ORIGINAL PAPER

THE GENERAL $F_{a,\lambda}(K,T)-$ STRUCTURE SATISFYING $aF^K+\lambda^r F^T=0$ ON M^n AND LIFTS PROBLEMS ASSOCIATED WITH THE STRUCTURE

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Abstract. *In this paper our aim is to get the general structure equation which covers previously acquired structures. In this context this paper consists of three main sections. In the first section, we define the general $F_{a,\lambda}(K,T)-$structure satisfying $aF^K+\lambda^r F^T=0,(F\neq 0, K\geq 3, T\geq 1$ and $(K\geq T), a$ and λ are non zero complex numbers, r some finite integer) on manifold M^n and studied to give some special examples. The second part, we find the integrability conditions by calculating Nijenhuis tensors of the horizontal lifts of the general $F_{a,\lambda}(K,T)-$structure satisfying $aF^K+\lambda^r F^T=0$. Later, we get the results of Tachibana operators applied to vector and covector fields according to the horizontal lifts of the general $F_{a,\lambda}(K,T)-$structure in cotangent bundle $T^*(M^n)$. In addition, we have studied to show the purity conditions of Sasakian metric with respect to the horizontal lifts of the structure. In the final section, all results obtained in the second section were investigated according to the complete and horizontal lifts of the general $F_{a,\lambda}(K,T)-$structure on tangent bundle $T(M^n)$.*

Keywords: *$F-$Structure; integrability conditions; Tachibana operators; lifts; Sasakian metric; tangent and cotangent bundles.*

1. INTRODUCTION

The investigation for the integrability of tensorial structures on manifolds and extension to the tangent or cotangent bundle, whereas the defining tensor field satisfies a polynomial identity has been an actively discussed research topic in the last 50 years, initiated by the fundamental works of Kentaro Yano and his collaborators, see for example [1]. There are a lot of structures on $n-$dimensional differentiable manifold M^n. Firstly, Ishihara and Yano [2] have obtained the integrability conditions of a structure F satisfying $F^3+F=0$. Gouli-Andreou [3] has studied the integrabilty conditions of a structure F satisfying $F^5+F=0$. Later, R. Nivas and C.S. Prasad [4] studied on the form $F_a(5,1)-$structure. Also $F_\lambda(7,1)-$structure extended in M^n to $T^*(M^n)$ by L. S. Das, R. Nivas and V. N. Pathak [5]. In 1989, V. C. Gupta [6] studied on more generalized form $F(K,1)-$structure satisfying $F^K+F=0$, where K is a positive integer $\geqslant 2$. In addition, manifolds with $F(2K+S,S)-$structure satisfying $F^{2K+S}+F^S=0,(F\neq 0$, fixed integer $K\geqslant 1$, fixed odd integer $S\geq 1)$ have been defined and studied by A. Singh [7] and the complete and horizontal lifts of $F(2K+S,S)-$structure extended in M^n to tangent bundle by A. Singh, R. K. Pandey and S. Khare [8]. In later times, M.D. Upadhyay and Ashwani Grag [9] and M. D. Upadhyay and

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V. C. Gupta studied on the $F(p,-(p-q))$ −structure satisfying $F^p - F^{p-q} = 0$ $(F \neq 0; p, q$ odd; $I)$ and $F(K,-(K-2))$ −structure satisfying $F^K - F^{K-2} = 0, (F \neq 0, I)$, respectively.

In 2004, R. Nivas and M. Saxena [10] studied on the complete and horizontal lifts of the $HSU-(4,2)$ −structure satisfying $F^4 + \lambda^r F^2 = 0$ $(F \neq 0, \lambda$ is non zero complex number, r some finite integer). Later, K. K. Dube [11] studied on the $F(2v+4,2)$ −structure satisfying $F^{2v+4} + F^2 = 0, F \neq 0$.

In this paper our aim is to get the general structure equation which covers previously acquired structures. In this context this paper consists of three main sections. In the first section, we define the general $F_{a,\lambda}(K,T)$ −structure satisfying $aF^K + \lambda^r F^T = 0, (F \neq 0, K \geq\geqslant 3, T \geqslant\geq 1$ and $(K \geqslant\geq T), a$ and λ are non zero complex numbers, r some finite integer) on manifold M and studied to give some special examples. The second part, we find the integrability conditions by calculating Nijenhuis tensors of the horizontal lifts of the general $F_{a,\lambda}(K,T)$ −structure satisfying $aF^K + \lambda^r F^T = 0$. Later, we get the results of Tachibana operators applied to vector and covector fields according to the horizontal lifts of the general $F_{a,\lambda}(K,T)-$ structure in cotangent bundle $T^*(M^n)$. In addition, we have studied to show the purity conditions of Sasakian metric with respect to the horizontal lifts of the structure. In the final section, all results obtained in the second section were investigated according to the complete and horizontal lifts of the general $F_{a,\lambda}(K,T)$ −structure on tangent bundle $T(M^n)$.

Let M^n be a differentiable manifold of class C^∞ and F be a non-null tensor field of type (1,1) satisfying

$$aF^K + \lambda^r F^T = 0, \tag{1.1}$$

where $F \neq 0, K \geqslant\geq 3, T \geq\geqslant 1$ and $(K \geq\geqslant T), a$ and λ are non zero complex numbers, r some finite integer. F is of constant rank r everywhere in M^n. We call such a structure the general $F_{a,\lambda}(K,T)$ −structure of rank r.

Let the operators l and m be defined as

$$l = -\frac{aF^{K-T}}{\lambda^r}, m = I + \frac{aF^{K-T}}{\lambda^r}, \tag{1.2}$$

where I denotes the identity operator on M^n.

The operators l and m defined by (1.2) satisfy the following:

$$\begin{aligned} l^2 &= \frac{a^2F^{2K-2T}}{\lambda^{2r}} = \frac{aF^{2K-T}}{\lambda^r}\frac{aF^{-T}}{\lambda^r} \\ &= -\frac{aF^{2K-T}}{\lambda^r}F^{-K} \\ &= -\frac{aF^{K-T}}{\lambda^r} \\ &= l, \end{aligned} \tag{1.3}$$

$$\begin{aligned} m^2 &= I^2 + 2\frac{aF^{K-T}}{\lambda^r} + \frac{a^2F^{2K-2T}}{\lambda^{2r}} \\ &= I + 2\frac{aF^{K-T}}{\lambda^r} + \frac{aF^{2K-T}}{\lambda^r}\frac{aF^{-T}}{\lambda^r} \\ &= I + 2\frac{aF^{K-T}}{\lambda^r} - \frac{aF^{2K-T}}{\lambda^r}F^{-K} \\ &= I + 2\frac{aF^{K-T}}{\lambda^r} - \frac{aF^{K-T}}{\lambda^r} \\ &= I + \frac{aF^{K-T}}{\lambda^r} \\ &= m, \end{aligned} \tag{1.4}$$

$$l + m = I, lm = ml = 0,$$
$$Fl = lF = F, Fm = mF = 0, \quad (1.5)$$

where I being the identity operator.

Consequently, if there is a tensor field $F \neq 0$ satisfying (1.1), then there exist on M^n two complementary distributions L and M. Corresponding to l and m respectively. Let the rank of F be constant and be equal to r ewerywhere, then the dimensions of L and M are r and $n - r$, respectively. We call such a structure a 'general $F_{a,\lambda}(K,T)$ −structure' and the manifold M^n with this structure a $F_{a,\lambda}(K,T)$ −manifold,where dim $M^n = n$.

In the manifold M^n endowed with $aF^K + \lambda^r F^T = 0, (F \neq 0, K \geq\gg 3, T \geq\gg 1$ and $(K \gg\geq T), a$ and λ are non zero complex numbers, r some finite integer) structure, the (1,1) tensor field ψ given by $\psi = l - m = -I - 2\frac{aF^{K-T}}{\lambda^r}$ gives an almost product structure.

