE. The coefficients of the ODE in $U(\xi) = F(Y)$then solely depend on $Y$ , because $\frac{d}{d\xi}$ and subsequent derivatives in (9) are now replaced b $(1 - Y^2)\frac{d}{dy}$, etc.. Therefore, it makes sense to attempt to find solution(s) as a finite power series in $Y$,

$$F(Y) = \sum_{n=0}^{N} a_n Y^n \quad (10)$$

To determine $N$ (highest order of $Y$), the following balancing procedure is used. At least two terms proportional to $Y^N$ must appear after substitution of ansatz (10) into the equation under study. As a result of this analysis, we definitely require $a_{N+1} = 0$ and $a_N \neq 0,$ for a particular $N$. It turns out that $N = 1$ or 2 in most cases. This balance (and thus $N$) is obtained by comparing the behavior of $Y^N$ in the highest derivative against its counterpart within the nonlinear term(s). As soon as $N$ is determined in this way, we get after substitution of (10) into (9) (transformed to the $Y$ variable) algebraic equations for $a_n(n = 0; 1; \dots; N)$. Depending on the problem under study, the wave number $c$ will remain fixed or undetermined. As already mentioned, the velocity $v$ of the traveling wave is always a function of $c$. If one is able to find nontrivial values for $a_n(n = 0; 1; \dots; N)$, in terms of known quantities, a solution is ultimately obtained [22].

## 4. THE EXACT SOLUTIONS OF BOUSSINESQ'S EQUATION BY TANH METHOD

Consider the Boussinesq equation in the form:

$$\frac{\partial^2 u(x,t)}{\partial t^2}-\frac{\partial^4 u(x,t)}{\partial x^4}-\frac{\partial^2 u(x,t)}{\partial x^2}+3\frac{\partial^2 u^2(x,t)}{\partial x^2}=0. \tag{11}$$

We consider the traveling wave transformation defined by:

$$U(\xi)=u(x,t), \qquad \xi=c(x-vt). \tag{12}$$

Using Eq. (12), then (11) we deduce the following ordinary differential equation:

$$c^2v^2\frac{d^2U}{d\xi^2}-c^2\frac{d^2U}{d\xi^2}+6c^2\frac{d^2U}{d\xi^2}U(\xi)+6c^2\left(\frac{dU}{d\xi}\right)^2-c^4\frac{d^4U}{d\xi^4}=0, \tag{13}$$

that is to say

$$(v^2-1)\frac{d^2U}{d\xi^2}+6\frac{d^2U}{d\xi^2}U(\xi)+6\left(\frac{dU}{d\xi}\right)^2-c^2\frac{d^4U}{d\xi^4}=0. \tag{14}$$

Now equilibrate the highest order derivative $\frac{d^4U}{d\xi^4}$ and the non-linear term $\frac{d^2U}{d\xi^2}U(\xi)$, on a $2N+2=N+4$ or equivalent to $N=2$.

So,

$$U(\xi)=a_0+a_1tanh(\xi)+a_2tanh^2U(\xi), \tag{15}$$

substituting Eq. (15) into (14) and using Mathematica software we get a polynomial of $tanh(\xi)$, $(k\;=\;0,1,2,\dots)$ Equating the coefficients of this polynomial of the same powers of $tanh(\xi)$ to zero, we obtain a system of algebraic equations for $a_0,\ a_1,\ a_2,\ v,\ c$.

$$\begin{gathered}16a_2c^2+2a_2v^2+6{a_1}^2+12a_0a_2-2a_2=0,\\ -16a_1c^2-2a_1v^2-12a_0a_1+2a_1+36a_1a_2=0,\\ -136a_2c^2-8a_2v^2-24{a_1}^2+36{a_2}^2-48a_0a_2+8a_2=0,\\ 40a_1c^2+2a_1v^2+12a_0a_1-2a_1-108a_1a_2=0,\\ 240a_2c^2+6a_2v^2+18{a_1}^2-96{a_2}^2+36a_0a_2-6a_2=0,\\ 72a_1a_2-24a_1c^2=0,\\ 60{a_2}^2-120a_2c^2=0,\end{gathered} \tag{16}$$

or $a_2\neq 0$.

