ORIGINAL PAPER

TRANSMUTED LOWER RECORD TYPE POWER FUNCTION DISTRIBUTION

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Abstract. *This study introduces a new lifetime distribution as an alternative to power function distribution and modified ones. This model is generated by using a method based on the distributions of lower records. We discuss some statistical inferences and mathematical properties of the suggested distribution. We consider five estimation methods for the point estimation of the proposed distribution. Then, a comprehensive Monte Carlo simulation study is carried out to assess the risk behavior of examined estimators.*

Keywords: *power function distribution; lower records; point estimation.*

1. INTRODUCTION

In the last decade, there are many lifetime distributions which newly generated by using the various method in the literature. There may be many reasons including having a more flexible structure than existing ones, being more useful, being able to more easily provide different statistical inferences, and having the potential to model data in different fields such as agriculture, actuarial, biology, engineering, and medical sciences to generate new distribution. The proposed distributions have emerged via different methods. One of these methods is the quadratic rank transmutation map (QTRM), which is based on order statistics. QRTM was proposed by [1] to generate a new distribution using the distribution of rank statistics order two. QRTM is described by

$$F_{QRTM}(x) = G(x)\left[1+\lambda(1-G(x))\right], \tag{1}$$

where, $\lambda \in (-1,1)$, $G(x)$ denotes the cumulative distribution function (CDF) of baseline distribution, and $F_{QRTM}(x)$ refers to the CDF of proposed distribution by the QTRM [1]. The generated distributions by using QRTM are called transmuted distributions. Recently, there are several transmuted distributions in the literature. Some of transmuted distributions can be listed as follows: Transmuted Weibull [2], transmuted Lindley [3], transmuted power function [4], transmuted complementary exponential power [5], transmuted exponential power [6]. Granzotto et al. [7] suggested a new method called cubic rank transmutation map (CRTM) to generate new distributions. The CRTM is designed based on the distributions of order statistics as QRTM. However, the QRTM includes the mixture of the distributions of the first two order statistics while the CRTM is constructed via the mixture of the distributions of the first three rank statistics. The CRTM is given by

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$$F_{CRTM}(x)=\lambda_1 G(x)+(\lambda_2-\lambda_1)G^2(x)+(1-\lambda_2)G^3(x), \tag{2}$$

where, $\lambda_1 \in [0,1], \lambda_2 \in [-1,1], G(x)$ denotes the CDF of baseline distribution, and $F_{CRTM}(x)$ refers to the CDF of proposed distribution by the CTRM [7]. The proposed distributions by using the CRTM are called cubic rank transmuted distributions. Some cubic rank transmuted distributions can be listed as follows: cubic rank transmuted Weibull [7], cubic rank transmuted log-logistic [7], cubic rank transmuted Kumaraswamy [8], cubic rank transmuted inverse Weibull [9], cubic rank transmuted modified Burr III [10]. Balakrishnan and He [11] suggested two new families based on distributions of lower and upper record values. The family of distributions based on the distributions of upper record values is given by

$$F_{UR}(x)=G(x)+p\left[(1-G(x))\log(1-G(x))\right], \tag{3}$$

where $p \in [0,1], G(x)$ denotes the CDF of baseline distribution, and $F_{UR}(x)$ refers to the CDF of the proposed distribution. It is referred to the family of distributions in (3) as the record-based transmuted family of distributions [11].

Balakrishnan and He [11] also provided the record-based family of distributions which is a mixture based on the distributions of the first two lower record values. This family of distributions is constructed by

Let $X_{L(1)}$ and $X_{L(2)}$ be the lower record values from a population with $G(x)$.