At the present time, we demonstrate some special examples for this general $F_{a,\lambda}(K,T)$ −structure satisfying $aF^K + \lambda^r F^T = 0, (F \neq 0, K \gg\geq 3, T \geq\gg 1, a$ and λ are non zero complex numbers, r some finite integer) on manifold M

Example 1. For $F \neq 0, K = 3, T = 1$ and $a = \lambda^r = 1$, we obtained the $F(3,1)$ −structure satisfying $F^3 + F = 0$ (see [12] for detail). In addition, using the (1.2) the operators l and m be defined as

$$l = -\frac{aF^{K-T}}{\lambda^r} = -F^2 \text{and } m = I + \frac{aF^{K-T}}{\lambda^r} = I + F^2,$$
$$l^2 = l, m^2 = m, lm = ml = 0, l + m = I,$$
$$Fl = lF = F, Fm = mF = 0.$$

The (1,1) tensor field ψ given by $\psi = l - m = -I - 2\frac{aF^{K-T}}{\lambda^r} = -I - 2F^2$ gives an almost product structure, where I denotes the identity operator on M^n.

Example 2. For $F \neq 0, K = K, T = 1$ and $a = \lambda^r = 1$, we obtained the $F(K,1)$ −structure satisfying $F^K + F = 0$ (see [6] for detail). In addition, using the (1.2) the operators l and m be defined as

$$l = -\frac{aF^{K-T}}{\lambda^r} = -F^{K-1} \text{and} m = I + \frac{aF^{K-T}}{\lambda^r} = I + F^{K-1},$$
$$l^2 = l, m^2 = m, lm = ml = 0, l + m = I,$$
$$Fl = lF = F, Fm = mF = 0.$$

The (1,1) tensor field ψ given by $\psi = l - m = -I - 2\frac{aF^{K-T}}{\lambda^r} = -I - 2F^{K-1}$ gives an almost product structure, where I denotes the identity operator on M^n.

Example 3. For $F \neq 0, K = 2v + 4 (v \geq\geqslant 0), T = 2$ and $a = \lambda^r = 1$, we obtained the $F(2v + 4,2)$ −structure satisfying $F^{2v+4} + F^2 = 0$ (see [6] for detail). In addition, using the (1.2) the operators l and m be defined as

$$l = -\frac{aF^{K-T}}{\lambda^r} = -F^{2v+2} \text{and} m = I + \frac{aF^{K-T}}{\lambda^r} = I + F^{2v+2},$$
$$l^2 = l, m^2 = m, lm = ml = 0, l + m = I,$$
$$Fl = lF = F, Fm = mF = 0.$$

The (1,1) tensor field ψ given by $\psi = l - m = -I - 2\frac{aF^{K-T}}{\lambda^r} = -I - 2F^{2v+2}$ gives an almost product structure, where I denotes the identity operator on M^n.

Example 4. For $F \neq 0, K = 4, T = 2$ and $a = 1$ and λ^r (λ is non zero complex numbers, r some finite integer), we obtained the $HSU - (4,2)$ −structure satisfying $F^4 + \lambda^r F^2 = 0$ (see [10] for detail). In addition, using the (1.2) the operators l and m be defined as

$$l = -\frac{aF^{K-T}}{\lambda^r} = -\frac{F^2}{\lambda^r} \text{and} m = I + \frac{aF^{K-T}}{\lambda^r} = I + \frac{F^2}{\lambda^r},$$
$$l^2 = l, m^2 = m, lm = ml = 0, l + m = I,$$
$$Fl = lF = F, Fm = mF = 0.$$

The (1,1) tensor field ψ given by $\psi = l - m = -I - 2\frac{aF^{K-T}}{\lambda^r} = -I - 2\frac{F^2}{\lambda^r}$ gives an almost product structure, where I denotes the identity operator on M^n.

Using the similary way, we can obtain the other structures [3-5, 8-10].

1.1. HORIZONTAL LIFT OF THE GENERAL $F_{a,\lambda}(K,T)$ −STRUCTURE SATISFYING $aF^K + \lambda^r F^T = 0$ ON COTANGENT BUNDLE

Let F, G be two tensor field of type (1,1) on the manifold M^n. If F^H denotes the horizontal lift of F, we have [1,5]

$$F^H G^H + G^H F^H = (FG + GF)^H. \tag{1.6}$$

Taking F and G identical, we get

$$(F^H)^2 = (F^2)^H,$$

Continuing the above process of replacing G in equation (1.6) by some higher powers of F, we obtain

$$(F^K)^H = (F^H)^K, \tag{1.7}$$
$$(F^T)^H = (F^H)^T,$$

where $F \neq 0, K \geq\gg 3, T \geq\gg 1$ and $(K \geq\gg T), a$ and λ are non zero complex numbers, r some finite integer.

Also, if G and H are tensors of the same type then we get

$$(G + H)^H = G^H + H^H. \tag{1.8}$$

Taking horizontal lift on both sides of equation $aF^K + \lambda^r F^T = 0$, we get

$$a(F^K)^H + \lambda^r (F^T)^H = 0. \tag{1.9}$$

In view of (1.7) and (1.8), we can write [5, 11]

$$a(F^H)^K + \lambda^r (F^H)^T = 0. \tag{1.10}$$

Proposition 1. *Let M^n be a Riemannian manifold with metric g, ∇ be the Levi-Civita connection and R be the Riemannian curvature tensor. Then the Lie bracket of the cotangent bundle $T^*(M^n)$ of M^n satisfies the following*

$i)[\omega^V,\theta^V]=0,$ (1.11)

$ii)[X^H,\omega^V]=(\nabla_X\omega)^V,$

$iii)[X^H,Y^H]=[X,Y]^H+\gamma R(X,Y)=[X,Y]^H+\left(pR(X,Y)\right)^V$

for all $X,Y\in\Im_0^1(M^n)$ and $\omega,\theta\in\Im_1^0(M^n)$. (See [1] p. 238, p. 277 for more details).

2. RESULTS

Definition 1. *Let F be a tensor field of type $(1,1)$ admitting $aF^K+\lambda^rF^T=0$ structure in M^n. The Nijenhuis tensor of a $(1,1)$ tensor field F of M^n is given by*

$$N_F=[FX,FY]-F[X,FY]-F[FX,Y]+F^2[X,Y] \tag{2.1}$$

for any $X,Y\in\Im_0^1(M^n)$ [12, 13, 14]. The condition of $N_F(X,Y)=N(X,Y)=0$ is essential to integrability condition in these structures.

The Nijenhuis tensor N_F is defined local coordinates by

$$N_{ij}^k\partial_k=(F_i^s\partial_s^kF_j^k-F_j^l\partial_lF_i^k-\partial_iF_j^lF_l^k+\partial_jF_i^sF_s^k)\partial_k \tag{2.2}$$

where $X=\partial_i, Y=\partial_j, F\in\Im_1^1(M^n)$.

2.1. THE NIJENHUIS TENSORS OF $(F^K)^H$ ON COTANGENT BUNDLE $T^*(M^n)$

Theorem 1. *The Nijenhuis tensors of $(F^K)^H$ and F^T denote by $\widetilde{N}$ and N, respectively. Thus, taking account of the definition of the Nijenhuis tensor, the formulas (1.11) stated in Proposition 1 and the structure $a(F^K)^H+\lambda^r(F^T)^H=0$, we find the following results of computation.*

$$\begin{aligned} i)\widetilde{N}_{(F^K)^H(F^K)^H}(X^H,Y^H)=\frac{\lambda^{2r}}{a^2}\{&\{[F^TX,F^TY]-F^T[F^TX,Y]-F^T[X,F^TY]\\ &+(F^T)^2[X,Y]\}^H+\gamma\{R(F^TX,F^TY)\\ &-R(F^TX,Y)F^T-R(X,F^TY)F^T+R(X,Y)(F^T)^2\}\}.\end{aligned}$$

$$ii)\widetilde{N}_{(F^K)^H(F^K)^H}(X^H,\omega^V)=\frac{\lambda^{2r}}{a^2}\{\omega\circ(\nabla_{F^TX}F^T)-(\omega\circ(\nabla_XF^T)F^T\}^V,$$

$$iii)\widetilde{N}_{(F^K)^H(F^K)^H}(\omega^V,\theta^V)=0.$$

Proof:

