ORIGINAL PAPER

THE RELATION BETWEEN FRENET FRAME OF THE NATURAL LIFT CURVE AND BISHOP FRAME OF THE CURVE

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Manuscript received: 16.10.2021;. Accepted paper: 03.12.2021;
Published online: 30.12.2021.

Abstract. *In this study, the relation between Frenet Frame of the natural lift curve* $\bar{\gamma}$ *of the curve* γ *and Bishop Frame vectors of* γ *is given in* IR^3 *and* IR_1^3.
Keywords: *Natural lift curve; Bishop Frame; Frenet Frame.*

1. INTRODUCTION

R. L. Bishop [1], put forward the best answer to this as "there are 3 more than one way to crack a curve". Bishop observed that parallel vector fields on a C^2 regular curve form a 3-dimensional vector space. He revealed the equations of the Bishop roof, which is named after him; hence it is sometimes referred to as the Relatively Parallel Adapted Frame (Bishop, [1])

Fenchel W. [2], stated that a point $\gamma(t)$ on a curve, when plotting the curve, the Frenet vectors $\{T, N, B\}$ change and thus spherical signs are formed.

Thorpe J.A. [3], together with the geodesic spray concepts, gave the theorem that "for a curve γ to be an integral curve for the geodesic spray X of the natural lift γ, and only if γ is a geodesic over " M. Çalışkan, Sivridağ and Hacısalihoğlu [4], using these concepts and theorem given by [3] in E^3, have given that the curve should be a curve when the natural lift curve of the spherical indicators of a curve is an integral curve of the geodesic spray. Ergün and Çalışkan [5], defined the concepts of the natural lift curve and geodesic spray in Minkowski 3-space.The anologue of the theorem of Thorpe was given in Minkowski 3-space by Ergün and Çalışkan [5].

Walrave [6], gave Frenet formulas of timelike, spacelike and null curves in IR_1^3 3-dimensional Minkowski space and characterized curves of constant curvature.

Let $\gamma : I \to \mathrm{IR}^3$ be a parametrized curve. We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve γ, where T, N and B are the tangent, the principal normal and the binormal vector fields of the curve γ, respectively.

Let γ be a reguler curve in IR^3. Then

$$T = \frac{\gamma'}{\|\gamma'\|},\ N = B \times T,\ B = \frac{\gamma' \times \gamma''}{\|\gamma' \times \gamma''\|},.$$

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If γ is a unit speed curve, then

$$T=\gamma^{'},\ N=\frac{\gamma^{''}}{\left\|\gamma^{''}\right\|},\ B=T\times N,.$$

Let γ be a unit speed space curve with curvature κ and torsion τ. Let Frenet vector fields of γ be $\{T,N,B\}$. Then, Frenet formulas are given by

$$T^{'}=\kappa N,\ N^{'}=-\kappa T+\tau B,\ B^{'}=-\tau N,$$

where $\kappa=\left\langle T^{'},N\right\rangle$ and $\tau=\left\langle N^{'},B\right\rangle$. For any unit speed curev $\gamma : I\rightarrow \mathrm{IR}^3$, we call $W(s)=\tau T(s)+\kappa B(s)$ the Darboux vector field of γ . θ being an angle between B and the Frenet instantaneous rotation vector W , we can write

$$\kappa=\|W\|\cos\theta,\tau=\|W\|\sin\theta.$$

Definition 1: *Let $\gamma : I\rightarrow \mathrm{IR}^3$ be a unit curve. Let $T=\dot{\gamma}$ be the tangent vector defined at each point of the curve. In this case, M_1 and M_2 vectors are perpendicular to the tangent vector T at each point and any two vector fields in the normal plane, on the curve γ , $\{T,N,B\}$, there is always a frame $\{T,M_1,M_2\}$, as an alternative to the moving frame. $\{T,M_1,M_2\}$ is Bishop frame to this alternative frame. Then, Frenet formulas are given by* [1],