Let us define a new random variable based on the distributions of these records.

$$Y=\begin{cases} X_{L(1)}, & U>p \\ X_{L(2)}, & U<p, \end{cases}$$

where U is the standard uniform random variable and $p\in(0,1)$. Then, the CDF of Y is given by

$$\begin{aligned} F_{LR}(x) &= (1-p)P\left(X_{L(1)}\le x\right)+pP\left(X_{L(2)}\le x\right) \\ &= (1-p)G(x)+p\left[G(x)\left(1-\log(G(x))\right)\right] \\ &= G(x)\left[1-p\log(G(x))\right]. \end{aligned} \tag{4}$$

Corresponding PDF and hazard function (HF) of Y are

$$f_{LR}(x)=g(x)\left[1-p\left(1+\log(G(x))\right)\right] \tag{5}$$

and

$$h_{LR}(x)=\frac{g(x)\left[1-p\left(1+\log(G(x))\right)\right]}{1-G(x)\left[1-p\log(G(x))\right]}, \tag{6}$$

respectively [11, 12]. Balakrishnan and He [11] noticed that the distribution with CDF (4) is called" dual record-transmuted distribution". The first special case of this family based on Frechet distribution is introduced by [12]. This new model is called as transmuted record type Frechet distribution in [12]. In this paper, we call this family of distributions as the family of transmuted lower record type distributions.

The main purpose of this paper to provide a new special case of the family of dual record-based transmuted distributions. We also aim to provide statistical inferences of the proposed distribution. This study is organized as follows: In Section 2, we introduce a new lifetime distribution as an alternative to power function distribution, and we examine some characteristics of the proposed distribution. In Section 3, we consider five estimation methods such as maximum likelihood, least squares, weighted least squares, Anderson-Darling, and Cramer von-Mises estimation methods to estimate the parameters of the suggested distribution.

2. TRANSMUTED LOWER RECORD TYPE POWER FUNCTION DISTRIBUTION

In this section, we introduce a new lifetime distribution from the family of transmuted lower record type distributions. Consider the baseline distribution power function distribution with CDF $G(x;\alpha)=x^{\alpha}$ and the probability density function (PDF) $g(x;\alpha)=\alpha x^{\alpha-1}$ in (4)-(5). Then, the PDF and CDF of transmuted lower record type power function distribution are

$$F(x;\alpha,p)=x^{\alpha}\left[1-p\log\left(x^{\alpha}\right)\right], \tag{7}$$

and

$$f(x;\alpha,p)=\alpha x^{\alpha-1}\left[1-p\left(1+\log\left(x^{\alpha}\right)\right)\right], \tag{8}$$

Respectively, where $\alpha>0$ is a shape parameter, and $p\in(0,1)$. In this study, the transmuted lower record type distribution is briefly denoted by $TLRT-PF(\alpha,p)$ .The HF of $TLRT-PF(\alpha,p)$ is given by

$$h_{LR}(x)=\frac{\alpha x^{\alpha-1}\left[1-p\left(1+\log\left(x^{\alpha}\right)\right)\right]}{1-x^{\alpha}\left[1-p\log\left(x^{\alpha}\right)\right]}. \tag{9}$$

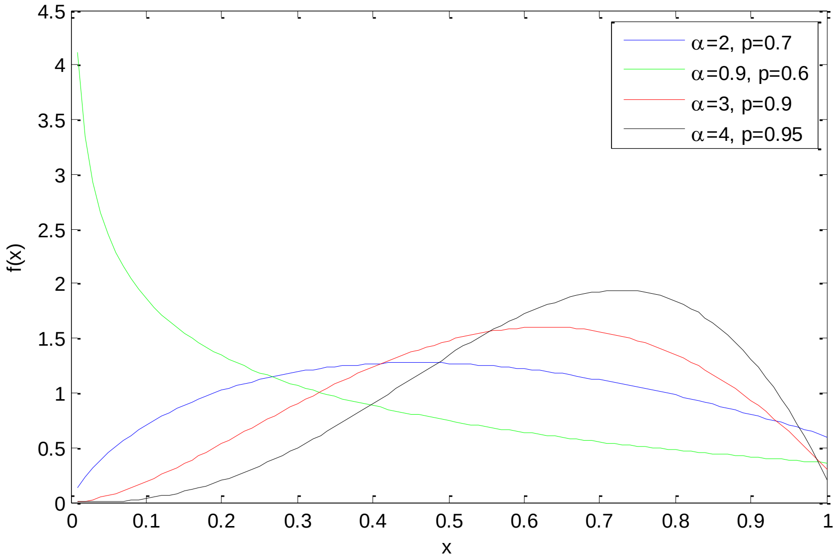

Figure 1. The PDFs of *TLRT – PF(α, p)* distribution for selected parameters.