$i)$ The Nijenhuis tensor $N_{(F^K)^H(F^K)^H}(X^H,Y^H)$ of the horizontal lift $(F^K)^H$ vanishes if F^T is an almost complex structure i.e., $(F^T)^2=-I$ and $R(F^TX,F^TY)=R(X,Y)$.

$$\begin{aligned} N_{(F^K)^H(F^K)^H}(X^H,Y^H)&=[(F^K)^HX^H,(F^K)^HY^H]-(F^K)^H[(F^K)^HX^H,Y^H]\\ &\quad-(F^K)^H[X^H,(F^K)^HY^H]+(F^K)^H(F^K)^H[X^H,Y^H]\\ &=\frac{\lambda^{2r}}{a^2}\{[(F^T)^HX^H,(F^T)^HY^H]-(F^T)^H[(F^T)^HX^H,Y^H]\end{aligned}$$

$$
\begin{aligned}
&-(F^T)^H[X^H,(F^T)^H Y^H]+((F^T)^H)^2[X^H,Y^H]\}\\
&=\frac{\lambda^{2r}}{a^2}\{[F^T X,F^T Y]^H+\gamma R(F^T X,F^T Y)\\
&-(F^T)^H[(F^T X),Y]^H-(F^T)^H\gamma R(F^T X,Y)\\
&-(F^T)^H[X,F^T Y]^H-(F^T)^H\gamma R(X,F^T Y)\\
&+((F^T)^H)^2[X,Y]^H+((F^T)^H)^2\gamma R(X,Y)\}\\
&=\frac{\lambda^{2r}}{a^2}\{\{[F^T X,F^T Y]-F^T[F^T X,Y]-F^T[X,F^T Y]\\
&+(F^T)^2[X,Y]\}^H+\gamma\{R(F^T X,F^T Y)-R(F^T X,Y)F^T\\
&-R(X,F^T Y)F^T+R(X,Y)(F^T)^2\}\}.
\end{aligned}
$$

$(F^K)^H$ is integrable if the curvature tensor R of ∇ satisfies $R(F^T X,F^T Y)=R(X,Y)$ and F^T is an almost complex structure, then we get $R(F^T X,Y)=-R(X,F^T Y)$. Hence using $(F^T)^2=-I$, we find $R(F^T X,F^T Y)-R(F^T X,Y)F^T-R(X,F^T Y)F^T+R(X,Y)(F^T)^2=0$. Therefore, it follows $N_{(F^K)^H(F^K)^H}(X^H,Y^H)=0$.

ii) The Nijenhuis tensor $N_{(F^K)^H(F^K)^H}(X^H,\omega^V)$ of the horizontal lift $(F^K)^H$ vanishes if $\nabla F^T=0$.

$$
\begin{aligned}
N_{(F^K)^H(F^K)^H}(X^H,\omega^V)&=[(F^K)^H X^H,(F^K)^H\omega^V]-(F^K)^H[(F^K)^H X^H,\omega^V]\\
&-(F^K)^H[X^H,(F^K)^H\omega^V]+(F^K)^H(F^K)^H[X^H,\omega^V]\\
&=\frac{\lambda^{2r}}{a^2}\{[(F^T)^H X^H,(F^T)^H\omega^V]-(F^T)^H[(F^T)^H X^H,\omega^V]\\
&-(F^T)^H[X^H,(F^T)^H\omega^V]+((F^T)^H)^2[X^H,\omega^V]\}\\
&=\frac{\lambda^{2r}}{a^2}\{[(F^T X)^H,(\omega\circ F^T)^V]-(F^T)^H[(F^T X)^H,\omega^V]\\
&-(F^T)^H[X^H,(\omega\circ F^T)^V]+((F^T)^H)^2(\nabla_X\omega)^V\}\\
&=\frac{\lambda^{2r}}{a^2}\{\omega\circ(\nabla_{F^T X}F^T)-(\omega\circ(\nabla_X F^T)F^T\}^V,
\end{aligned}
$$

where $F^T\in\Im_1^1(M^n)$, $X\in\Im_0^1(M^n)$, $\omega\in\Im_1^0(M^n)$. The theorem is proved.

iii) The Nijenhuis tensor $\widetilde{N}_{(F^K)^H(F^K)^H}(\omega^V,\theta^V)$ of the horizontal lift $(F^K)^H$ vanishes.

Because of $[\omega^V,\theta^V]=0$ for $\omega\circ F^T,\theta\circ F^T,\omega,\theta\in\Im_1^0(M^n)$ on $T^*(M^n)$, the Nijenhuis tensor $\widetilde{N}_{(F^K)^H(F^K)^H}(\omega^V,\theta^V)$ of the horizontal lift $(F^K)^H$ vanishes.

2.2. TACHIBANA OPERATORS APPLIED TO VECTOR AND COVECTOR FIELDS ACCORDING TO LIFTS OF THE STRUCTURE $aF^K+\lambda^r F^T=0$ ON $T^*(M^n)$

Definition 2. *Let $\varphi\in\Im_1^1(M^n)$, and $\Im(M^n)=\sum_{r,s=0}^{\infty}\Im_s^r(M^n)$ be a tensor alebra over R. A map $\phi_\varphi\big|_{r+s\rangle 0}:\overset{*}{\Im}(M^n)\to\Im(M^n)$ is called as Tachibana operatör or $\phi_\varphi$operatör on M^n if*

$a)\phi_\varphi$ is linear with respect to constant coefficient,

$b)\phi_\varphi:\overset{*}{\Im}(M^n)\to\Im_{s+1}^r(M^n)$ for all r and s,

$c)\phi_\varphi(K\overset{C}{\otimes}L)=(\phi_\varphi K)\otimes L+K\otimes\phi_\varphi L$ for all $K,L\in\overset{*}{\Im}(M^n)$,

$d)\phi_{\varphi X}Y=-(L_Y\varphi)X$ for all $X,Y\in\Im_0^1(M^n)$,where $L_Y$is the Liederivation with respect to Y (see [3, 5, 9]),

$e)$

$$
\begin{aligned}
(\phi_{\varphi X}\eta)Y&=(d(\iota_Y\eta))(\varphi X)-(d(\iota_Y(\eta o\varphi)))X+\eta((L_Y\varphi)X) \qquad (2.3)\\
&=\phi X(\iota_Y\eta)-X(\iota_{\varphi Y}\eta)+\eta((L_Y\varphi)X)
\end{aligned}
$$

for all $\eta \in \Im_1^0(M^n)$ and $X, Y \in \Im_0^1(M^n)$, where $\iota_Y\eta = \eta(Y) = \eta \overset{C}{\otimes} Y$, $\overset{*}{\Im}{}_s^r(M^n)$ the module of all pure tensor fields of type (r,s) on M^n with respect to the affinor field, $\overset{C}{\otimes}$ is a tensor product with a contraction C [12,13,15] (see [14] for applied to pure tensor field).

Remark 1. *If $r = s = 0$, then from $c), d)$ and $e)$ of Definition 2 we have $\phi_{\varphi X}(\iota_Y\eta) = \phi X(\iota_Y\eta) - X(\iota_{\varphi Y}\eta)$ for $\iota_Y\eta \in \Im_0^0(M^n)$, which is not well-defined ϕ_φ −operator. Different choices of Y and η leading to same function $f = \iota_Y\eta$ do get the same values. Consider $M^n = R^2$ with standard coordinates x, y. Let $\varphi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Consider the function $f = 1$. This may be written in many different ways as $\iota_Y\eta$. Indeed taking $\eta = dx$, we may choose $Y = \frac{\partial}{\partial_x}$ or $Y = \frac{\partial}{\partial_x} + x\frac{\partial}{\partial_y}$. Nov the right-hand side of $\phi_{\varphi X}(\iota_Y\eta) = \phi X(\iota_Y\eta) - X(\iota_{\varphi Y}\eta)$ is $(\phi X)1 - 0 = 0$ in the first case, and $(\phi X)1 - Xx = -Xx$ in the second case. For $X = \frac{\partial}{\partial_x}$, the latter expression is $-1 \neq 0$. Therefore, we put $r + s > 0$ [13].*

Remark 2 *From $d)$ of Definition 2 we have*

$$\phi_{\varphi X}Y = [\varphi X, Y] - \varphi X, Y]. \tag{2.4}$$

By virtue of

$$fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X \tag{2.5}$$

for any $f, g \in \Im_0^0(M^n)$, we see that $\phi_{\varphi X}Y$ is linear in X, but not Y [13].