Solving them using Mathematica gives:

$$\left\{a_0 \to \frac{1}{6}(-8c^2 - v^2 + 1), a_1 \to 0, a_2 \to 2c^2\right\}, \tag{17}$$

substitution in (15), result:

$$u(x,t) = \frac{1}{6}(-8c^2 - v^2 + 1) + 2c^2 tanh^2\big(c(x - tv)\big). \tag{18}$$

## 5. NEW ITERATIVE METHOD

The principle of new iterative method (NIM) is as follows (see [8, 11] for more details).

Consider the following general functional equation

$$u = N(u) + f, \tag{19}$$

where $N$a nonlinear operator from a Banach space $B \to B$ and $f$ is a known function.

We are looking for a solution $u$ of Eq. (19) having the series form:

$$u = \sum_{i=0}^{+\infty} u_i. \tag{20}$$

The nonlinear operator $N$ is decomposed as

$$N\left(\sum_{i=0}^{+\infty} u_i\right) = N(u_0) + \sum_{i=1}^{+\infty}\left\{N\left(\sum_{J=0}^{i} u_J\right) - N\left(\sum_{J=0}^{i-1} u_J\right)\right\}. \tag{21}$$

From equations (20) and (21), Eq.(19) is equivalent to

$$\sum_{i=0}^{+\infty} u_i = f + N(u_0) + \sum_{i=1}^{+\infty}\left\{N\left(\sum_{J=0}^{i} u_J\right) - N\left(\sum_{J=0}^{i-1} u_J\right)\right\}. \tag{22}$$

We definie the recurrence relation:

$$\begin{cases} u_0 = f, \\ u_1 = N(u_0), \\ u_{m+1} = N(u_0 + \cdots + u_m) - N(u_0 + \cdots + u_{m-1}), \qquad m = 1, 2, \dots . \end{cases} \tag{23}$$

Then

$$(u_1 + \cdots + u_{m+1}) = N(u_0 + \cdots + u_m), \qquad m = 1, 2, \dots, \tag{24}$$

and

$$u = f + \sum_{i=1}^{+\infty} u_i. \tag{25}$$

If $N$ is a contraction, i.e $\|N(x) - N(y)\| \leq K\|x - y\|, 0 < K < 1$, then

$$u_{m+1} = \|N(u_0 + \cdots + u_m) - N(u_0 + \cdots + u_{m-1})\| \ll K\|u_m\| \leq K^{n+1}\|u_0\|, m = 1, 2, ...,$$

and the series $\sum_{i=0}^{+\infty} u_i$ absolutely and uniformly converges to a solution of Eq.(19) [23], which is unique, in view of the Banach fixed point theorem [24].

## 6. NEW TRANSFORM ITERATIVE METHOD

To illustrate the basic idea of the (NTIM), we consider the following equation:

$$D_t^{\boldsymbol{n}} u(x,t) + Ru(x,t) = g(x,t), \quad n \in \mathbb{N}, \tag{26}$$

subject to the initial conditions

$$u(x,0) = h^0(x), \quad u_t^k(x,0) = h^k(x), \quad k \in \{1, \dots, n-1\}, \tag{27}$$

were $D_t^n = \frac{\partial^n}{\partial t^n}$, $R$ a general nonlinear operator and $g$ as a continuous function.

Taking the Laplace transform on both sides of (26), we get:

$$\mathcal{L}\{u(x,t)\} = s^{-n} \sum_{k=0}^{n-1} s^{n-k-1}\, h^k(x) - s^{-n}\mathcal{L}\{Ru(x,t) - g(x,t)\}. \tag{28}$$

Applying the inverse Laplace transform on both sides of (28) and using the initial conditions (27), leads to

$$u(x,t) = \mathcal{L}^{-1}\left\{s^{-n} \sum_{k=0}^{n-1} s^{n-k-1}\, h^k(x)\right\} - \mathcal{L}^{-1}\{s^{-n}\mathcal{L}\{Ru(x,t) - g(x,t)\}\}. \tag{29}$$