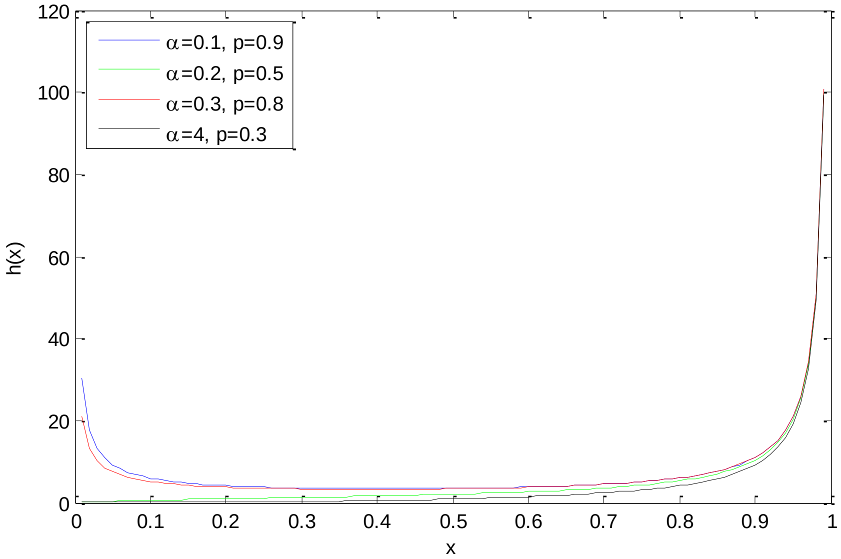

Figure 2. The HFs of *TLRT – PF(α, p)* distribution for selected parameters.

Figs. 1-2 illustrate the possible shapes of PDF and HF of $TLRT-PF\ \alpha,p$ distribution for different parameter values, respectively. From Fig. 2, The HF of $TLRT-PF\ \alpha,p$ distribution can be increasing and bathtub shaped. Thus, it can be said that $TLRT-PF\ \alpha,p$ distribution has the potential to model data sets having different-shaped HF.

2.1. MOMENTS

The r^{th} moment of $TLRT-PF\ \alpha,p$ distribution is

$$E\left(X^r\right)=\alpha\left[\lim_{u\to\infty}\frac{(1-p)r+\alpha+e^{-u(\alpha+r)}\left\{(p-p\alpha u-1)r-\alpha-p\alpha^2u\right\}}{(\alpha+r)^2}\right], \tag{10}$$

where $r\in\mathbb{N}$.

The variance of $TLRT-PF\ \alpha,p$ distribution is given by

$$Var\left(X\right)=\mu_2-\mu_1^{\,2}, \tag{11}$$

The coefficient of skewness (CS) and the coefficient of kurtosis (CK) can be computed by

$$CS=\frac{\mu_3-3\mu_2+2\mu_1^{\,3}}{\left(\mu_2-\mu_1^{\,2}\right)^{3/2}} \tag{12}$$

and

$$CK=\frac{\mu_4-4\mu_1\mu_3+6\mu_2\mu_1^{\,2}-3\mu_1^{\,4}}{\left(\mu_1-\mu_2^{\,2}\right)^2}, \tag{13}$$

respectively, μ_i denotes i^{th} moment of $TLRT-PF\ \alpha, p$ distribution for $i=1,2,3,4$.

Table 1 provides the mean (μ), variance σ^2, CS, and CK of $TLRT-PF\ \alpha, p$ distribution for selected parameter values.

Table 1. The mean, variance, CS, and CK for some parameter values

α	p	μ	σ^2	CS	CK
0.3	0.7	0.1065	0.0396	-14.8750	8.4668
0.7	0.7	0.2422	0.0661	-15.5282	3.3261
1	0.7	0.3250	0.0721	-17.8486	2.4628
2	0.7	0.5111	0.0637	-29.5592	2.0028
3	0.7	0.6187	0.0491	-45.6336	2.2326
4	0.7	0.6880	0.0377	-65.6730	2.5411
5	0.7	0.7361	0.0295	-89.6063	2.8365
1.2	0.1	0.5206	0.0804	-22.2125	1.8281
1.2	0.2	0.4958	0.0822	-21.0148	1.8076
1.2	0.3	0.4710	0.0827	-20.1758	1.8180
1.2	0.4	0.4462	0.0820	-19.6410	1.8617
1.2	0.5	0.4214	0.0801	-19.3873	1.9409
1.2	0.6	0.3966	0.0770	-19.4198	2.0569
1.2	0.7	0.3719	0.0726	-19.7765	2.2093
1.2	0.8	0.3471	0.0670	-20.5436	2.3915
1.2	0.9	0.3223	0.0601	-21.8930	2.5779