Theorem 2. *Let $(F^K)^H$ be a tensor field of type $(1,1)$ on $T^*(M^n)$. If the Tachibana operator ϕ_φ applied to vector fields according to horizontal lifts of $aF^K + \lambda^r F^T = 0$ structure defined by (1.9) on $T^*(M^n)$, then we get the following results.*

$$i)\phi_{(F^K)^H X^H}Y^H = \frac{\lambda^r}{a}\{((L_Y F^T)X)^H + (pR(Y, F^T X))^V - ((pR(Y,X)) \circ F^T)^V\},$$

$$ii)\phi_{(F^K)^H X^H}\omega^V = \frac{\lambda^r}{a}(((\nabla_X\omega) \circ F^T)^V - \left(\nabla_{(F^T X)}\omega)^V\right),$$

$$iii)\phi_{(F^K)^H \omega^V}X^H = \frac{\lambda^r}{a}(\omega \circ (\nabla_X F^T))^V,$$

$$iv)\ \phi_{(F^K)^H \omega^V}\theta^V = 0,$$

where horizontal lifts $X^H, Y^H \in \Im_0^1(T^*(M^n))$ of $X, Y \in \Im_0^1(M^n)$ and the vertical lift $\omega^V, \theta^V \in \Im_0^1(T^*(M^n))$ of $\omega, \theta \in \Im_1^0(M^n)$ are given, respectively.

Proof:

$$\begin{aligned} i)\phi_{(F^K)^H X^H}Y^H &= -(L_{Y^H}(F^K)^H)X^H \\ &= -L_{Y^H}(F^K)^H X^H + (F^K)^H L_{Y^H}X^H \\ &= \frac{\lambda^r}{a}L_{Y^H}(F^T)^H X^H - \frac{\lambda^r}{a}(F^T)^H([Y,X]^H + (pR(Y,X))^V) \\ &= \frac{\lambda^r}{a}\{(L_Y F^T X)^H + (pR(Y, F^T X))^V - (F^T L_Y X)^H - ((pR(Y,X)) \circ F^T)^V\} \\ &= \frac{\lambda^r}{a}\{((L_Y F^T)X)^H + (pR(Y, F^T X))^V - ((pR(Y,X)) \circ F^T)^V\} \end{aligned}$$

$$\begin{aligned} ii)\phi_{(F^K)^H X^H}\omega^V &= -(L_{\omega^V}(F^K)^H)X^H \\ &= -L_{\omega^V}(F^K)^H X^H + (F^K)^H L_{\omega^V}X^H \\ &= \frac{\lambda^r}{a}L_{\omega^V}(F^T X)^H + \frac{\lambda^r}{a}(F^T)^H(\nabla_X\omega)^V \\ &= \frac{\lambda^r}{a}(\nabla_{(F^T X)}\omega)^V + \frac{\lambda^r}{a}((\nabla_X\omega)\circ F^T)^V \\ &= \frac{\lambda^r}{a}(((\nabla_X\omega)\circ F^T)^V - (\nabla_{(F^T X)}\omega)^V) \end{aligned}$$

$$\begin{aligned} iii)\phi_{(F^K)^H\omega^V}X^H &= -(L_{X^H}(F^K)^H)\omega^V \\ &= -L_{X^H}(F^K)^H\omega^V + (F^K)^H L_{X^H}\omega^V \\ &= \frac{\lambda^r}{a}L_{X^H}(\omega\circ F^T)^V - \frac{\lambda^r}{a}(F^T)^H(\nabla_X\omega)^V \\ &= \frac{\lambda^r}{a}(\nabla_X(\omega\circ F^T))^V - \frac{\lambda^r}{a}((\nabla_X\omega)\circ F^T)^V \\ &= \frac{\lambda^r}{a}(\omega\circ(\nabla_X F^T))^V \end{aligned}$$

$$\begin{aligned} iv)\phi_{(F^K)^H\omega^V}\theta^V &= -(L_{\theta^V}(F^K)^H)\omega^V \\ &= -L_{\theta^V}(F^K)^H\omega^V + (F^K)^H(L_{\theta^V}\omega^V) \\ &= \frac{\lambda^r}{a}L_{X^H}(\omega\circ F^T)^V - \frac{\lambda^r}{a}(F^T)^H(\nabla_X\omega)^V \\ &= 0 \end{aligned}$$

2.3. THE PURITY CONDITIONS OF SASAKIAN METRIC WITH RESPECT TO $(F^K)^H$

Definition 3. *A Sasakian metric* $^S g$ *is defined on* $T^*(M^n)$ *by the three equations*

$$^S g\,(\omega^V,\theta^V) = (g^{-1}(\omega,\theta))^V = g^{-1}(\omega,\theta)o\pi, \tag{2.6}$$

$$^S g\,(\omega^V, Y^H) = 0, \tag{2.7}$$

$$^S g\,(X^H, Y^H) = (g(X,Y))^V = g(X,Y)\circ\pi. \tag{2.8}$$

For each $x \in M^n$ the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space $\pi^{-1}(x) = T_x^*(M^n)$ by

$$g^{-1}(\omega,\theta) = g^{ij}\omega_i\theta_j,$$

where $X, Y \in \Im_0^1(M^n)$ and $\omega,\theta \in \Im_1^0(M^n)$. Since any tensor field of type (0,2) on $T^*(M^n)$ is completely determined by its action on vector fields of type X^H and ω^V (see [1], p.280), it follows that $^S g$ is completely determined by equations (2.6), (2.7) and (2.8).

Theorem 3. *Let* $(T^*(M^n), {}^S g)$ *be the cotangent bundle equipped with Sasakian metric* $^S g$ *and a tensor field* $(F^K)^H$ *of type* (1,1) *defined by (1.9). Sasakian metric* $^S g$ *is pure with respect to* $(F^K)^H$ *if* $F^T = I$ *(I = identity tensor field of type* (1,1)).

Proof: We put

$$S(\tilde X,\tilde Y) = {}^S g((F^K)^H\tilde X,\tilde Y) - {}^S g(\tilde X,(F^K)^H\tilde Y).$$

If $S(\tilde{X},\tilde{Y}) = 0$, for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form ω^V, θ^V or X^H, Y^H, then $S = 0$. By virtue of $a(F^H)^K + \lambda^r (F^H)^T = 0$ and (2.6), (2.7), (2.8), we get

$$i) S(\omega^V,\theta^V) = {}^S g((F^K)^H \omega^V,\theta^V) - {}^S g(\omega^V,(F^K)^H \theta^V)$$
$$= {}^S g(-\frac{\lambda^r}{a}(F^T)^H \omega^V,\theta^V) - {}^S g(\omega^V, -\frac{\lambda^r}{a}(F^T)^H \theta^V)$$
$$= -\frac{\lambda^r}{a}({}^S g((\omega \circ F^T)^V,\theta^V) - {}^S g(\omega^V,(\theta \circ F^T)^V)).$$

$$ii) S(X^H,\theta^V) = {}^S g((F^K)^H X^H,\theta^V) - {}^S g(X^H,(F^K)^H \theta^V)$$
$$= {}^S g(-\frac{\lambda^r}{a}(F^T)^H X^H,\theta^V) - {}^S g(X^H, -\frac{\lambda^r}{a}(F^T)^H \theta^V)$$
$$= -\frac{\lambda^r}{a}({}^S g((F^T X)^H,\theta^V) - {}^S g(X^H,(\omega \circ F^T)^V).$$
$$= 0$$

$$iii) S(X^H,Y^H) = {}^S g((F^K)^H X^H,Y^H) - {}^S g(X^H,(F^K)^H Y^H)$$
$$= {}^S g(-\frac{\lambda^r}{a}(F^T)^H X^H,Y^H) - {}^S g(X^H, -\frac{\lambda^r}{a}(F^T)^H Y^H)$$
$$= -\frac{\lambda^r}{a}({}^S g((F^T X)^H,Y^H) - {}^S g(X^H,(F^T Y)^H)).$$

thus, $F^T = I$, then ${}^S g$ is pure with respect to $(F^K)^H$.

