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ROUGH ABEL STATISTICAL QUASI CAUCHY OF TRIPLE SEQUENCES

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Abstract. *In this paper, we investigated some basic properties of rough I-convergence of a triple sequence spaces of fuzzy in three dimensional matrix spaces which are not earlier. In addition, it was studied the set of all rough I-limits of a triple sequence spaces and also the relation between analytic ness and rough I-core of a triple sequence spaces.*
Keywords: *Abel series method; convergence and divergence of triple sequences; quasi-Cauchy.*

1. INTRODUCTION

The concept of continuity and any concept involving continuity play a very important role not only in pure mathematics but also in other branches of science involving mathematics especially in computer sciences, information theory, biological science, economics and dynamical systems.

Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ will denote the set of non negative integers and the set of real numbers, respectively. A function $f: \mathbb{R}^3 \to \mathbb{R}^3$ is continuous if and only if it preserves Cauchy sequences, lacunary statistical quasi-Cauchy sequences, ρ-statistical quasi-Cauchy sequences ideal quasi-Cauchy sequences, strongly lacunary quasi-Cauchy sequences, slowly oscillating sequences. The triple sequence $\theta_{r,s,t} = \{(m_r, n_s, k_t)\}$ is called triple lacunary if there exist three increasing sequences of integers such that

$$m_0 = 0, h_r = m_r - m_{r-1} \to \infty \text{ as } r \to \infty,$$

$$n_0 = 0, \overline{h_s} = n_s - n_{s-1} \to \infty \text{ as } s \to \infty,$$

$$k_0 = 0, \overline{h_t} = k_t - k_{t-1} \to \infty \text{ as } t \to \infty.$$

Let $m_{r,s,tj} = m_r n_s k_t, h_{r,s,t} = h_r \overline{h_s h_t}$, and $\theta_{r,s,t}$ is determine by:

$$I_{r,s,t} = \{(m,n,k): m_{r-1} < m \leq m_r \quad \text{and} \quad n_{s-1} < n \leq n_s \quad \text{and} \quad k_{t-1} < k \leq k_t\},$$

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$$q_r = \frac{m_r}{m_{r-1}}, \overline{q_s} = \frac{n_s}{n_{s-1}}, \overline{q_t} = \frac{k_t}{k_{t-1}}.$$

Throughout the paper β be a non negative real number. A real triple sequences (α_{mnk}) of points in $\mathbb{R}^3$ is called rough statistically Δ^3-convergent to an $\ell \in \mathbb{R}$ if $\lim_{r,s,t\to\infty} \frac{1}{rst} |\{m \le r, n \le s, k \le t: |\Delta^3 \alpha_{mnk} - \ell| \ge \beta + \epsilon\}| = 0$ for every $\epsilon > 0$, and this is denoted by $st - \lim \Delta^3 \alpha_{mnk} = \ell$. A triple sequence (α_{mnk}) is called lacunary rough statistically Δ^3-convergent to an $\ell \in \mathbb{R}$ if $\lim_{r,s,t\to\infty} \frac{1}{h_{rst}} |\{(m,n,k) \in I_{rst}: |\Delta^3 \alpha_{mnk} - \ell| \ge \beta + \epsilon\}| = 0$ for every $\epsilon \ge 0$, and this is denoted by $S_\theta - \lim \Delta^3 \alpha_{mnk} = \ell$.

Throughout this paper we assume that $\liminf_{r,s,t} \frac{k_{rst}}{k_{r-1,s-1,t-1}} > 1$. A triple sequence (α_{mnk}) is slowly rough oscillating if for any given $\beta, \epsilon > 0$ there exists a $\delta = \delta(\beta + \epsilon) > 0$ and $N = N(\beta + \epsilon)$ such that $|\alpha_{uvw} - \alpha_{rst}| < \beta + \epsilon$ whenever $r, s, t \ge N(\beta + \epsilon)$ and $r \le u \le (1 + \delta)(rst)$, $s \le v \le (1 + \delta)(rst)$, $t \le w \le (1 + \delta)(rst)$.

A triple sequence (α_{mnk}) is slowly rough oscillating if $(\Delta^3 \alpha_{mnk})$ is slowly rough oscillating, where $\Delta^3 \alpha_{mnk} = \alpha_{mnk} - \alpha_{m,n+1,k} - \alpha_{m,n,k+1} + \alpha_{m,n+1,k+1} - \alpha_{m+1,n,k} + \alpha_{m+1,n+1,k} + \alpha_{m+1,n,k+1} - \alpha_{m+1,n+1,k+1}$, for every $m, n, k \in \mathbb{N}$.