Thus, we obtain the following typical form

$$u(x,t) = f(x,t) + N[u(x,t)], \tag{30}$$

with

$$f(x,t) = \mathcal{L}^{-1}\left\{s^{-n} \sum_{k=0}^{n-1} s^{n-k-1}\, h^k(x)\right\}, \tag{31}$$

and

$$N[u(x,t)] = -\mathcal{L}^{-1}\{s^{-n}L\{Ru(x,t) - g(x,t)\}\}. \tag{32}$$

The function $f$ depends on initial conditions and $N$ represents the nonlinear part of equation. Applying the iterative method discussed in to the problem (30), we obtain the following algorithm:

$$\begin{cases} u_0 = f, \\ \quad u_1 = N(u_0), \\ u_{m+1} = N(u_0 + \cdots + u_m) - N(u_0 + \cdots + u_{m-1}), \qquad m = 1, 2, \dots, \end{cases} \tag{33}$$

then

$$(u_1 + \cdots + u_{m+1}) = N(u_0 + \cdots + u_m), \qquad m = 1, 2, \dots, \tag{34}$$

and

$$u = f + \sum_{i=1}^{+\infty} u_i. \tag{35}$$

## 7. THE APPROXIMATE SOLUTION OF THE BOUSSINESQ EQUATION BY (NTIM)

Consider the Boussinesq equation in the form:

$$u_{tt}(x,t) - u_{4x}(x,t) - u_{2x}(x,t) + 3(u^2)_{xx}(x,t) = 0, \tag{36}$$

subject to the initial conditions

$$u(x,0) = \frac{1}{24} + \frac{1}{8} tanh^2\left(\frac{x}{4}\right), \tag{37}$$

$$u_t(x,0) = \frac{-1}{32} tanh\left(\frac{x}{4}\right) sech^2\left(\frac{x}{4}\right). \tag{38}$$

Taking the Laplace transform on both sides of (36), we get:

$$\mathcal{L}\{u(x,t)\} = s^{-1}\left(\frac{1}{24} + \frac{1}{8} tanh^2\left(\frac{x}{4}\right)\right) + s^{-2}\left(\frac{-1}{32} tanh\left(\frac{x}{4}\right) sech^2\left(\frac{x}{4}\right)\right) + s^{-2}\mathcal{L}\{u_{4x} + u_{2x} - 3(u^2)_{xx}\}. \tag{39}$$

Applying the inverse Laplace transform on both sides of (39), leads to

$$u(x,t) = \frac{1}{24} + \frac{1}{8} tanh^2\left(\frac{x}{4}\right) - \frac{1}{32} t\, tanh\left(\frac{x}{4}\right) sech^2\left(\frac{x}{4}\right) + \mathcal{L}^{-1}\left\{s^{-2}\mathcal{L}\{u_{4x} + u_{2x} - 3(u^2)_{xx}\}\right\}. \tag{40}$$

Thus, we obtain the following typical form

$$u(x,t) = f(x,t) + N[u(x,t)], \tag{41}$$

with

$$f(x,t) = \frac{1}{24} + \frac{1}{8} tanh^2\left(\frac{x}{4}\right) - \frac{1}{32} t\, tanh\left(\frac{x}{4}\right) sech^2\left(\frac{x}{4}\right), \tag{42}$$

and

$$N[u(x,t)] = \mathcal{L}^{-1}\left\{s^{-2} L\{u_{4x} + u_{2x} - 3(u^2)_{xx}\}\right\}. \tag{43}$$

Applying the new iterative method discussed in to the problem (41), we obtain the following algorithm:

$$u_0 = \frac{1}{24} + \frac{1}{8} tanh^2\left(\frac{x}{4}\right) - \frac{1}{32} t\, tanh\left(\frac{x}{4}\right) sech^2\left(\frac{x}{4}\right),$$