2.4. COMPLETE LIFT OF $aF^K + \lambda^r F^T = 0$ −StRUCTURE ON TANGENT BUNDLE $T(M^n)$

Let M^n be an n −dimensional differentiable manifold of class C^∞ and $T_P(M^n)$ the tangent space at a point P of M^n and

$$T(M^n) = \underset{P \in M^n}{U} T_P(M^n) \tag{2.9}$$

is the tangent bundle over the manifold M^n.

Let us denote by $T_s^r(M^n)$, the set of all tensor fields of class C^∞ and of type (r,s) in M^n and $T(M^n)$ be the tangent bundle over M^n. The complete lift of F^C of an element of $T_1^1(M^n)$ with local components F_i^h has components of the form [16]

$$F^C = \begin{bmatrix} F_i^h & 0 \\ \delta_i^h & F_i^h \end{bmatrix}. \tag{2.10}$$

Now we obtain the following results on the complete lift of F satisfying $F_{a,\lambda}(K,T)$ −structure satisfying $aF^K + \lambda^r F^T = 0, (F \neq 0, K \geq\gg 3, T \geq\gg 1$ and $(K \geq\gg T), a$ and λ are non zero complex numbers, r some finite integer).

Let $F, G \in T_1^1(M^n)$. Then we have [16]

$$(FG)^C = F^C G^C. \tag{2.11}$$

Replacing G by F in (2.11) we obtain

$$(FF)^C = F^C F^C \text{or} (F^2)^C = (F^C)^2. \tag{2.12}$$

Now putting $G = F^4$ in (2.11) since G is $(1,1)$ tensor field therefore F^4 is also $(1,1)$ so we obtain $(FF^4)^C = F^C(F^4)^C$ which in view of (2.12) becomes

$$(F^5)^C = (F^C)^5.$$

Continuing the above process of replacing G in equation (2.11) by some higher powers of F, we obtain

$$(F^K)^C = (F^C)^K,\ (F^T)^C = (F^C)^T,$$

where $F \neq 0, K \gneqq 3, T \gneqq 1$ and $(K \gneqq T)$. Also if G and H are tensors of the same type then

$$(G+H)^C = G^C + H^C \tag{2.13}$$

Taking complete lift on both sides of equation $aF^K + \lambda^r F^T = 0$, we get

$$(aF^K + \lambda^r F^T)^C = 0$$

Using (2.13) and $I^C = I$, we get

$$a(F^K)^C + \lambda^r (F^T)^C = 0 \tag{2.14}$$

$$a(F^C)^K + \lambda^r (F^C)^T = 0.$$

Let F satisfying $(1,1)$ be an $F-$structure of rank r in M^n. Then the complete lifts $l^C = -\frac{a}{\lambda^r}(F^{K-T})^C$ of l and $m^C = I + \frac{a}{\lambda^r}(F^{K-T})^C$ of m are complementary projection tensors in $T(M^n)$. Thus there exist in $T(M^n)$ two complementary distributions L^C and M^C determined by l^C and m^C, respectively.

Proposition 2. *The $(1,1)$ tensor field $\tilde{\psi}$ given by $\tilde{\psi} = l^C - m^C = -2\frac{a}{\lambda^r}(F^{K-T})^C - I$ gives an almost product structure on $T(M^n)$.*

Proof: For $l^C = -\frac{a}{\lambda^r}(F^{K-T})^C$, $m^C = I + \frac{a}{\lambda^r}(F^{K-T})^C$ and $\tilde{\psi} = l^C - m^C = -2\frac{a}{\lambda^r}(F^{K-T})^C - I$, we have

$$\begin{aligned}
\tilde{\psi}^2 &= 4\frac{a^2}{\lambda^{2r}}(F^{2K-2T})^C + 4\frac{a}{\lambda^r}(F^{K-T})^C + I \\
&= 4\frac{a}{\lambda^r}(F^{2K-T})^C(\frac{a}{\lambda^r}F^{-T})^C + 4\frac{a}{\lambda^r}(F^{K-T})^C + I \\
&= 4\frac{a}{\lambda^r}(F^{2K-T})^C(-F^{-K})^C + 4\frac{a}{\lambda^r}(F^{K-T})^C + I \\
&= -4\frac{a}{\lambda^r}(F^{K-T})^C + 4\frac{a}{\lambda^r}(F^{K-T})^C + I \\
&= I,
\end{aligned}$$

where $\tilde{\psi} \in \mathfrak{I}_1^1(T(M^n)), I$ = identity tensor field of type $(1,1)$.

2.5. INTEGRABILITY CONDITIONS OF THE STRUCTURE $a(F^K)^C + \lambda^r(F^T)^C = 0$ ON TANGENT BUNDLE $T(M^n)$

Definition 4. *Let X and Y be any vector fields on a Riemannian manifold (M^n, g), we have [20]*

$$[X^H, Y^H] = [X,Y]^H - (R(X,Y)u)^V,$$

$$[X^H, Y^V] = (\nabla_X Y)^V,$$
$$[X^V, Y^V] = 0,$$

where R is the Riemannian curvature tensor of g defined by

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

In particular, we have the vertical spray u^V and the horizontal spray u^H on $T(M^n)$ defined by

$$u^V = u^i(\partial_i)^V = u^i\partial_{\bar{i}}, u^H = u^i(\partial_i)^H = u^i\delta_i,$$

where $\delta_i = \partial_i - u^j\Gamma_{ji}^s\partial_{\bar{s}}. u^V$ is also called the canonical or Liouville vector field on $T(M^n)$.

Let $F \in \mathfrak{I}_1^1(M^n)$ and suppose that F satisfies (1.1). Then the Nijenhuis tensor N_F of F is a tensor field of type (1,2) given by [1]

$$N(X,Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X,Y] \qquad (2.15)$$

Let N^C be the Nijenhuis tensor of F^C in $T(M^n)$, where F^C is the complete lift of F in M^n. Then we have

$$N^C(X^C, Y^C) = [F^C X^C, F^C Y^C] - F^C[F^C X^C, Y^C] - F^C[X^C, F^C Y^C] + (F^2)^C[X^C, Y^C].$$

for any $X, Y \in \mathfrak{I}_0^1(M^n)$ and $F \in \mathfrak{I}_1^1(M^n)$ we have [1]

$$[X^C, Y^C] = [X,Y]^C, F^C X^C = (FX)^C \qquad (2.16)$$
$$(X+Y)^C = X^C + Y^C$$

From (1.5) and (2.16) we have

$$F^C l^C = (Fl)^C = F^C \qquad (2.17)$$
$$F^C m^C = (Fm)^C = 0$$

Theorem 4. *The following identities hold:*
$a)m^C N^C(l^C X^C, l^C Y^C) = m^C[F^C X^C, F^C Y^C]$
b)$m^C N^C(X^C, Y^C) = m^C[F^C X^C, F^C Y^C]$
$c)N^C(m^C X^C, m^C Y^C) = (F^C)^2[m^C X^C, m^C Y^C]$
$d)m^C N^C((aF^K + \lambda^r F^T)^C X^C, (aF^K + \lambda^r F^T)^C Y^C)) = m^C N^C[l^C X^C, l^C Y^C]$

Proof: Using the equalites of (1.2),(2.15) and (2.17), easily we get the results.

Theorem 5. *For any* $X, Y \in \mathfrak{I}_0^1(M^n)$ *the following conditions are equivalent.*
$i)m^C N^C(X^C, Y^C) = 0$
$ii)m^C N^C(l^C X^C, l^C Y^C) = 0$
$iii)m^C N^C((aF^K + \lambda^r F^T)^C X^C, (aF^K + \lambda^r F^T)^C Y^C)) = 0$
Proof: In consequence of the equation $d)$ in the above theorem, we have, $N^C(l^C X^C, l^C Y^C) = 0$ if and only if $N^C((aF^K + \lambda^r F^T)^C X^C, (aF^K + \lambda^r F^T)^C Y^C)) = 0$ for all $X, Y \in \mathfrak{I}_0^1(M^n)$. The right hand sides of the equations $a)$ and $b)$ are equal in the Theorem 4 and in view of the last equation shows that conditions $(i), (ii)$ and (iii) are equivalent.