2.2. STOCHASTIC ORDERING

Stochastic ordering is useful for evaluating the comparative features for a non-negative continuous random variable [12]. The following theorem illustrates that the TLRT-PF random variables can be ordered for the likelihood ratio.

Theorem 2.2. Let $X \sim TLRT-PF(\alpha, p_1)$ and $Y \sim TLRT-PF(\alpha, p_2)$. If $p_1 > p_2$ then X is smaller than Y in the likelihood ratio order, i.e., the ratio function of corresponding pdfs is decreasing in *x*.

Proof: For any $x > 0$, the ratio of densities is given by

$$g(x) = \frac{1-p_1\left(1+\log\left(x^{\alpha}\right)\right)}{1-p_2\left(1+\log\left(x^{\alpha}\right)\right)}$$

Consider the derivative of $\log\left(g(x)\right)$ in x

$$\frac{d\log\left(g(x)\right)}{dx} = -\frac{\alpha\left(p_1-p_2\right)}{x\left(p_1+p_1\log\left(x^{\alpha}\right)-1\right)\left(p_2+p_2\log\left(x^{\alpha}\right)-1\right)} < 0$$

for $p_1 > p_2$ and thus proof is completed.

Corollary 2.2. It follows from [13] that X is also smaller than Y in the hazard ratio, mean residual life and stochastic orders under the conditions given in Theorem 2.2.

2.3. RANDOM NUMBERS GENERATION

In this subsection, we provide random generation from $TLRT-PF\ \alpha, p$ distribution. In this regard, an acceptance-rejection (AR) sampling method is given in the following algorithm to generate the data from $TLRT-PF\ \alpha, p$ distribution, In this algorithm, the Beta distribution is chosen as a proposal distribution. The AR algorithm is given as follows:

Algorithm 1

Step 1. Generate data on random variable Y variable from the Beta distribution with PDF g given as follows:

$$g\left(y;\beta,\lambda\right)=\frac{1}{B\left(\beta,\lambda\right)}y^{\beta-1}\left(1-y\right)^{\lambda-1}.$$

Step 2. Generate U from the standard uniform distribution (independent of Y)

Step 3. If

$$U<\frac{f\left(Y;\alpha,p\right)}{k\times g\left(Y;\beta,\lambda\right)},$$

then set $X=Y$ (“accept”); else go back to Step 1 (“reject”), where PDF $f\left(.\right)$ is given in (8) and

$$k=\max_{y\in(0,1)}\frac{f\left(y;\alpha,p\right)}{g\left(y;\beta,\lambda\right)}.$$

We notice that Algorithm 1 is used for all Monte Carlo simulations in this paper.

3. POINT ESTIMATION

In this section, we provide five different estimators of the parameters of the $TLRT-PF\ \alpha, p$ distribution such as maximum likelihood estimator (MLE), least squares estimator (LSE), weighted least squares estimator (WLSE), Anderson-Darling estimator (ADE), and Cramér–von Mises estimator (CvME).