1.1. ABEL STATISTICAL WARD CONTINUITY

A triple sequence (α_{mnk}) of real numbers is called Abel convergent (or Abel summable) to ℓ if the series $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \alpha_{mnk} x^m y^n z^k$ is convergent for $0 \le x, y, z \le 1$ and $\lim_{x\to1^-, y\to1^-, z\to1^-} (1-x)(1-y)(1-z) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \alpha_{mnk} x^m y^n z^k = \ell$.

In this case, we write Abel-$\lim \alpha_{mnk} = \ell$. The concept of a Cauchy sequence involves far more than that the distance between successive terms is tending to 0.

A triple sequence (α_{mnk}) of points in $\mathbb{R}^3$ is called Abel Cauchy if $(\Delta^3 \alpha_{mnk})$ is Abel convergent to 0, i.e. the series $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \Delta^3 \alpha_{mnk} x^m y^n z^k$ is convergent for $0 \le x, y, z < 1$ and

$$\lim_{x\to1^-, y\to1^-, z\to1^-} (1-x)(1-y)(1-z) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \Delta^3 \alpha_{mnk} x^m y^n z^k = 0.$$

A and $\Delta^3 A$ will denote the set of Abel convergent triple sequences and the set of Abel Cauchy sequences, respectively.

A triple sequence (α_{mnk}) is called Abel rough statistically convergent to a real number ℓ if

$$\lim_{x\to1^-, y\to1^-, z\to1^-} (1-x)(1-y)(1-z) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0: |\alpha_{mnk} - \ell| \ge \beta + \epsilon}^{\infty} \alpha_{mnk} x^m y^n z^k = 0 \text{ for every } \epsilon > 0,$$

and denoted by $Abel_{st} - \lim \alpha_{mnk} = \ell$.

Now we give some definitions as follows.

Definition 1. *A rough triple sequence of points in a subset E of $\mathbb{R}^3$ is called Abel rough statistically Cauchy if $Abel_{st} - lim \Delta^3 \alpha_{mnk} = 0$, (i.e.)*

$$\lim_{x\to 1^-, y\to 1^-, z\to 1^-}(1-x)(1-y)(1-z)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0:|\Delta^3\alpha_{mnk}-\ell|\geq\beta+\epsilon}^{\infty}\Delta^3\alpha_{mnk}x^m y^n z^k = 0$$

for every $\epsilon > 0$, where $\Delta^3\alpha_{mnk} = \alpha_{mnk} - \alpha_{m,n+1,k} - \alpha_{m,n,k+1} + \alpha_{m,n+1,k+1} - \alpha_{m+1,n,k} + \alpha_{m+1,n+1,k} + \alpha_{m+1,n,k+1} - \alpha_{m+1,n+1,k+1}$, for every $m, n, k \in \mathbb{N}$.

For any fixed constant $c \in \mathbb{R}$, the triple sequence $(c\alpha_{mnk})$ is Abel statistically Cauchy whenever (α_{mnk}) is, and the sum of two Abel statistically Cauchy sequence is Abel rough statistically Cauchy. Thus the set of all Abel rough statistically Cauchy sequences is a vector space of the space of all triple sequences. The product of two Abel statistical Cauchy sequences need not be Abel rough statistically Cauchy as it can be seen by considering the product of the triple sequence $\left(\sqrt{mnk}\right)$ itself. Cauchy sequences have the property that any subsequence of a Cauchy sequence is Cauchy. The analogous property fails for Abel rough statistical Cauchy sequences. A counter example is the triple sequence $\left(\sqrt{mnk}\right)$ with the sub-sequence (mnk).