$$\begin{aligned}
&\dot{T}=k_1M_1+k_2M_2,\\
&\dot{M}_1=k_1T,\\
&\dot{M}_2=k_2T,\\
&\kappa(t)=\sqrt{k_1^2+k_2^2},\ \phi(t)=\arctan\left(\frac{k_1}{k_2}\right),\ \tau(t)=\dot{\phi},\\
&k_1=\kappa\cos\phi,\ k_2=\kappa\sin\phi,\\
&T=T,\\
&M_1=\cos\phi N-\sin\phi B,\\
&M_2=\sin\phi N+\cos\phi B
\end{aligned}$$

where the differentiable functions k_1 and k_2 are the Bishop curvatures.

Definition 2: *Let M be a hypersurface in IR^3 and let $\gamma : I\rightarrow M$ be a parametrized curve. γ is called an integral curve of X if*

$$\frac{d}{ds}(\gamma(s))=X(\gamma(s))\ (\text{for all } t\in I),[3].$$

where X *is a smooth tangent vector field on* M. *We have*

$$TM = \bigcup_{P\in M} T_P M = \chi(M)$$

where $T_P M$ *is the tangent space of* M *at* P *and* $\chi(M)$ *is the space of vector fields on* M.

Definition 3: *For any parametrized curve* $\gamma : I \to M$, $\bar{\gamma} : I \to TM$ *given by*

$$\bar{\gamma}(s) = \left(\gamma(s), \dot{\gamma}(s)\right) = \dot{\gamma}(s)|_{\gamma(s)},$$

is called the natural lift of γ *on* TM. *Thus, we can write*

$$\frac{d\bar{\gamma}}{ds} = \frac{d}{ds}\left(\dot{\gamma}(s)|_{\gamma(s)}\right) = D_{\dot{\gamma}(s)}\dot{\gamma}(s)$$

where D *is the Levi-Civita connection on* IR^3, [3].

Definition 4: *A* $X \in \chi(TM)$ *is called a geodesic spray if for* $V \in TM$

$$X(V) = -\langle S(V), V\rangle N,\ [3].$$

Theorem 5: *The natural lift* $\bar{\gamma}$ *of the curve* γ *is an integral curve of geodesic spray* X *if and only if* γ *is a geodesic on* M, [3].

We denote by $\left\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\right\}$ the moving Frenet frame along the curve $\bar{\gamma}$, where $\bar{T}, \bar{N}$ and $\bar{B}$ are the tangent, the principal normal and the binormal vector of the curve $\bar{\gamma}$, respectively.

Corollary 6: *Let* $\bar{\gamma}$ *be the natural lift of* γ *in* IR^3 *and be a reguler curve. Then*

$$\bar{T}(s) = N(s)$$
$$\bar{N}(s) = -\cos\theta T(s) + \sin\theta B(s)$$
$$\bar{B}(s) = \sin\theta T(s) + \cos\theta B(s), [7].$$

Corollary 7: *Let* $\bar{\gamma}$ *be the natural lift of* γ *with curvature* $\bar{\kappa}$ *and torsion* $\bar{\tau}$. *Then*

$$\bar{\kappa}(s) = \frac{1}{\cos\theta}, \bar{\tau}(s) = \frac{\dot{\theta}}{\|W\|\cos\theta}, [7].$$

Let Minkowski 3-space IR_1^3 be the vector space IR^3 equipped with the Lorentzian inner product g given by

$$g(X,X) = -x_1^2 + x_2^2 + x_3^2,$$

where $X = (x_1, x_2, x_3) \in \mathrm{IR}^3$.