Theorem 6. *The complete lift M^C in $T(M^n)$ of a distribution M in M^n is integrable if M is integrable in M^n.*

Proof: The distribution M is integrable if and only if [1]

$$l(mX, mY) = 0, \tag{2.18}$$

where $X, Y \in \mathfrak{I}_0^1(M^n)$ and $l = I - m$.

Taking complete lift of both sides and using (2.16), we get

$$l^C(m^C X^C, m^C Y^C) = 0$$

for all $X, Y \in \mathfrak{I}_0^1(M^n)$, where $l^C = (I - m)^C = I - m^C$ is the projection tensor complementary to m^C.

Theorem 7. *The complete lift M^C in $T(M^n)$ of a distribution M in M^n is integrable if $l^C N^C(m^C X^C, m^C Y^C) = 0$, or equivalently $N^C(m^C X^C, m^C Y^C) = 0$ for all $X, Y \in \mathfrak{I}_0^1(M^n)$.*

Proof: The distribution M is integrable in M^n if and only if $N(mX, mY) = 0$ for any $X, Y \in \mathfrak{I}_0^1(M^n)$ [1]. By virtue of condition $m^C N^C(l^C X^C, l^C Y^C) = m^C[F^C X^C, F^C Y^C]$, we get

$$N^C(m^C X^C, m^C Y^C) = (F^C)^2[m^C X^C, m^C Y^C]$$

Multiplying throughout by l^C , we get

$$l^C N^C(m^C X^C, m^C Y^C) = (F^C)^2 l^C[m^C X^C, m^C Y^C]$$

In view of (1.2) the above relation becomes

$$l^C N^C(m^C X^C, m^C Y^C) = 0. \tag{2.19}$$

Also we have

$$m^C N^C(m^C X^C, m^C Y^C) = 0 \tag{2.20}$$

Adding (2.19) and (2.20), we get $(l^C + m^C)N^C(m^C X^C, m^C Y^C) = 0$. Since $l^C + m^C = I^C = I$, we have $N^C(m^C X^C, m^C Y^C) = 0$.

Theorem 8. *Let the distribution L be integrable in M^n, that is $mN(X, Y) = 0$ for all $X, Y \in \mathfrak{I}_0^1(M^n)$.Then the distribution L^C is integrable in $T(M^n)$ if and only if any one of the conditions of Theorem 5 is satisfied , for all $X, Y \in \mathfrak{I}_0^1(M^n)$.*

Proof: The distribution L is integrable in M^n if and only if $mN(lX, lY) = 0$. Thus distribution L^C is integrable in $T(M^n)$ if and only if $m^C N^C(l^C X^C, l^C Y^C) = 0$. Hence the theorem follows by making use of the eqution $d)$ of the Theorem 4.

Theorem 9. *The complete lift F^C of an $(aF^K + \lambda^r F^T)$ −structure F in M^n is partially integrable in $T(M^n)$ if and only if F is partially integrable in M^n.*

Proof: The $(aF^K + \lambda^r F^T)$ −structure F in M^n is partially integrable if and only if

$$N(lX, lY) = 0 \tag{2.21}$$

for any $X,Y \in \Im_0^1(M^n)$. In view of the equations (1.1) and (2.1), we obtain $N^C(l^C X^C, l^C Y^C) = (N(lX,lY))^C$ which implies $N^C(m^C X^C, m^C Y^C) = 0$ if and only if $N(lX,lY) = 0$. Also from the Theorem 5, $N^C(l^C X^C, l^C Y^C) = 0$ is equivalent to $N^C((F^{2K+S} + F^S)^C X^C, (F^{2K+S} + F^S)^C Y^C)) = 0$.

Theorem 10. *The complete lift F^C of an $(aF^K + \lambda^r F^T)$ −structure F is partially integrable in $T(M^n)$ if and only if F is partially integrable in M^n.*

Proof: A necessary and sufficient condition for an $(aF^K + \lambda^r F^T)$ −structure in M^n to be integrable is that

$$N(X,Y) = 0 \tag{2.22}$$

for any $X,Y \in \Im_0^1(M^n)$. In view of Theorem 4, we get $N^C(X^C, Y^C) = (N(X,Y))^C$. Therefore with the help of (2.22) we obtain the result.

2.6. THE PURITY CONDITIONS OF SASAKIAN METRIC WITH RESPECT TO THE STRUCTURE $a(F^K)^C + \lambda^r (F^T)^C = 0$ ON $T(M^n)$

Definition 5. *The Sasaki metric ${}^S g$ is a (positive definite) Riemannian metric on the tangent bundle $T(M^n)$ which is derived from the given Riemannian metric on M^n as follows [13]:*

$$\begin{aligned} {}^S g\,(X^H, Y^H) &= g(X,Y), \\ {}^S g\,(X^H, Y^V) &= {}^S g(X^V, Y^H) = 0, \\ {}^S g\,(X^V, Y^V) &= g(X,Y) \end{aligned} \tag{2.23}$$

for all $X,Y \in \Im_0^1(M^n)$.

Theorem 11. *The Sasaki metric ${}^S g$ is pure with respect to $(F^K)^C$ if $\nabla F^T = 0$ and $F^T = \frac{\lambda^r}{a} I$, where I=ıdentity tensor field of type* $(1,1)$.

Proof: $S(\tilde{X},\tilde{Y}) = {}^S g((F^K)^C \tilde{X}, \tilde{Y}) - {}^S g(\tilde{X}, (F^K)^C \tilde{Y})$ if $S(\tilde{X},\tilde{Y}) = 0$ for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form X^V, Y^V or X^H, Y^H then $S = 0$.

$$\begin{aligned} i)\, S(X^V,Y^V) &= {}^S g((F^K)^C X^V, Y^V) - {}^S g(X^V, (F^K)^C Y^V) \\ &= -\frac{\lambda^r}{a}\{{}^S g((F^T X)^V, Y^V) - {}^S g(X^V, (F^T Y)^V)\} \\ &= -\frac{\lambda^r}{a}\{\big(g(F^T X, Y)\big)^V - \big(g(X, F^T Y)\big)^V\} \end{aligned}$$

$$\begin{aligned} ii)\, S(X^V,Y^H) &= {}^S g((F^K)^C X^V, Y^H) - {}^S g(X^V, (F^K)^C Y^H) \\ &= \frac{\lambda^{r S}}{a} g(X^V, (F^T Y)^H + (\nabla_\gamma F^T) Y^H) \\ &= \frac{\lambda^{r S}}{a} g\big(X^V, (\nabla_\gamma F^T) Y^H\big) \\ &= \frac{\lambda^r}{a}\big(g\big(X, \big((\nabla F^T) u\big) Y\big)^V\big) \end{aligned}$$

$$\begin{aligned} iii)\, S(X^H,Y^H) &= {}^S g((F^K)^C X^H, Y^H) - {}^S g(X^H, (F^K)^C Y^H) \\ &= -\frac{\lambda^{r S}}{a} g((F^T)^C X^H, Y^H) + \frac{\lambda^{r S}}{a} g(X^H, (F^T)^C Y^H) \end{aligned}$$

$$= -\frac{\lambda^{rS}}{a} g\big((F^T X)^H + (\nabla_\gamma F^T)X^H, Y^H\big)$$
$$+\frac{\lambda^{rS}}{a} g(X^H, (F^T Y)^H + (\nabla_\gamma F^T)Y^H)$$
$$= -\frac{\lambda^r}{a}\{g\big((F^T X), Y\big)^V - g\big(X, (F^T Y)\big)^V\}$$

Theorem 12. *Let ϕ_φ be the Tachibana operator and the structure $a(F^K)^C + \lambda^r (F^T)^C = 0$ defined by Definition 2 and (2.14), respectively. If $L_Y F^T = 0$,*
then all results with respect to $(F^K)^C$ is zero, where $X, Y \in \Im_0^1(M^n)$, the complete lifts $X^C, Y^C \in \Im_0^1\big(T(M^n)\big)$ and the vertical lift $X^V, Y^V \in \Im_0^1\big(T(M^n)\big)$.