$$u_1 = N(u_0) = \frac{-t^4 sech^8\left(\frac{x}{4}\right)}{32768} + \frac{3t^4 tanh^2\left(\frac{x}{4}\right) sech^6\left(\frac{x}{4}\right)}{8192} - \frac{t^4 tanh^4\left(\frac{x}{4}\right) sech^4\left(\frac{x}{4}\right)}{4096}$$
$$- \frac{t^3 tanh\left(\frac{x}{4}\right) sech^6\left(\frac{x}{4}\right)}{768} - \frac{t^3 tanh^3\left(\frac{x}{4}\right) sech^4\left(\frac{x}{4}\right)}{1536} + \frac{t^3 tanh\left(\frac{x}{4}\right) sech^4\left(\frac{x}{4}\right)}{512}$$
$$+ \frac{t^3 tanh^5\left(\frac{x}{4}\right) sech^2\left(\frac{x}{4}\right)}{1536} - \frac{t^3 tanh^3\left(\frac{x}{4}\right) sech^2\left(\frac{x}{4}\right)}{1024} - \frac{t^2 sech^6\left(\frac{x}{4}\right)}{256}$$
$$+ \frac{3t^2 sech^4\left(\frac{x}{4}\right)}{512} + \frac{t^2 tanh^2\left(\frac{x}{4}\right) sech^4\left(\frac{x}{4}\right)}{256} + \frac{t^2 tanh^4\left(\frac{x}{4}\right) sech^2\left(\frac{x}{4}\right)}{128}$$
$$- \frac{3}{256} t^2 tanh^2\left(\frac{x}{4}\right) sech^2\left(\frac{x}{4}\right),$$

$$u_2 = N(u_0 + u_1) - N(u_0)$$
$$= -\frac{1}{43293270343680} t^4 sech^{18}\left(\frac{x}{4}\right) ( -417312000 + 391039488t^2 + 4503600t^4$$
$$-3249288t^6 + 6(-84627200 + 41571264t^2 + 160275t^4 + 785176t^6) cosh\left(\frac{x}{2}\right)$$
$$-24(-2208640 + 14637952t^2 + 206635t^4 + 73570t^6) cosh(x)$$
$$+263692800 cosh\left(\frac{3x}{2}\right) - 202511232t^2 cosh\left(\frac{3x}{2}\right) - 433350t^4 cosh\left(\frac{3x}{2}\right)$$
$$+323568t^6 cosh\left(\frac{3x}{2}\right) + 148915200 cosh(2x) + 34438656t^2 cosh(2x)$$
$$+859440t^4 cosh(2x) - 25704t^6 cosh(2x) + 25536000 cosh\left(\frac{5x}{2}\right)$$
$$+23899008t^2 cosh\left(\frac{5x}{2}\right) - 121290t^4 cosh\left(\frac{5x}{2}\right) + 672t^6 cosh\left(\frac{5x}{2}\right) - 4838400 cosh(3x)$$
$$-3290112t^2 cosh(3x) + 3240t^4 cosh(3x) - 1666560\, cosh\left(\frac{7x}{2}\right)$$
$$+61824t^2 cosh\left(\frac{7x}{2}\right) + 30t^4 cosh\left(\frac{7x}{2}\right) + 26880 cosh(4x) + 31194240t sinh\left(\frac{x}{2}\right)$$
$$-39954240t^3 sinh\left(\frac{x}{2}\right) + 1461600t^5 sinh\left(\frac{x}{2}\right) + 36398208t sinh(x)$$
$$-14259840t^3 sinh(x) - 232365t^5 sinh(x) + 20829312t sinh\left(\frac{3x}{2}\right)$$

$$+10065600t^3 sinh\left(\frac{3x}{2}\right) - 422730t^5 sinh\left(\frac{3x}{2}\right) + 5738880t sinh(2x)$$

$$+3448320t^3 sinh(2x) + 147350t^5 sinh(2x) + 67200t sinh\left(\frac{5x}{2}\right)$$

$$-974400t^3 sinh\left(\frac{5x}{2}\right) - 13370t^5 sinh\left(\frac{5x}{2}\right) - 362880t sinh(3x)$$

$$+17280t^3 sinh(3x) + 315t^5 sinh(3x) - 61824t sinh\left(\frac{7x}{2}\right)$$

$$+960t^3 sinh\left(\frac{7x}{2}\right) + 1344t sinh(4x))$$

$u(x,t) \approx u_0 + u_1 + u_2$

$$= -\frac{1}{43293270343680} sech^{18}\left(\frac{x}{4}\right)(-275526451200 - 19926466560t^2$$