A vector $X = (x_1, x_2, x_3) \in \mathrm{IR}^3$ is said to be timelike if $g(X,X) < 0$, spacelike if $g(X,X) > 0$ and lightlike (or null) if $g(X,X) = 0$. Similarly, an arbitrary curve $\gamma = \gamma(t)$ in IR_1^3 where t is a pseudo-arclength parameter, can be locally timelike, spacelike or null (lightlike), if all of its velocity vectors $\dot{\gamma}(t)$ are respectively timelike, spacelike or null (lightlike), for every $t \in I \subset \mathrm{IR}$. A lightlike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ (*resp.* $x_1 < 0$) and a timelike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ (*resp.* $x_1 < 0$).The norm of a vector X is defined by [8].

$$\|X\|_{IL} = \sqrt{|g(X,X)|}.$$

Lemma 8: *Let X and Y be nonzero Lorentz orthogonal vectors in IR_1^3. If X is timelike, then Y is spacelike [9].*

Lemma 9: *Let X and Y be pozitive (negative) timelike vectors in IR_1^3. Then*

$$g(X,Y) \le \|X\|\|Y\|$$

whit equality if and only if X and Y are linearly dependent [9].

Lemma 10: i) *Let X and Y be pozitive (negative) timelike vectors in IR_1^3. By the Lemma 9, there is unique nonnegative real number $\phi(X,Y)$ such that*

$$g(X,Y) = \|X\|\|Y\|\cosh\phi(X,Y)$$

the Lorentzian timelike angle between X and Y is defined to be $\phi(X,Y)$.

ii) *Let X and Y be spacelike vektors in IR_1^3 that span a spacelike vector subspace. Then we have*

$$|g(X,Y)| \le \|X\|\|Y\|$$

Hence, there is a unique real number $\phi(X,Y)$ between 0 and π such that

$$g(X,Y) = \|X\|\|Y\|\cos\phi(X,Y)$$

the Lorentzian spacelike angle between X and Y is defined to be $\phi(X,Y)$.

iii) *Let* X *and* Y *be spacelike vectors in* IR_1^3 *that span a timelike vector subspace. Then we have*

$$g(X,Y) > \|X\|\|Y\| .$$

Hence, there is a unique pozitive real number $\phi(X,Y)$ *between* 0 *and* π *such that*

$$|g(X,Y)| = \|X\|\|Y\|\cosh\phi(X,Y)$$

the Lorentzian timelike angle between X *and* Y *is defined to be* $\phi(X,Y)$

iv) *Let* X *be a spacelike vector and* Y *be a pozitive timelike vector in* IR_1^3 *. Then there is a unique nonnegative reel number* $\phi(X,Y)$ *such that*

$$|g(X,Y)| = \|X\|\|Y\|\sinh\phi(X,Y)$$

the Lorentzian timelike angle between X *and* Y *is defined to be* $\phi(X,Y)$, [9].

We denote the moving Frenet frame along the curve γ by $\{T(t), N(t), B(t)\}$, where T, N and B are the tangent, the principal normal and the binormal vector of the curve γ , respectively.

i) Let γ be a unit speed timelike space curve with curvature κ and torsion τ and Frenet vector fields of γ be $\{T, N, B\}$. In this trihedron, T is a timelike vector field, N and B are spacelike vector fields. Then, Frenet formulas are given by [6],

$$\dot{T} = \kappa N,$$

$$\dot{N} = \kappa T + \tau B,$$

$$\dot{B} = -\tau N.$$

ii) Let γ be a unit speed spacelike space curve with a spacelike binormal. For the Frenet vector fields we assume that T and B are spacelike vector fields and N is a timelike vector field . Then, Frenet formulas are given by [6],

$$\dot{T} = \kappa N,$$

$$\dot{N} = \kappa T + \tau B,$$

$$\dot{B} = \tau N.$$

iii) Let γ be a unit speed spacelike space curve with a timelike binormal. We assume that T and N are spacelike vector fields and B is a timelike vector field. Then, Frenet formulas are given by [6],

$$\dot{T} = \kappa N,$$
$$\dot{N} = -\kappa T + \tau B,$$
$$\dot{B} = \tau N.$$

Definition 11: *Let $\gamma : I \to \mathrm{IR}_1^3$ be a unit speed spacelike or timelik space curve. Let $T = \dot{\gamma}$ be the tangent vector defined at each point of the curve. In this case, M_1 and M_2 vectors are perpendicular to the tangent vector T at each point and any two vector fields in the normal plane, on the curve γ , $\{T, N, B\}$, there is always a frame $\{T, M_1, M_2\}$,as an alternative to the moving frame. $\{T, M_1, M_2\}$ is Bishop frame to this alternative frame* [10].