Let $X_1, X_2, \ldots, X_n$ be a random sample from the $TLRT-PF\ \alpha, p$ distribution and $X_{(1)}<X_{(2)}<\ldots<X_{(n)}$ denote the corresponding order statistics. Also, $x_{(i)}$ denotes the observed value of $X_{(i)}$. In this regard, the log-likelihood function of the $TLRT-PF\ \alpha, p$ distribution is

$$\ell(\boldsymbol{\theta}) = n\log(\alpha) + (\alpha - 1)\sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}\log\left[1 - p\left(1 + \log\left(x_i^{\alpha}\right)\right)\right], \tag{14}$$

where $\boldsymbol{\theta} = (\alpha, p)$ is a parameter vector. Then, maximum likelihood estimator (MLE) of $\boldsymbol{\theta}$ is given as follows:

$$\hat{\boldsymbol{\theta}}_{MLE} = \arg\max_{\theta}\left\{\ell(\boldsymbol{\theta})\right\}. \tag{15}$$

Let us define the following four functions which are used to obtain the different type of estimates:

$$Q_{LS}(\boldsymbol{\theta}) = \sum_{i=1}^{n}\left(x_{(i)}^{\alpha}\left[1 - p\log\left(x_{(i)}^{\alpha}\right)\right] - \frac{i}{n+1}\right)^2,$$

$$Q_{WLS}(\boldsymbol{\theta}) = \sum_{i=1}^{n}\frac{(n+2)(n+1)^2}{i(n-i+1)}\left(x_{(i)}^{\alpha}\left[1 - p\log\left(x_{(i)}^{\alpha}\right)\right] - \frac{i}{n+1}\right)^2,$$

$$Q_{CvM}(\boldsymbol{\theta}) = \frac{1}{12n} + \sum_{i=1}^{n}\left(x_{(i)}^{\alpha}\left[1 - p\log\left(x_{(i)}^{\alpha}\right)\right] - \frac{2i-1}{2n}\right)^2$$

and

$$Q_{AD}(\boldsymbol{\theta}) = -n - \frac{1}{n}\sum_{i=1}^{n}\left((2i-1)\log\left[x_{(i)}^{\alpha}\left[1 - p\log\left(x_{(i)}^{\alpha}\right)\right]\right]\right)$$
$$+ \frac{1}{n}\sum_{i=1}^{n}\left(\log\left(1 - \left\{x_{(i)}^{\alpha}\left[1 - p\log\left(x_{(i)}^{\alpha}\right)\right]\right\}\right)\right).$$

the LSEs, WLSEs, CvMEs, and ADEs of the parameters $\boldsymbol{\theta} = (\alpha, p)$ are given, respectively, by

$$\hat{\boldsymbol{\theta}}_{LSE} = \arg\min_{\theta}\left\{Q_{LS}(\boldsymbol{\theta})\right\}, \tag{16}$$

$$\hat{\boldsymbol{\theta}}_{WLSE} = \arg\min_{\theta}\left\{Q_{WLS}(\boldsymbol{\theta})\right\}, \tag{17}$$

$$\hat{\boldsymbol{\theta}}_{CvME} = \arg\min_{\theta}\left\{Q_{CvM}(\boldsymbol{\theta})\right\}, \tag{18}$$

$$\hat{\boldsymbol{\theta}}_{ADE} = \arg\min_{\theta}\left\{Q_{AD}(\boldsymbol{\theta})\right\}. \tag{19}$$

The estimators given in (15)-(19) can be obtained by **optim ()** function in R with Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm. This algoritm is firstly studied by Fletcher [14].

4. SIMULATION STUDY

In this section, we consider a comprehensive Monte Carlo simulation study to assess the performances of MLEs, of risk measures according to biases and MSEs. In the simulation study is performed based on 5000 repetitions. We consider the sample size 25, 50, 100, 200, 500 and two parameter settings as follows:

$$(\alpha = 0.5,\ \beta = 1.5,\ \lambda = 0.2),\ (\alpha = 1,\ \beta = 2,\ \lambda = 0.5).$$

Table 2. Average biases of MLEs, LSEs, WLSEs, ADEs, and CvMEs of *α* and *p* parameters