$$+279417600t^4 + 391039488t^6 + 4503600t^8 - 3249288t^{10}$$

$$+6(-86594027520 - 5658132480t^2 + 73427200t^4 + 41571264t^6$$

$$+160275t^8 + 785176t^{10})\, cosh\left(\frac{x}{2}\right) - 24(17891328000 + 861020160t^2$$

$$-8229760t^4 + 14637952t^6 + 206635t^8 + 73570t^{10}) cosh(x)$$

$$-300574310400 cosh\left(\frac{3x}{2}\right) - 8453652480t^2 cosh\left(\frac{3x}{2}\right) + 19837440t^4 cosh\left(\frac{3x}{2}\right)$$

$$-202511232t^6 cosh\left(\frac{3x}{2}\right) - 433350t^8 cosh\left(\frac{3x}{2}\right) + 323568t^{10} cosh\left(\frac{3x}{2}\right)$$

$$-171096145920 cosh(2x) - 1878589440t^2 cosh(2x) - 31718400t^4 cosh(2x)$$

$$+34438656t^6 cosh(2x) + 859440t^8 cosh(2x) - 25704t^{10} cosh(2x)$$

$$-75969331200 cosh\left(\frac{5x}{2}\right) + 72253440t^2 cosh\left(\frac{5x}{2}\right) - 19622400t^4 cosh\left(\frac{5x}{2}\right)$$

$$+23899008t^6 cosh\left(\frac{5x}{2}\right) - 121290t^8 cosh\left(\frac{5x}{2}\right) + 672t^{10} cosh\left(\frac{5x}{2}\right)$$

$$-25102909440 cosh(3x) + 185794560t^2 cosh(3x) - 4838400t^4 cosh(3x)$$

$$-3290112t^6 cosh(3x) + 3240t^8 cosh(3x) - 5780275200\, cosh\left(\frac{7x}{2}\right)$$

$$+51609600t^2 cosh\left(\frac{7x}{2}\right) - 376320t^4 cosh\left(\frac{7x}{2}\right) + 61824t^6 cosh\left(\frac{7x}{2}\right) + 30t^8 cosh\left(\frac{7x}{2}\right)$$

$$-825753600cosh(4x) + 5160960t^2cosh(4x) + 26880t^4cosh(4x)$$

$$-55050240cosh\left(\frac{9x}{2}\right) + 59041382400tsinh\left(\frac{x}{2}\right) - 1599037440t^3sinh\left(\frac{x}{2}\right)$$

$$+31194240t^5sinh\left(\frac{x}{2}\right) - 39954240t^7sinh\left(\frac{x}{2}\right) + 1461600t^9sinh\left(\frac{x}{2}\right)$$

$$+82657935360tsinh(x) - 2091048960t^3sinh(x) + 36398208t^5\ sinh(x)$$

$$-14259840t^7sinh(x) - 232365t^9sinh(x) + 67629219840t\ sinh\left(\frac{3x}{2}\right)$$

$$-1509580800t^3sinh\left(\frac{3x}{2}\right) + 20829312t^5sinh\left(\frac{3x}{2}\right) + 10065600t^7sinh\left(\frac{3x}{2}\right)$$

$$-422730t^9sinh\left(\frac{3x}{2}\right) + 37571788800tsinh(2x) - 682106880t^3sinh(2x)$$

$$+5738880t^5sinh(2x) + 3448320t^7sinh(2x) + 147350t^9sinh(2x)$$

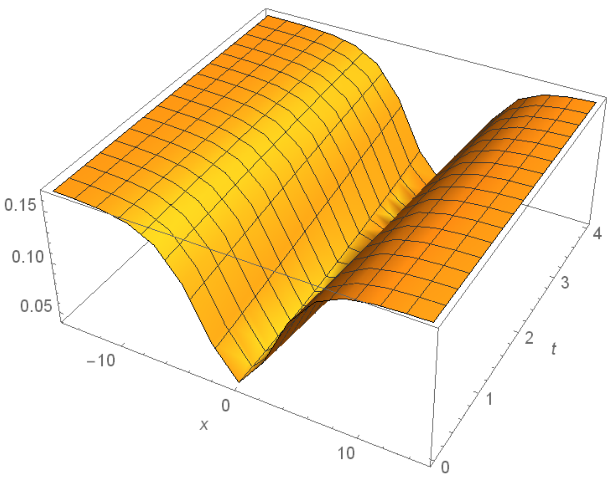
$$+14450688000tsinh\left(\frac{5x}{2}\right) - 184934400t^3sinh\left(\frac{5x}{2}\right) + 67200t^5sinh\left(\frac{5x}{2}\right)$$