Let γ be a unit speed timelike space curve. In this trihedron, T is a timelike vector field, M_1 and M_2 are spacelike vector fields. Then, Frenet formulas are given by [10],

$$\dot{T} = k_1 M_1 + k_2 M_2,$$
$$\dot{M}_1 = k_1 T,$$
$$\dot{M}_2 = k_2 T,$$
$$\kappa(t) = \sqrt{\left|k_1^2 + k_2^2\right|},\ \phi(t) = \arctan\left(\frac{k_1}{k_2}\right),\ \tau(t) = \dot{\phi},$$
$$k_1 = \kappa \cos\phi,\ k_2 = \kappa \sin\phi,$$
$$T = T,$$
$$M_1 = \cos\phi N - \sin\phi B,$$
$$M_2 = \sin\phi N + \cos\phi B$$

where the differentiable functions k_1 and k_2 are the Bishop curvatures.

Let γ be a unit speed spacelike space curve with a spacelike binormal. In this trihedron, M_1 is a timelike vector field, T and M_2 are spacelike vector fields. Then, Frenet formulas are given by [10],

$$\dot{T} = k_1 M_1 - k_2 M_2$$
$$\dot{M}_1 = k_1 T$$
$$\dot{M}_2 = k_2 T$$
$$\kappa(t) = \sqrt{\left|k_1^2 - k_2^2\right|},\ \phi(t) = \arg\tanh\left(\frac{k_1}{k_2}\right),\ \tau(t) = \dot{\phi},$$
$$k_1 = \kappa \cosh\phi,\ k_2 = \kappa \sinh\phi,$$
$$T = T,$$
$$M_1 = \cosh\phi N - \sinh\phi B,$$
$$M_2 = -\sinh\phi N + \cosh\phi B$$

where the differentiable functions k_1 and k_2 are the Bishop curvatures.

Let γ be a unit speed spacelike space curve with a timelike binormal. In this trihedron, M_2 is a timelike vector field, T and M_1 are spacelike vector fields. Then, Frenet formulas are given by [10],

$$\dot{T} = k_1 M_1 - k_2 M_2$$
$$\dot{M}_1 = -k_1 T$$
$$\dot{M}_2 = -k_2 T$$
$$\kappa(t) = \sqrt{\left|k_1^2 - k_2^2\right|},\ \phi(t) = \arg\tanh\left(\frac{k_1}{k_2}\right),\ \tau(t) = \dot{\phi},$$
$$k_1 = \kappa\cosh\phi,\ k_2 = \kappa\sinh\phi,$$
$$T = T,$$
$$M_1 = \cosh\phi N - \sinh\phi B,$$
$$M_2 = -\sinh\phi N + \cosh\phi B$$

where the differentiable functions k_1 and k_2 are the Bishop curvatures.

Definition 12: *Let M be a hypersurface in IR_1^3 and let $\gamma : I \to M$ be a parametrized curve. γ is called an integral curve of X if*

$$\frac{d}{dt}(\gamma(t)) = X(\gamma(t)) \ (\text{for all } t \in I)$$

where X is a smooth tangent vector field on M [8]. *We have*

$$TM = \bigcup_{P\in M} T_P M = \chi(M)$$

where $T_P M$ is the tangent space of M at P and $\chi(M)$ is the space of vector fields of M .

Definition 13: *For any parametrized curve $\gamma : I \to M$, $\bar{\gamma} : I \to TM$ given by*

$$\bar{\gamma}(t) = \left(\gamma(t), \dot{\gamma}(t)\right) = \dot{\gamma}(t)|_{\gamma(t)}$$

is called the natural lift of γ on TM, [5]. *Thus, we can write*

$$\frac{d\bar{\gamma}}{dt} = \frac{d}{dt}\left(\dot{\gamma}(t)|_{\gamma(t)}\right) = \nabla_{\dot{\gamma}(t)} \dot{\gamma}(t)$$

where ∇ *is the Levi-Civita connection on* IR_1^3 .