Any convergent triple sequence is Abel rough statistically Cauchy: Let (α_{mnk}) be a rough convergent triple sequence with limit ℓ and $\epsilon > 0$. Then there exists an $I, J, L \in \mathbb{N}$ such that $|\alpha_{mnk} - \ell| < \frac{\beta+\epsilon}{2}$ for all $m, n, k \geq \mathbb{N}$. Hence $\left\{m, n, k \in \mathbb{N}: |\alpha_{mnk} - \ell| \geq \frac{\beta+\epsilon}{2}\right\} \subseteq \{1,2,3, \ldots, N\}$ for every $\beta, \epsilon > 0$. Therefore $\sum_{m\in I}\sum_{n\in J}\sum_{k\in L:|\alpha_{mnk}-\ell|\geq\frac{\beta+\epsilon}{2}}\alpha_{mnk}x^m y^n z^k \leq \sum_{m=1}^{I}\sum_{n=1}^{J}\sum_{k=1}^{L}\alpha_{mnk}x^m y^n z^k$ for every $\epsilon > 0$. On the other hand

$$\begin{aligned}&\sum_m\sum_n\sum_{k:|\alpha_{mnk}|\geq\beta+\epsilon}\alpha_{mnk}x^m y^n z^k \leq \\ &\sum_m\sum_n\sum_{k:|\alpha_{m+1n+1k+1}-\ell|\geq\frac{\beta+\epsilon}{2}}\alpha_{mnk}x^m y^n z^k + \sum_m\sum_n\sum_{k:|\ell-\alpha_{mnk}|<\frac{\beta+\epsilon}{2}}\alpha_{mnk}x^m y^n z^k \\ &\leq 2\sum_{m=1}^{I}\sum_{n=1}^{J}\sum_{k=1}^{L}\alpha_{mnk}x^m y^n z^k\end{aligned}$$

for every $\epsilon > 0$. Therefore

$$\begin{aligned}&\lim_{x\to 1^-, y\to 1^-, z\to 1^-}(1-x)(1-y)(1-z)\sum_m\sum_n\sum_{k:|\alpha_{mnk}|\geq\beta+\epsilon}\alpha_{mnk}x^m y^n z^k \\ &\leq 2\lim_{x\to 1^-, y\to 1^-, z\to 1^-}(1-x)(1-y)(1-z)\sum_{m=1}^{I}\sum_{n=1}^{J}\sum_{k=1}^{L}\alpha_{mnk}x^m y^n z^k = 0\end{aligned}$$

for every $\epsilon > 0$.

Definition 2. *A subset E of $\mathbb{R}^3$ is called Abel statistically ward compact if any triple sequence of points in E has an Abel rough statistical Cauchy subsequence, (i.e.) whenever $\alpha = (\alpha_{rst})$ is a triple sequence of points in E, there is an Abel rough statistical Cauchy subsequence $\xi = (\xi_{mnk}) = \left(\alpha_{r_m s_n t_k}\right)$ of α.*

Definition 3. *A function f is called Abel statistically ward continuous on a subset E of $\mathbb{R}^3$ if it preserves Abel rough statistical Cauchy sequences points in E, (i.e.) $\left(f(\alpha_{mnk})\right)$ is Abel statistically Cauchy whenever (α_{mnk}) is an Abel rough statistical Cauchy sequence of points in E.*

The idea of rough convergence was first introduced by Phu [1-3] in finite dimensional normed spaces. He showed that the set LIM_x^r is bounded, closed and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of $*LIM_x^r$ on the roughness of degree r.

Aytar [4] studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [5] studied that the r-limit set of the sequence is equal to intersection of these sets and that r-core of the sequence is equal to the union of these sets. Dündar and Cakan [6] investigated of rough ideal convergence and defined the set of rough ideal limit points of a sequence The notion of I-convergence of a triple sequence spaces which is based on the structure of the ideal I of subsets of $\mathbb{N}^3$, where $\mathbb{N}$ is the set of all natural numbers, is a natural generalization of the notion of convergence and statistical convergence.

Our purpose in this paper is to investigate some basic properties of rough I-convergence of a triple sequence spaces of fuzzy in three dimensional matrix spaces which are not earlier. We also study the set of all rough I-limits of a triple sequence spaces and the relation between analytic ness and rough I-core of a triple sequence spaces.

Let K be a subset of the set of positive integers $\mathbb{N}^3$ and let us denote the set $K_{ij\ell} = \{(m,n,k) \in K: m \leq i, n \leq j, k \leq \ell\}$. Then the natural density of K is given by $\delta_3(K) = \lim_{i,j,\ell \to \infty} \frac{|K_{ij\ell}|}{ij\ell}$, where $|K_{ij\ell}|$ denotes the number of elements in $K_{ij\ell}$. The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

A triple sequence (real or complex) can be defined as a function $x: \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by Sahiner et al. [7, 8], Esi et al. [9-12], Dutta et al. [13], Subramanian et al. [14], Debnath et al. [15] and many others. For more information about the quasi Cauchy sequences, one can refer to Çakalli [16], Taylan [17], Çakalli et al. [18],Çanak and Dik [19].