		$\hat{\alpha}$					$\hat{p}$				
Parameters	**n**	**MLE**	**LSE**	**WLSE**	**ADE**	**CvME**	**MLE**	**LSE**	**WLSE**	**ADE**	**CvME**
$\alpha = 2, p = 0.9$	25	-0.0255	-0.1832	-0.1561	-0.1197	-0.0293	-0.1560	-0.2389	-0.2264	-0.2043	-0.1325
	50	0.0168	-0.0533	-0.0323	-0.0271	0.0186	-0.0841	-0.1166	-0.1081	-0.1055	-0.0673
	100	0.0303	0.0000	0.0115	0.0092	0.0338	-0.0445	-0.0595	-0.0536	-0.0549	-0.0359
	250	0.0359	0.0214	0.0276	0.0251	0.0377	-0.0219	-0.0284	-0.0253	-0.0269	-0.0171
	500	0.0270	0.0215	0.0241	0.0229	0.0279	-0.0092	-0.0119	-0.0105	-0.0113	-0.0075
	1000	0.0192	0.0164	0.0179	0.0173	0.0196	-0.0062	-0.0082	-0.0071	-0.0075	-0.0059
$\alpha = 0.5, p = 0.7$	25	0.1161	0.0532	0.0590	0.0734	0.0995	0.0040	-0.0846	-0.0945	-0.0617	0.0177
	50	0.1175	0.0821	0.0901	0.0960	0.1039	0.0596	0.0214	0.0196	0.0363	0.0699
	100	0.1140	0.0919	0.1023	0.1014	0.1026	0.0752	0.0611	0.0706	0.0695	0.0851
	250	0.1135	0.0984	0.1075	0.1043	0.1036	0.0793	0.0795	0.0864	0.0798	0.0911
	500	0.1133	0.1014	0.1092	0.1064	0.1034	0.0787	0.0846	0.0890	0.0836	0.0892
	1000	0.1124	0.1018	0.1090	0.1062	0.1028	0.0785	0.0868	0.0900	0.0847	0.0890
$\alpha = 1, p = 0.8$	25	0.0264	-0.0664	-0.0633	-0.0268	0.0201	-0.1279	-0.2266	-0.2368	-0.1817	-0.1154
	50	0.0299	-0.0135	-0.0031	0.0040	0.0282	-0.0695	-0.1161	-0.1056	-0.0969	-0.0628
	100	0.0263	0.0073	0.0145	0.0150	0.0267	-0.0371	-0.0555	-0.0480	-0.0472	-0.0304
	250	0.0221	0.0127	0.0171	0.0167	0.0222	-0.0205	-0.0309	-0.0257	-0.0260	-0.0185
	500	0.0119	0.0079	0.0102	0.0098	0.0116	-0.0140	-0.0186	-0.0159	-0.0163	-0.0138
	1000	0.0081	0.0059	0.0071	0.0069	0.0077	-0.0103	-0.0128	-0.0113	-0.0116	-0.0104
$\alpha = 0.8, p = 0.5$	25	0.1072	0.0171	0.0150	0.0503	0.1007	-0.0030	-0.1209	-0.1285	-0.0784	-0.0136
	50	0.0698	0.0159	0.0234	0.0377	0.0622	-0.0054	-0.0794	-0.0720	-0.0501	-0.0175
	100	0.0411	0.0111	0.0181	0.0257	0.0356	-0.0122	-0.0543	-0.0465	-0.0341	-0.0207
	250	0.0186	-0.0015	0.0067	0.0085	0.0120	-0.0273	-0.0567	-0.0449	-0.0417	-0.0378
	500	0.0134	0.0058	0.0097	0.0088	0.0108	-0.0125	-0.0231	-0.0176	-0.0191	-0.0162
	1000	0.0077	0.0042	0.0060	0.0060	0.0067	-0.0121	-0.0170	-0.0144	-0.0144	-0.0135

Table 3. Average MSEs of MLEs, LSEs, WLSEs, ADEs, and CvMEs of α and p parameters