Definition 14: *A* $X \in \chi(TM)$ *is called a geodesic spray if for* $V \in TM$

$$X(V) = \varepsilon g(S(V),V)N, \varepsilon = g(N,N), [5].$$

Theorem 15: *The natural lift* $\bar{\gamma}$ *of the curve* γ *is an integral curve of geodesic spray* X *if and only if* γ *is a geodesic on* M, [5].

We denote by $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ the moving Frenet frame along the curve $\bar{\gamma}$, where $\bar{T}, \bar{N}$ and $\bar{B}$ are the tangent, the principal normal and the binormal vector of the curve $\bar{\gamma}$, respectively.

Corollary 16: *Let* γ *be a unit speed timelike space curve and* $\bar{\gamma}$ *be the natural lift of* γ *If* W *is a spacelike vector field, then*

$$\bar{T}(s) = N(s)$$
$$\bar{N}(s) = -\cosh\theta T(s) - \sinh\theta B(s)$$
$$\bar{B}(s) = -\sinh\theta T(s) - \cosh\theta B(s), [7].$$

Corollary 17: *Let* γ *be a unit speed timelike space curve and the natural lift* $\bar{\gamma}$ *of the curve* γ *be a space curve with curvature* $\bar{\kappa}$ *and torsion* $\bar{\tau}$ *. If* W *is a spacelike vector field,then*

$$\bar{\kappa}(s) = \frac{1}{\cosh\theta}, \bar{\tau}(s) = -\frac{\dot{\theta}}{\|W\|\cosh\theta}, [7].$$

Corollary 18: *Let* γ *be a unit speed timelike space curve and* $\bar{\gamma}$ *be the natural lift of* γ *If* W *is a timelike vector field, then*

$$\bar{T}(s) = N(s)$$
$$\bar{N}(s) = -\sinh\theta T(s) - \cosh\theta B(s)$$
$$\bar{B}(s) = -\cosh\theta T(s) - \sinh\theta B(s), [7].$$

Corollary 19: *Let* γ *be a unit speed timelike space curve and the natural lift* $\bar{\gamma}$ *of the curve* γ *be a space curve with curvature* $\bar{\kappa}$ *and torsion* $\bar{\tau}$ *. If* W *is a timelike vector field, then*

$$\bar{\kappa}(s) = \frac{1}{\sinh\theta}, \bar{\tau}(s) = \frac{\dot{\theta}}{\|W\|\sinh\theta}, [7].$$

Corollary 20: *Let* γ *be a unit speed spacelike space curve with a spacelike binormal and* $\bar{\gamma}$ *be the natural lift of* γ*. Then*

$$\overline{T}(s)=N(s)$$
$$\overline{N}(s)=\cos\theta T(s)+\sin\theta B(s)$$
$$\overline{B}(s)=\sin\theta T(s)-\cos\theta B(s)\ ,[7].$$

Corollary 21: *Let* γ *be a unit speed spacelike space curve with a spacelike binormal and the natural lift* $\overline{\gamma}$ *of the curve* γ *be a space curve with curvature* $\overline{\kappa}$ *and torsion* $\overline{\tau}$ *. Then*

$$\overline{\kappa}(s)=\frac{1}{\cos\theta},\overline{\tau}(s)=-\frac{\dot{\theta}}{\|W\|\cos\theta},[7].$$

Corollary 22: *Let* γ *be a unit speed spacelike space curve with a timelike binormal and* $\overline{\gamma}$ *be the natural lift of* γ*. If* W *is a spacelike vector field, then*

$$\overline{T}(s)=N(s)$$
$$\overline{N}(s)=\sinh\theta T(s)-\cosh\theta B(s)$$
$$\overline{B}(s)=\cosh\theta T(s)-\sinh\theta B(s),[7].$$