2. MAIN RESULTS

Theorem 1. *Any triple sequence of Abel statistical ward continuous function on an triple sequence of Abel rough statistical ward compact subset E of $\mathbb{R}^3$ is uniformly continuous.*

Proof. Let f be a function on a triple sequence of Abel rough statistical ward compact subset E of $\mathbb{R}^3$ into $\mathbb{R}^3$. Suppose that f is not uniformly continuous on E so that there exist an $\beta_0, \epsilon_0 > 0$ and two triple sequence (α_{rst}) and (η_{rst}) of points in E such that $|\alpha_{rst} - \eta_{rst}| < \frac{1}{rst}$ and $|f(\alpha_{rst}) - f(\eta_{rst})| \geq \beta_0 + \epsilon_0$ for all $r, s, t \in \mathbb{N}$. Since E is Abel statistically ward compact, there is a triple subsequence $\left(\alpha_{r_m s_n t_k}\right)$ of (α_{rst}) that is Abel statistically Cauchy. On the other hand there is a triple subsequence $\left(\eta_{r_{m_i} s_{n_j} t_{k_\ell}}\right)$ of $\left(\eta_{r_m s_n t_k}\right)$ that is Abel rough statistically Cauchy. The corresponding triple subsequence $\left(\alpha_{r_{m_i} s_{n_j} t_{k_\ell}}\right)$ is also Abel rough statistically Cauchy which follows from the following inclusion:

$$\left\{(m,n,k) \in \mathbb{N}^3 : \left|\alpha_{r_{m_{i+1}} s_{n_{j+1}} t_{k_{\ell+1}}} - \alpha_{r_{m_i} s_{n_j} t_{k_\ell}}\right| \geq \beta + \epsilon\right\}$$

$$\subseteq \left\{(m,n,k) \in \mathbb{N}^3 : \left|\alpha_{r_{m_{i+1}} s_{n_{j+1}} t_{k_{\ell+1}}} - \eta_{r_{m_{i+1}} s_{n_{j+1}} t_{k_{\ell+1}}}\right| \geq \frac{\beta+\epsilon}{3}\right\} \cup$$
$$\left\{(m,n,k) \in \mathbb{N}^3 : \left|\eta_{r_{m_{i+1}} s_{n_{j+1}} t_{k_{\ell+1}}} - \eta_{r_{m_i} s_{n_j} t_{k_\ell}}\right| \geq \frac{\beta+\epsilon}{3}\right\} \cup$$
$$\left\{(m,n,k) \in \mathbb{N}^3 : \left|\eta_{r_{m_i} s_{n_j} t_{k_\ell}} - \alpha_{r_{m_i} s_{n_j} t_{k_\ell}}\right| \geq \frac{\beta+\epsilon}{3}\right\}$$

for every $\beta, \epsilon > 0$, that implies

$$\lim_{x\to1^-,y\to1^-,z\to1^-}(1-x)(1-y)(1-z)\sum_m \sum_n \sum_{k:\left|\eta_{r_{m_{i+1}} s_{n_{j+1}} t_{k_{\ell+1}}} - \eta_{r_{m_i} s_{n_j} t_{k_\ell}}\right| \geq \beta+\epsilon} \eta_{mnk} x^m y^n z^k$$
$$+ \lim_{x\to1^-,y\to1^-,z\to1^-}(1-x)(1-y)(1-z)\sum_m \sum_n \sum_{k:\left|\eta_{r_{m_i} s_{n_j} t_{k_\ell}} - \alpha_{r_{m_i} s_{n_j} t_{k_\ell}}\right| \geq \beta+\epsilon} \eta_{mnk} x^m y^n z^k +$$
$$\lim_{x\to1^-,y\to1^-,z\to1^-}(1-x)(1-y)(1-z)\sum_m \sum_n \sum_{k:\left|\alpha_{r_{m_i} s_{n_j} t_{k_\ell}} - \alpha_{r_{m_{i+1}} s_{n_{j+1}} t_{k_{\ell+1}}}\right| \geq \frac{\beta+\epsilon}{3}} \alpha_{mnk} x^m y^n z^k$$
$$= 0 + 0 + 0 = 0,$$