$i)\phi_{(F^K)^C X^C} Y^C = +\frac{\lambda^r}{a}\big((L_Y F^T)X\big)^C$

$ii)\phi_{(F^K)^C X^C} Y^V = +\frac{\lambda^r}{a}\big((L_Y F^T)X\big)^V$

$iii)\phi_{(F^K)^C X^V} Y^C = +\frac{\lambda^r}{a}\big((L_Y F^T)X\big)^V$

$iv)\phi_{(F^K)^C X^V} Y^V = 0$

Proof:

$i)$ $\phi_{(F^K)^C X^C} Y^C = -(L_{Y^C}(F^K)^C)X^C$
$$= -\frac{\lambda^r}{a}\{-L_{Y^C}(F^T X)^C + (F^T)^C L_{Y^C} X^C\}$$
$$= \frac{\lambda^r}{a}\big((L_Y F^T)X\big)^C$$

$ii)\phi_{(F^K)^C X^C} Y^V = -(L_{Y^V}(F^K)^C)X^C$
$$= -L_{Y^V}(F^K)^C X^C + (F^K)^C L_{Y^V} X^C$$
$$= -\frac{\lambda^r}{a}\{-L_{Y^V}(F^T X)^C + (F^T)^C L_{Y^V} X^C\}$$
$$= \frac{\lambda^r}{a}\big((L_Y F^T)X\big)^V$$

$iii)\phi_{(F^K)^C X^V} Y^C = -(L_{Y^C}(F^K)^C)X^V$
$$= -L_{Y^C}(F^K)^C X^V + (F^K)^C L_{Y^C} X^V$$
$$= -\frac{\lambda^r}{a}\{-L_{Y^C}(F^T X)^V + (F^T)^C L_{Y^C} X^V\}$$
$$= \frac{\lambda^r}{a}\big((L_Y F^T)X\big)^V$$

$iv)\phi_{(F^K)^C X^V} Y^V = -(L_{Y^V}(F^K)^C)X^V$
$$= -L_{Y^V}(F^K)^C X^V + (F^K)^C L_{Y^V} X^V$$
$$= 0$$

Theorem 13. *If $L_Y F^T = 0$ for $Y \in M$, then its complete lift Y^C to the tangent bundle is an almost holomorfic vector field with respect to the structure $a(F^K)^C + \lambda^r (F^T)^C = 0$.*

Proof:

$i)(L_{Y^C}(F^K)^C)X^C = L_{Y^C}(F^K)^C X^C - (F^K)^C L_{Y^C} X^C$
$$= -\frac{\lambda^r}{a}\{L_{Y^C}(F^T X)^C - (F^T)^C L_{Y^C} X^C\}$$
$$= -\frac{\lambda^r}{a}\big((L_Y F^T)X\big)^C$$

$$ii)(L_{Y^C}(F^K)^C)X^V = L_{Y^C}(F^K)^C X^V - (F^K)^C L_{Y^C} X^V$$
$$= -\frac{\lambda^r}{a}\{L_{Y^C}(F^T X)^V - (F^T)^C L_{Y^C} X^V\}$$
$$= -\frac{\lambda^r}{a}\big((L_Y F^T)X\big)^V$$

2.7. HORIZONTAL LIFT OF $F_{a,\lambda}(K,T)$ −STRUCTURE ON TANGENT BUNDLE $T(M^n)$

Let F_i^h be the component of F at A in the coordinate neighbourhood U of M^n. Then the horizontal lift F^H of F is also a tensor field of type (1,1) in $T(M^n)$ whose components $\tilde{F}_B^A$ in $\pi^{-1}(U)$ are given by

$$F^H = F^C - \gamma(\nabla F) = \begin{pmatrix} F_i^h & 0 \\ -\Gamma_t^h F_i^t + \Gamma_i^t F_t^h & F_i^h \end{pmatrix}. \tag{2.24}$$

Let F, G be two tensor fields of type (1,1) on the manifold M. If F^H denotes the horizontal lift of F, we have

$$(FG)^H = F^H G^H. \tag{2.25}$$

Taking F and G identical, we get

$$(F^H)^2 = (F^2)^H. \tag{2.26}$$

Multiplying both sides by F^H and making use of the same (2.26) , we get

$$(F^H)^3 = (F^3)^H$$

Thus it follows that

$$(F^H)^4 = (F^4)^H, (F^H)^5 = (F^5)^H \tag{2.27}$$

and so on. Taking horizontal lift on both sides of equation $aF^K + \lambda^r F^T = 0$ we get

$$a(F^K)^H + \lambda^r (F^T)^H = 0 \tag{2.28}$$

view of (2.27), we can write

$$a(F^H)^K + \lambda^r (F^H)^T = 0.$$

2.8. INTEGRABILITY CONDITIONS OF THE STRUCTURE $a(F^K)^H + \lambda^r(F^T)^H = 0$ ON TANGENT BUNDLE $T(M^n)$

Theorem 14. *The Nijenhuis tensor $N_{(F^K)^H (F^K)^H}(X^H, Y^H)$ of the horizontal lift of F^K vanishes if the Nijenhuis tensor of the F^T is zero and* $\{-(\hat{R}(F^T X, F^T Y)u) + (F^T(\hat{R}(F^T X, Y)u)) + \big(F^T(R(X, F^T Y)u)\big) - ((F^T)^2(\hat{R}(X,Y)u))\}^V = 0.$

Proof:

$$N_{(F^K)^H (F^K)^H}(X^H, Y^H) = [(F^K)^H X^H, (F^K)^H Y^H] - (F^K)^H[(F^K)^H X^H, Y^H]$$
$$-(F^K)^H[X^H, (F^K)^H Y^H] + (F^K)^H (F^K)^H [X^H, Y^H]$$

$$
\begin{aligned}
&= \frac{\lambda^{2r}}{a^2}\{([F^TX,F^TY]-(F^T)[F^TX,Y]\\
&\quad -(F^T)[X,F^TY]-(F^T)(F^T)[X,Y])^H\\
&\quad -(\hat{R}(F^TX,F^TY)u)^V+(F^T(\hat{R}(F^TX,Y)u))^V\\
&\quad +(F^T(\hat{R}(X,F^TY)u))^V-((F^T)^2(\hat{R}(X,Y))u)^V\}\\
&= \frac{\lambda^{2r}}{a^2}\{\left(N_{F^TF^T}(X,Y)\right)^H-(\hat{R}(F^TX,F^TY)u)^V\\
&\quad +(F(\hat{R}(F^TX,Y)u))^V+(F(\hat{R}(X,F^TY)u))^V\\
&\quad -((F^T)^2(\hat{R}(X,Y)u))^V\},
\end{aligned}
$$

where $\hat{R}$ denotes the curvature tensor of the affine connection $\hat{\nabla}$ defined by $\hat{\nabla}_XY=\nabla_YX+[X,Y]$ (see [1] p.88-89) and u is a vector field defined by Definition 4.