Corollary 23: *Let* γ *be a unit speed spacelike space curve with a timelike binormal and the natural lift* $\overline{\gamma}$ *of the curve* γ *be a space curve with curvature* $\overline{\kappa}$ *and torsion* $\overline{\tau}$*. If* W *is a spacelike vector field, then*

$$\overline{\kappa}(s)=\frac{1}{\sinh\theta},\overline{\tau}(s)=-\frac{\dot{\theta}}{\|W\|\sinh\theta},[7].$$

Corollary 24: *Let* γ *be a unit speed spacelike space curve with a timelike binormal and* $\overline{\gamma}$ *be the natural lift of* γ *If* W *is a timelike vector field, then*

$$\overline{T}(s)=N(s)$$
$$\overline{N}(s)=\cosh\theta T(s)-\sinh\theta B(s)$$
$$\overline{B}(s)=\sinh\theta T(s)-\cosh\theta B(s),[7].$$

Corollary 25: *Let* γ *be a unit speed spacelike space curve with a timelike binormal and the natural lift* $\overline{\gamma}$ *of the curve* γ *be a space curve with curvature* $\overline{\kappa}$ *and torsion* $\overline{\tau}$*. If* W *is a timelike vector field, then*

$$\overline{\kappa}(s)=\frac{1}{\cosh\theta},\overline{\tau}(s)=\frac{\dot{\theta}}{\|W\|\cosh\theta},[7].$$

2. THE RELATION BETWEEN FRENET FRAME OF THE NATURAL LIFT CURVE AND BISHOP FRAME OF THE CURVE

In this section, the relations between the two frames are given.

Corollary 26: *Let* $\overline{\gamma}$ *be the natural lift of* γ *in* IR^3 *and be a reguler curve. The relation between the* $\{\overline{T}(s),\overline{N}(s),\overline{B}(s)\}$ *and the* $\{T(s),M_1(s),M_2(s)\}$ *of is as follows.*

$$\overline{T}(s)=\cos\theta M_1+\sin\theta M_2$$
$$\overline{N}(s)=-\cos\phi T(s)-\sin\phi\sin\theta M_1+\sin\phi\cos\theta M_2$$
$$\overline{B}(s)=\sin\phi T(s)-\cos\phi\sin\theta M_1+\cos\phi\cos\theta M_2.$$

Corollary 27: *Let* γ *be a unit speed timelike space curve and* $\overline{\gamma}$ *be the natural lift of* γ*. If* $\overline{\gamma}$ *is a spacelike space curve with a timelike binormal and* W *is a spacelike vector field. The relation between the* $\{\overline{T}(s),\overline{N}(s),\overline{B}(s)\}$ *and the* $\{T(s),M_1(s),M_2(s)\}$ *of is as follows,*

$$\overline{T}(s)=\cos\theta M_1+\sin\theta M_2$$
$$\overline{N}(s)=\cosh\phi T(s)-\sinh\phi\sin\theta M_1+\sinh\phi\cos\theta M_2$$
$$\overline{B}(s)=\sinh\phi T(s)-\cosh\phi\sin\theta M_1+\cosh\phi\cos\theta M_2.$$

Corollary 28: *Let* γ *be a unit speed timelike space curve and* $\overline{\gamma}$ *be the natural lift of* γ*. If* $\overline{\gamma}$ *is a spacelike space curve with a timelike binormal and* W *is a timelike vector field. The relation between the* $\{\overline{T}(s),\overline{N}(s),\overline{B}(s)\}$ *and the* $\{T(s),M_1(s),M_2(s)\}$ *of is as follows,*

$$\overline{T}(s)=\cos\theta M_1+\sin\theta M_2$$
$$\overline{N}(s)=\sinh\phi T(s)-\cosh\phi\sin\theta M_1+\cosh\phi\cos\theta M_2$$
$$\overline{B}(s)=\cosh\phi T(s)-\sinh\phi\sin\theta M_1+\sinh\phi\cos\theta M_2.$$