		$\hat{\alpha}$					$\hat{p}$				
Parameters	**n**	**MLE**	**LSE**	**WLSE**	**ADE**	**CvME**	**MLE**	**LSE**	**WLSE**	**ADE**	**CvME**
$\alpha = 2, p = 0.9$	25	0.1267	0.2319	0.1915	0.1521	0.1971	0.0666	0.1524	0.1282	0.1098	0.1012
	50	0.0595	0.0985	0.0753	0.0661	0.0933	0.0272	0.1043	0.0427	0.0365	0.0863
	100	0.0286	0.0431	0.0346	0.0306	0.0435	0.0109	0.0220	0.0166	0.0138	0.0186
	250	0.0158	0.0216	0.0174	0.0164	0.0225	0.0052	0.0095	0.0068	0.0064	0.0089
	500	0.0067	0.0090	0.0074	0.0072	0.0093	0.0022	0.0038	0.0028	0.0027	0.0037
	1000	0.0034	0.0045	0.0037	0.0037	0.0046	0.0011	0.0019	0.0014	0.0014	0.0019
$\alpha = 0.5, p = 0.7$	25	0.0258	0.0201	0.0220	0.0203	0.0273	0.0435	0.1014	0.1236	0.0889	0.0856
	50	0.0202	0.0157	0.0183	0.0166	0.0196	0.0263	0.0451	0.0622	0.0357	0.0462
	100	0.0161	0.0129	0.0144	0.0136	0.0149	0.0189	0.0295	0.0271	0.0206	0.0313
	250	0.0144	0.0117	0.0132	0.0127	0.0127	0.0128	0.0180	0.0163	0.0162	0.0196
	500	0.0135	0.0111	0.0126	0.0120	0.0115	0.0091	0.0116	0.0114	0.0104	0.0124
	1000	0.0129	0.0108	0.0122	0.0116	0.0110	0.0075	0.0097	0.0098	0.0088	0.0101
$\alpha = 1, p = 0.8$	25	0.0408	0.0652	0.0645	0.0448	0.0620	0.0695	0.1530	0.1878	0.1020	0.1048
	50	0.0216	0.0339	0.0284	0.0253	0.0335	0.0354	0.0720	0.0614	0.0522	0.0581
	100	0.0114	0.0154	0.0126	0.0119	0.0157	0.0176	0.0281	0.0221	0.0207	0.0248
	250	0.0058	0.0075	0.0062	0.0060	0.0076	0.0089	0.0131	0.0100	0.0098	0.0118
	500	0.0023	0.0030	0.0025	0.0025	0.0030	0.0035	0.0050	0.0040	0.0040	0.0048
	1000	0.0011	0.0014	0.0012	0.0012	0.0014	0.0017	0.0023	0.0019	0.0019	0.0023
$\alpha = 0.8, p = 0.5$	25	0.0594	0.0606	0.0554	0.0532	0.0791	0.0823	0.1242	0.1439	0.1048	0.1162
	50	0.0344	0.0386	0.0379	0.0344	0.0433	0.0635	0.0887	0.0916	0.0779	0.0812
	100	0.0198	0.0246	0.0239	0.0212	0.0254	0.0427	0.0578	0.0594	0.0490	0.0531
	250	0.0095	0.0145	0.0122	0.0112	0.0141	0.0244	0.0382	0.0332	0.0299	0.0344
	500	0.0036	0.0051	0.0042	0.0045	0.0051	0.0087	0.0121	0.0104	0.0114	0.0116
	1000	0.0017	0.0024	0.0020	0.0019	0.0023	0.0044	0.0059	0.0051	0.0048	0.0056

From Tables 2-3, It is seen that as the sample size increases, the MSEs and biases of all estimators decrease and approach zero. We also observed that the MSEs and bias of the estimators are very close to each other. Therefore, someone who will study the point estimation for $TLRT - PF \;\; \alpha, p$ distribution can use any of examined methods.

5. CONCLUSIONS

In this study, we introduce a new lifetime distribution as an alternative to power function distribution and modified ones. We genarete this new model by using the method based on the distributions of first two lower records. Thus, we provide a new sub model of the family of dual record-based transmuted distributions suggested by Balakrishnan and He (2021). We examine some statistical properties of $TRLT - PF(\alpha, p)$ distribution such as moments, variance, CS, CK, stochastic ordering, random generation, hazard function and its graphs. We provide five estimators including MLE, LSE, WLSE, ADE, and CvME of the parameters for $TRLT - PF(\alpha, p)$ distribution. Then, Monte Carlo simulations are performed to assess the performance of the estimators according to MSE and bias. The results of the simulation study show that five estimators provide estimation procedures. It is quite difficult to choose between five predictors according to MSE and bias due to the MSEs and biases of the estimators are very close to each other in the simulation study. However, we recommend the maximum likelihood method to estimate the parameters of $TLRT-PF\ \alpha, p$ distribution according to simulation results.

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