for every $\epsilon > 0$. Now the sequence

$$(\mu_{ij\ell}) = \begin{pmatrix} \alpha_{r_{m_1} s_{n_1} t_{k_1}} \eta_{r_{m_1} s_{n_1} t_{k_1}} & \cdots \\ \alpha_{r_{m_2} s_{n_2} t_{k_2}} \eta_{r_{m_2} s_{n_2} t_{k_2}} & \cdots \\ \vdots & \\ \alpha_{r_{m_i} s_{n_j} t_{k_\ell}} \eta_{r_{m_i} s_{n_j} t_{k_\ell}} & \cdots \end{pmatrix}$$

is Abel statistical quasi Cauchy while the triple sequence

$$\left(f(\mu_{ij\ell})\right) = \begin{pmatrix} f\left(\alpha_{r_{m_1} s_{n_1} t_{k_1}}\right) f\left(\eta_{r_{m_1} s_{n_1} t_{k_1}}\right) & \cdots \\ f\left(\alpha_{r_{m_2} s_{n_2} t_{k_2}}\right) f\left(\eta_{r_{m_2} s_{n_2} t_{k_2}}\right) & \cdots \\ \vdots & \\ f\left(\alpha_{r_{m_i} s_{n_j} t_{k_\ell}}\right) f\left(\eta_{r_{m_i} s_{n_j} t_{k_\ell}}\right) & \cdots \end{pmatrix}$$

is not Abel statistically quasi Cauchy since

$$\lim_{x\to1^-,y\to1^-,z\to1^-}(1-x)(1-y)(1-z)\sum_m \sum_n \sum_{k:\left|\left(f(\mu_{ij\ell})\right) - \left(f(\mu_{i+1 j+1 \ell+1})\right)\right| \geq \beta_0+\epsilon} f(\mu_{mnk}) x^m y^n z^k$$
$$= \lim_{x\to1^-,y\to1^-,z\to1^-}(1-x)(1-y)(1-z)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f(\mu_{mnk}) x^m y^n z^k \neq 0.$$

Hence this establishes a contradiction.

Theorem 2. *A triple sequence of Abel statistical ward continuous image of any triple sequence of Abel statistical ward compact subset of* $\mathbb{R}^3$ *is triple sequence of Abel statistically ward compact.*

Proof: Let $f: E \to \mathbb{R}^3$ be an Abel statistical ward continuous function and B be an Abel statistical ward compact subset of E. Take any triple sequence $\eta = (\eta_{mnk})$ of points in $f(B)$. Write $\eta_{mnk} = f(\alpha_{mnk})$ for each $m, n, k \in \mathbb{N}, \alpha = (\alpha_{mnk})$. Since B is Abel statistically ward compact there exists an Abel rough statistical Cauchy sequence $\xi = (\xi_{mnk})$ of the triple sequence α. Since f is Abel statistically ward continuous $f(\xi) = \left(f(\xi_{mnk})\right)$ is Abel rough statistically Cauchy which is a subsequence of the triple sequence η.

Theorem 3. *A triple sequence in* $\mathbb{R}^3$ *is Abel statistically ward compact if and only if it is triple analytic.*

Proof: Abel statistical sequential method is regular, any triple analytic subset of $\mathbb{R}^3$ is Abel statistically ward compact. Suppose now that E is not an triple analytic. First pick an element α_{000} of E so that $\alpha_{000} > 1$. Then choose an element α_{111} of E so that $\alpha_{111} > \alpha_{000} + 3$. Similarly choose an element α_{222} of E so that $\alpha_{222} > \alpha_{111} + 8$. We can inductively choose elements of E so that $\alpha_{m+1n+1k+1} > 2^{m+n+k}$ for each $m, n, k \in \mathbb{N}$. Take any triple subsequence $\alpha_{r_m s_n t_k}$ of the triple sequence (α_{rst}). Thus

$$\sum_{m\in\mathbb{N}} \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}:|\Delta^3\alpha_{mnk}|\geq 1} \Delta^3\alpha_{mnk} x^{\frac{m}{m+n+k}} y^{\frac{n}{m+n+k}} z^{\frac{k}{m+n+k}} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \Delta^3\alpha_{mnk} x^{\frac{m}{m+n+k}} y^{\frac{n}{m+n+k}} z^{\frac{k}{m+n+k}} = \left(\frac{1}{1-x}\right)^{m+n+k}.$$