Corollary 29: *Let* γ *be a unit speed timelike space curve and* $\overline{\gamma}$ *be the natural lift of* γ*. If* $\overline{\gamma}$ *is a spacelike space curve with a spacelike binormal and* W *is a spacelike vector field. The relation between the* $\{\overline{T}(s),\overline{N}(s),\overline{B}(s)\}$ *and the* $\{T(s),M_1(s),M_2(s)\}$ *of is as follows,*

$$\overline{T}(s)=\cos\theta M_1+\sin\theta M_2$$
$$\overline{N}(s)=\cosh\phi T(s)-\sinh\phi\sin\theta M_1+\sinh\phi\cos\theta M_2$$
$$\overline{B}(s)=-\sinh\phi T(s)-\cosh\phi\sin\theta M_1+\cosh\phi\cos\theta M_2.$$

Corollary 30: *Let* γ *be a unit speed timelike space curve and* $\overline{\gamma}$ *be the natural lift of* γ*. If* $\overline{\gamma}$ *is a spacelike space curve with a spacelike binormal and* W *is a timelike vector field. The relation between the* $\{\overline{T}(s),\overline{N}(s),\overline{B}(s)\}$ *and the* $\{T(s),M_1(s),M_2(s)\}$ *of is as follows,*

$$\overline{T}(s)=\cos\theta M_1+\sin\theta M_2$$
$$\overline{N}(s)=\sinh\phi T(s)-\cosh\phi\sin\theta M_1+\cosh\phi\cos\theta M_2$$
$$\overline{B}(s)=-\cosh\phi T(s)+\sinh\phi\sin\theta M_1-\sinh\phi\cos\theta M_2.$$

Corollary 31: *Let* γ *be a unit speed spacelike space curve with a spacelike binormal and* $\overline{\gamma}$ *be the natural lift of* γ. *The relation between the* $\{\overline{T}(s),\overline{N}(s),\overline{B}(s)\}$ *and the* $\{T(s),M_1(s),M_2(s)\}$ *of is as follows,*

$$\overline{T}(s)=\cosh\theta M_1+\sinh\theta M_2$$
$$\overline{N}(s)=\cos\phi T(s)+\sin\phi\sinh\theta M_1+\sin\phi\cosh\theta M_2$$
$$\overline{B}(s)=\sin\phi T(s)-\cos\phi\sinh\theta M_1-\cos\phi\cosh\theta M_2.$$

Corollary 32: *Let* γ *be a unit speed spacelike space curve with a timelike binormal and* $\overline{\gamma}$ *be the natural lift of* γ. *If* $\overline{\gamma}$ *is a spacelike space curve with a timelike binormal and* W *is a spacelike vector field. The relation between the* $\{\overline{T}(s),\overline{N}(s),\overline{B}(s)\}$ *and the* $\{T(s),M_1(s),M_2(s)\}$ *of is as follows,*

$$\overline{T}(s)=\cosh\theta M_1+\sinh\theta M_2$$
$$\overline{N}(s)=-\sinh\phi T(s)+\cosh\phi\sinh\theta M_1+\cosh\phi\cosh\theta M_2$$
$$\overline{B}(s)=-\cosh\phi T(s)+\sinh\phi\sinh\theta M_1+\sinh\phi\cosh\theta M_2.$$

Corollary 33: *Let* γ *be a unit speed spacelike space curve with a timelike binormal and* $\overline{\gamma}$ *be the natural lift of* γ. *If* $\overline{\gamma}$ *is a spacelike space curve with a timelike binormal and* W *is a timelike vector field. The relation between the* $\{\overline{T}(s),\overline{N}(s),\overline{B}(s)\}$ *and the* $\{T(s),M_1(s),M_2(s)\}$ *of is as follows,*

$$\overline{T}(s)=\cosh\theta M_1+\sinh\theta M_2$$
$$\overline{N}(s)=-\cosh\phi T(s)+\sinh\phi\sinh\theta M_1+\sinh\phi\cosh\theta M_2$$
$$\overline{B}(s)=-\sinh\phi T(s)+\cosh\phi\sinh\theta M_1+\cosh\phi\cosh\theta M_2.$$