Theorem 15. *The Nijenhuis tensor $N_{(F^K)^H(F^K)^H}(X^H,Y^V)$ of the horizontal lift of F^K vanishes if the Nijenhuis tensor of the F^T is zero and $\nabla F^T=0$.*

Proof:

$$
\begin{aligned}
N_{(F^K)^H(F^K)^H}(X^H,Y^V) &= [(F^K)^HX^H,(F^K)^HY^V]-(F^K)^H[(F^K)^HX^H,Y^V]\\
&\quad -(F^K)^H[X^H,(F^K)^HY^V]+(F^K)^H(F^K)^H[X^H,Y^V]\\
&= \frac{\lambda^{2r}}{a^2}\{[F^TX,F^TY]^V-(F^T[F^TX,Y])^V-(F^T[X,F^TY])^V\\
&\quad +((F^T)^2[X,Y])^V+\left(\nabla_{F^TY}F^TX\right)^V-\left(F^T(\nabla_YF^TX)\right)^V\\
&\quad -\left(F^T\left(\nabla_{F^TY}X\right)\right)^V+((F^T)^2\nabla_YX)^V\}\\
&= \frac{\lambda^{2r}}{a^2}\{\left(N_{F^TF^T}(X,Y)\right)^V+\left(\nabla_{F^TY}F^T\right)X-\left(F^T\left((\nabla_YF^T)X\right)\right)^V\}
\end{aligned}
$$

Theorem 16. *The Nijenhuis tensor $N_{(F^K)^H(F^K)^H}(X^V,Y^V)$ of the horizontal lift of F^K vanishes.*

Proof: Because of $[X^V,Y^V]=0$ for $X,Y\in M$, easily we get

$$N_{(F^K)^H(F^K)^H}(X^V,Y^V)=0.$$

Theorem 17. *The Sasakian metric ${}^Sg$ is pure with respect to $(F^K)^H$ if $F^T=\frac{\lambda^r}{a}I$, where I =ıdentity tensor field of type $(1,1)$.*

Proof: $S(\tilde{X},\tilde{Y})={}^Sg((F^K)^H\tilde{X},\tilde{Y})-{}^Sg(\tilde{X},(F^K)^H\tilde{Y})$ if $S(\tilde{X},\tilde{Y})=0$ for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form X^V,Y^V or X^H,Y^H then $S=0$.

$$
\begin{aligned}
i)S(X^V,Y^V) &= {}^Sg((F^K)^HX^V,Y^V)-{}^Sg(X^V,(F^K)^HY^V)\\
&= -\frac{\lambda^r}{a}\{{}^Sg((F^TX)^V,Y^V)-{}^Sg(X^V,(F^TY)^V)\}\\
&= -\frac{\lambda^r}{a}\{\left(g(F^TX,Y)\right)^V-\left(g(X,F^TY)\right)^V\}
\end{aligned}
$$

$$
\begin{aligned}
ii)S(X^V,Y^H) &= {}^Sg((F^K)^HX^V,Y^H)-{}^Sg(X^V,(F^K)^HY^H)\\
&= \frac{\lambda^r}{a}{}^Sg(X^V,(F^TY)^H)\\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
iii)S(X^H,Y^H) &= {}^Sg((F^K)^HX^H,Y^H)-{}^Sg(X^H,(F^K)^HY^H)\\
&= -\frac{\lambda^r}{a}\{({}^Sg(F^TX)^H,Y^H)-{}^Sg(X^H,(F^TY)^H)\}\\
&= -\frac{\lambda^r}{a}\{(g(F^TX),Y)^V-(g(X,(F^TY)^H))^V\}
\end{aligned}
$$

Theorem 18. *Let ϕ_φ be the Tachibana operator and the structure $a(F^K)^H+\lambda^r(F^T)^H=0$ defined by Definition 2 and (2.28), respectively. if $L_YF^T=0$ and $F=\frac{\lambda^r}{a}I$, then all results with respect to $(F^K)^H$ is zero, where $X,Y\in\Im_0^1(M)$, the horizontal lifts $X^H,Y^H\in\Im_0^1\big(T(M^n)\big)$ and the vertical lift $X^V,Y^V\in\Im_0^1\big(T(M^n)\big)$*

$i)\phi_{(F^K)^H X^H}Y^H=\frac{\lambda^r}{a}\{-\big((L_YF^T)X\big)^H+(\hat{R}(Y,F^TX)u)^V-(F^T(\hat{R}(Y,X)u))^V\},$

$ii)\phi_{(F^K)^H X^H}Y^V=-\frac{\lambda^r}{a}\{-\big(((L_YF^T)X\big)^V+\big((\nabla_YF^T)X\big)^V\},$

$iii)\phi_{(F^K)^H X^V}Y^H=-\frac{\lambda^r}{a}\{-\big((L_YF^T)X\big)^V-(\nabla_{F^TX}Y)^V+\big(F^T(\nabla_XY)\big)^V\},$

$iv)\phi_{(F^K)^H X^V}Y^V=0,$

Proof:

$$\begin{aligned} i)\ \phi_{(F^K)^H X^H}Y^H &= -(L_{Y^H}(F^K)^H)X^H \\ &= -L_{Y^C}(F^K)^H X^H+(F^K)^H L_{Y^H}X^H \\ &= \frac{\lambda^r}{a}[Y,F^TX]^H-\frac{\lambda^r}{a}\gamma\hat{R}[Y,F^TX] \\ &\quad -\frac{\lambda^r}{a}(F^T[Y,X])^H+\frac{\lambda^r}{a}(F^T)^H(\hat{R}(Y,X)u)^V \\ &= \frac{\lambda^r}{a}\{-\big((L_YF^T)X\big)^H+(\hat{R}(Y,F^TX)u)^V \\ &\quad -(F^T(\hat{R}(Y,X)u))^V\} \end{aligned}$$

$$\begin{aligned} ii)\phi_{(F^K)^H X^H}Y^V &= -(L_{Y^V}(F^K)^H)X^H \\ &= -L_{Y^V}(F^KX)^H+(F^K)^H L_{Y^V}X^H \\ &= \frac{\lambda^r}{a}[Y,F^TX]^V-\frac{\lambda^r}{a}(\nabla_YF^TX)^V \\ &\quad -\frac{\lambda^r}{a}(F^T[Y,X])^V+\frac{\lambda^r}{a}\big(F^T(\nabla_YX)\big)^V \\ &= -\frac{\lambda^r}{a}\{-\big((L_YF^T)X\big)^V+\big((\nabla_YF^T)X\big)^V\} \end{aligned}$$

$$\begin{aligned} iii)\phi_{(F^K)^H X^V}Y^H &= -(L_{Y^H}(F^K)^H)X^V \\ &= -L_{Y^H}(F^KX)^V+(F^K)^H L_{Y^H}X^V \\ &= -\frac{\lambda^r}{a}[Y,F^TX]^V+\frac{\lambda^r}{a}\big(\nabla_{F^TX}Y\big)^V \\ &\quad -\frac{\lambda^r}{a}(F^T[Y,X])^H-\frac{\lambda^r}{a}\big(F^T(\nabla_XY)\big)^V \\ &= -\frac{\lambda^r}{a}\{-\big((L_YF^T)X\big)^V-\big(\nabla_{F^TX}Y\big)^V+\big(F^T(\nabla_XY)\big)^V\} \end{aligned}$$

$$\begin{aligned} iv)\phi_{(F^K)^H X^V}Y^V &= -(L_{Y^V}(F^K)^H)X^V \\ &= \frac{\lambda^r}{a}L_{Y^V}(F^TX)^V-\frac{\lambda^r}{a}(F^T)^H L_{Y^V}X^V \\ &= 0 \end{aligned}$$

3. CONCLUSIONS

In this paper we define the general $\mathrm{F}_{a,\lambda}(\mathrm{K},\mathrm{T})$ −structure satisfying $\mathrm{aF^K}+\lambda^r\mathrm{F^T}=0,(\mathrm{F}\neq 0,\mathrm{K}\geq 3,\mathrm{T}\geq 1$ and $(\mathrm{K}\geq \mathrm{T}),\mathrm{a}$ and λ are non zero complex numbers, r some finite integer) on manifold $\mathrm{M^n}$ and studied to give some special examples. The second part, we find the integrability conditions by calculating Nijenhuis tensors of the horizontal lifts of the general $F_{a,\lambda}(K,T)$ −structure satisfying $aF^K + \lambda^r F^T = 0$. Later, we get the results of Tachibana operators applied to vector and covector fields according to the horizontal lifts of the general $F_{a,\lambda}(K,T)$ −structure in cotangent bundle $T^*(M^n)$. In addition, we have studied to show the purity conditions of Sasakian metric with respect to the horizontal lifts of the structure. In the final section, all results obtained in the second section were investigated according to the complete and horizontal lifts of the general $F_{a,\lambda}(K,T)$ −structure on tangent bundle $T(M^n)$. In the future, if we get different structure, we can find the features of this structure.

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