Hence

$$\lim_{x\to 1^-, y\to 1^-, z\to 1^-} (1-x)^{m+n+k}(1-y)^{m+n+k}(1-z)^{m+n+k} \sum_{m\in\mathbb{N}} \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}:|\Delta^3\alpha_{mnk}|\geq 1} \Delta^3\alpha_{mnk} x^{\frac{m}{m+n+k}} y^{\frac{n}{m+n+k}} z^{\frac{k}{m+n+k}}$$
$$= \lim_{x\to 1^-, y\to 1^-, z\to 1^-} (1-x)^{m+n+k}(1-y)^{m+n+k}(1-z)^{m+n+k} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \Delta^3\alpha_{mnk} x^{\frac{m}{m+n+k}} y^{\frac{n}{m+n+k}} z^{\frac{k}{m+n+k}} = 1 \neq 0.$$

Thus the triple sequence α_{mnk} has no Abel statistical Cauchy subsequence. If it is not triple analytic then similarly we construct a subsequence of points in E which has no Abel statistical Cauchy subsequence.

Theorem 4. *The sum of two triple sequence of Abel statistical ward continuous functions is Abel statistically ward continuous.*

Proof: Let f and g be two triple sequence be Abel statistical ward continuos functions on a subset E of $\mathbb{R}^3$, and (α_{mnk}) be an Abel rough statistical Cauchy sequence of points in E. Take any $\epsilon > 0$. Since f is triple sequence of Abel statistically ward continuous on E, we have

$$\lim_{x\to 1^-, y\to 1^-, z\to 1^-} (1-x)(1-y)(1-z) \sum_m \sum_n \sum_{k:|\Delta^3 f(\alpha_{mnk})|\geq\frac{\beta+\epsilon}{2}} \Delta^3 f(\alpha_{mnk}) x^m y^n z^k = 0;$$

since g is triple sequence of Abel rough statistically ward continuous on E,

$$\lim_{x\to 1^-, y\to 1^-, z\to 1^-}(1-x)(1-y)(1-z)\sum_m\sum_n\sum_{k:|\Delta^3 g(\alpha_{mnk})|\geq\frac{\beta+\epsilon}{2}}\Delta^3 g(\alpha_{mnk})x^m y^n z^k = 0.$$

Now it follows from the inequality

$$\sum_m\sum_n\sum_{k:|f+g(\alpha_{mnk}-\alpha_{m,n+1,k}-\alpha_{m,n,k+1}+\alpha_{m,n+1,k+1}-\alpha_{m+1,n,k}+\alpha_{m+1,n+1,k}+\alpha_{m+1,n,k+1}-\alpha_{m+1,n+1,k+1})|\geq\beta+\epsilon}\left(f+\Delta^3 g(\alpha_{mnk})\right)x^m y^n z^k$$

$$\leq \lim_{x\to 1^-, y\to 1^-, z\to 1^-}(1-x)(1-y)(1-z)\sum_m\sum_n\sum_{k:|\Delta^3 f(\alpha_{mnk})|\geq\frac{\beta+\epsilon}{2}}\Delta^3 f(\alpha_{mnk})x^m y^n z^k + \lim_{x\to 1^-, y\to 1^-, z\to 1^-}(1-x)(1-y)(1-z)\sum_m\sum_n\sum_{k:|\Delta^3 g(\alpha_{mnk})|\geq\frac{\beta+\epsilon}{2}}\Delta^3 g(\alpha_{mnk})x^m y^n z^k = 0+0 = 0.$$

Theorem 5. *If triple sequence of a function is Abel statistically ward continuous on a subset E of $\mathbb{R}^3$, then it is continuous on E.*