Corollary 34: *Let* γ *be a unit speed spacelike space curve with a timelike binormal and* $\overline{\gamma}$ *be the natural lift of* γ. *If* $\overline{\gamma}$ *is a spacelike space curve with a spacelike binormal and* W *is a spacelike vector field. The relation between the* $\{\overline{T}(s),\overline{N}(s),\overline{B}(s)\}$ *and the* $\{T(s),M_1(s),M_2(s)\}$ *of is as follows,*

$$\overline{T}(s)=\cosh\theta M_1+\sinh\theta M_2$$
$$\overline{N}(s)=-\sinh\phi T(s)+\cosh\phi\sinh\theta M_1+\cosh\phi\cosh\theta M_2$$
$$\overline{B}(s)=\cosh\phi T(s)-\sinh\phi\sinh\theta M_1-\sinh\phi\cosh\theta M_2.$$

Corollary 35: *Let* γ *be a unit speed spacelike space curve with a timelike binormal and*

$\bar{\gamma}$ *be the natural lift of* γ. *If* $\bar{\gamma}$ *is a spacelike space curve with a spacelike binormal and* W *is a timelike vector field. The relation between the* $\{\bar{T}(s),\bar{N}(s),\bar{B}(s)\}$ *and the* $\{T(s),M_1(s),M_2(s)\}$ *of is as follows,*

$$\bar{T}(s)=\cosh\theta M_1+\sinh\theta M_2$$

$$\bar{N}(s)=-\cosh\phi T(s)+\sinh\phi\sinh\theta M_1+\sinh\phi\cosh\theta M_2$$

$$\bar{B}(s)=\sinh\phi T(s)-\cosh\phi\sinh\theta M_1-\cosh\phi\cosh\theta M_2.$$

3. CONCLUSION

In this article, the relationship between the Bishop Frame of the curve and the Frenet Frame of the natural lift of the curve is given. As a result, the transition matrix between the two frames can also be calculated.

REFERENCES

[1] Bishop, R.L., *The American Mathematical Monthly*, **3**(82), 246, 1975.
[2] Fenchel, W., *Bulletin of the American Mathematical Society*, **57**, 44, 1951.
[3] Thorpe, J.A., *Elementary Topics In Differential Geometry,*Springer-Verlag, New York, Heidelberg-Berlin, 1979.
[4] Çalışkan, M., Sivridag, A.I., Hacisalihoglu, H.H., *Communications Faculty of Sciences University of Ankara*, **33**, 235, 1984.
[5] Ergün, E., Çalışkan, M, *International Journal of Contemporary Mathematical Sciences*, **6**, 39, 1929, 2011.
[6] Walrave J., PhD. Thesis *Curves and surfaces in Minkowski space*, , K.U. Leuven, 1995.
[7] Ergün, E., Bilici, M. Çalışkan, M., *Bulletin of Society for Mathematical Services & Standards*, **1**(2), 59, 2012.
[8] O'Neill, B., *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.
[9] Ratcliffe, J.G., *Foundations of Hyperbolic Manifold*, Springer-Verlag, New York, 1994.
[10] Bükcü, B., Karacan, M.K., *Tamkang Journal of Mathematics*, **3**(39), 255, 2008.
[11] Silva, L.C.B., PhD Thesis *Differential geometry of rotation minimizing frames, spherical curves, and quantum mechanics of a constrained particle*, Universidade Federal De Pernambuco, 2017.
[12] Arreaga, G., Capovilla, R., Guven, J., *Classical Quantum Gravity*, **18**(23), 5065, 2001.
[13] Do Carmo, M.P., *Differential Geometry of Curves and Surfaces*, Pearson Education, 1976.