$$-974400t^7sinh\left(\frac{5x}{2}\right) - 13370t^9sinh\left(\frac{5x}{2}\right) + 3715891200tsinh(3x)$$

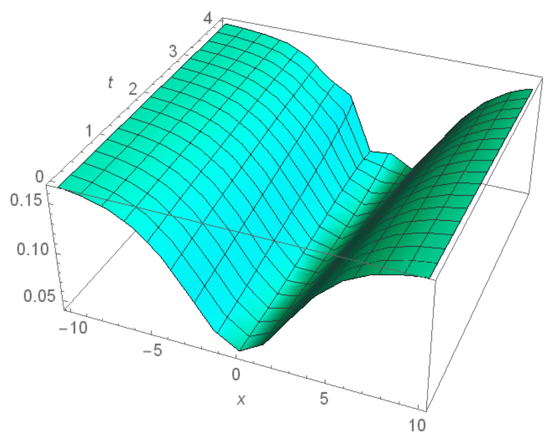
$$-23224320t^3sinh(3x) - 362880t^5sinh(3x) + 17280t^7sinh(3x) + 315t^9sinh(3x)$$

$$+578027520tsinh\left(\frac{7x}{2}\right) + 860160t^3sinh\left(\frac{7x}{2}\right) - 61824t^5sinh\left(\frac{7x}{2}\right) + 960t^7sinh\left(\frac{7x}{2}\right)$$

$$+41287680tsinh(4x) + 430080t^3sinh(4x) + +1344t^5sinh(4x)) \qquad (44)$$



**a) $U_{Exact}(x,t)$**



**b) $U_{Approx}(x,t)$**

**Figure 1. Plot of the Exact solution (18) of Eq. (11) when $c=\frac{1}{4}$and $v=\frac{1}{2}$ and approximate solution (44) of Eq. (36), for $(x,t)\in[-10,10]\times[0,4]$.**

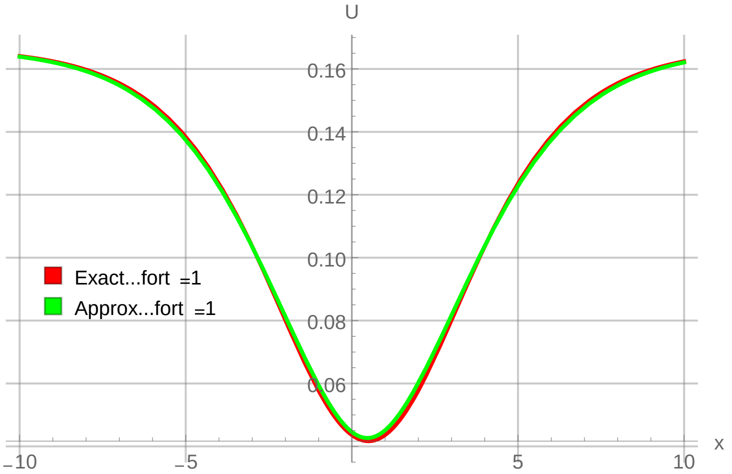



**Figure 2. Plot of the Exact solution (18) of Eq. (11) when $c = \frac{1}{4}$and$v = \frac{1}{2}$and approximate solution (44) of Eq. (36), for $t = 1$ and $x \in [-10, 10]$.**

## 8. CONCLUSION

Our search for the approximate solution of this equation is to show the importance of the new transform iterative method (NTIM) used and its efficiency, by comparing and observingthe compatibility of the solution approximate with the exact solution, and this is illustratedby the graphical representation of the two solutions.

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