Proof: Suppose that a function f is not continuous on E so that there exist a triple convergent sequence (α_{rst}) with $\lim_{r,s,t\to\infty}\alpha_{rst} = \ell$ such that $\left(f(\alpha_{rst})\right)$ is not convergent to $f(\ell)$. If $\left(f(\alpha_{rst})\right)$ is bounded, then either $\left(f(\alpha_{rst})\right)$ has a limit different from $f(\ell)$, or there are at least two rough triple convergent subsequences of $\left(f(\alpha_{rst})\right)$ with different limits, which is a contradiction. If $\left(f(\alpha_{rst})\right)$ is unbounded above. Then we can find an $(r_1 s_1 t_1)$ such that $\left(f\left(\alpha_{r_1 s_1 t_1}\right)\right) > f(\alpha_{000}) + 1$. There exists a positive integer an $(r_2 s_2 t_2) > (r_1 s_1 t_1)$ such that $f\left(\alpha_{r_2 s_2 t_2}\right) > f\left(\alpha_{r_1 s_1 t_1}\right) + 2$. Suppose that we have chosen an $(r_{m-1} s_{n-1} t_{k-1}) > (r_{m-2} s_{n-2} t_{k-2})$ such that $f\left(\alpha_{r_{m-1} s_{n-1} t_{k-1}}\right) > f\left(\alpha_{r_{m-2} s_{n-2} t_{k-2}}\right) + 2^{(m-2)+(n-2)+(k-2)}$. Then we can choose an $(r_m s_n t_k) > r_{m-1} s_{n-1} t_{k-1}$ such that $f\left(\alpha_{r_m s_n t_k}\right) > f\left(\alpha_{r_{m-1} s_{n-1} t_{k-1}}\right) + 2^{(m-1)+(n-1)+(k-1)}$. Inductively we can construct a subsequence $\left(f\left(\alpha_{r_m s_n t_k}\right)\right)$ of $\left(f(\alpha_{rst})\right)$ such that $\left(f\left(\alpha_{r_{m+1} s_{n+1} t_{k+1}}\right)\right) > \left(f\left(\alpha_{r_m s_n t_k}\right)\right) + 2^{m+n+k}$, for each $m, n, k \in \mathbb{N}$. Since the rough triple sequence $\left(\alpha_{r_m s_n t_k}\right)$ is a subsequence of (α_{rst}), the subsequence $\left(\alpha_{r_m s_n t_k}\right)$ is convergent so is Abel rough statistically Cauchy. But $\left(f\left(\alpha_{r_m s_n t_k}\right)\right)$ is not Abel rough statistically Cauchy. For each $m, n, k \in \mathbb{N}$ we have $\Delta^3 f\left(\alpha_{r_m s_n t_k}\right) > 2^{m+n+k}$. The series $\sum_m\sum_n\sum_{k:|\Delta^3 f(\alpha_{mnk})|\geq 1}\Delta^3 f(\alpha_{mnk})x^m y^n z^k$ is convergent and equal to $\frac{1}{(1-x)(1-y)(1-z)}$ for any x, y, z satisfying $0 < x, y, z < 1$, so

$$\lim_{x\to 1^-, y\to 1^-, z\to 1^-}(1-x)(1-y)(1-z)\sum_m\sum_n\sum_{k:|\Delta^3 f(\alpha_{mnk})|\geq 1}\Delta^3 f(\alpha_{mnk})x^m y^n z^k = 1 \neq 0.$$

Thus the rough triple sequence $\left(f(\alpha_{mnk})\right)$ is not Abel statistically Cauchy. If $\left(f(\alpha_{mnk})\right)$ is unbounded below, similarly

$$\lim_{x\to 1^-, y\to 1^-, z\to 1^-}(1-x)(1-y)(1-z)\sum_m\sum_n\sum_{k:|\Delta^3 f(\alpha_{mnk})|\geq 1}\Delta^3 f(\alpha_{mnk})x^m y^n z^k \neq 0.$$

The contradiction for all possible cases to the Abel statistical ward continuity of f.

Remark 1. *The converse of the preceding theorem is not always true.*

Example 1. *Consider the function defined by* $f(x) = x^6$ *and the Abel rough statistical Cauchy sequence defined by* $\sqrt{rst}$.

Note 1. *Abel statistical ward continuity implies not only ordinary continuity, but also statistical continuity implies lacunary statistical sequential continuity implies* λ *statistical continuity implies* ρ*-statistical continuity and* G*-sequential continuity for any regular subsequential method* G.

Note 2. *Any continuous function on a compact subset* E *of* $\mathbb{R}^3$ *is uniformly continuous on* E. *We have an analogous theorem for an Abel statistical ward continuous function defined on an Abel statistical ward compact subset of* $\mathbb{R}^3$.

CONCLUSION

In this article, are explored some key features of rough I-convergence of a fuzzy triple array spaces in three-dimensional matrix spaces and the relationship between analyticity and the whole set of rough I-limits of triple array spaces. This article will be useful for future research, as well.

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