ORIGINAL PAPER

PREŠIĆ TYPE OPERATORS ON ORDERED VECTOR METRIC SPACES

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Abstract. *In this paper we present a fixed point theorem for order-preserving Prešić type operators on ordered vector metric spaces. This result extends many results in the literature obtained for Prešić type operators both on metric spaces and partially ordered metric spaces. We also emphasize the relationships between our work and the previous ones in the literature. Finally we give an example showing the fact that neither results for Prešić type contractions on metric spaces nor the results for ordered Prešić type contractions on ordered metric space is applicable to it.*

Keywords: *Fixed point; Riesz space; ordered metric spaces; vector metric space; Prešić type operators.*

1. INTRODUCTION

In different branches of mathematics and other sciences, fixed point theory has a great role to find a solution for a wide range of problems. Increasing numbers of authors have interested in this theory till Banach [1] and many generalizations have been made such as [2-4]. Ran and Reuring [5] pioneered to the trend weakening the requirement on the contraction with partial order. Nieto and Lopez followed the footprints of them and gave fixed point results for nondecreasing, nonincreasing and even not monotone contractions [6, 7]. Later Cevik and Altun [8] present vector metric space and generalized the results in [6, 7]. After that many authors have given fixed point results on vector metric space [9-14]. On the other hand, Prešić [15, 16] presented a contraction condition on a finite product space and gave the following result.

Theorem 1. *Let (X,ρ) be a complete metric space, k be a positive integer and the mapping $f\colon X^k \to X$ satisfies the following contractive type condition*

$$\rho\big(f(x_0,\dots,x_{k-1}),f(x_1,\dots,x_k)\big) \le \sum_{i=1}^{k}\beta_i\rho(x_{i-1},x_i)$$

for every $x_0,\dots,x_{k-1},x_k \in X$, where $\beta_i \in \mathbb{R}^+$ with $\sum_{i=1}^{k}\beta_i < 1$. Then there exists a unique point $x \in X$ such that $f(x,\dots,x) = x$. In addition to that if $x_0,\dots,x_{k-1}$ arbitrary points in X and the sequence (x_n) is defined as

$$x_{n+k} = f(x_n,,x_{n+1},\dots,x_{n+k-1})$$

then (x_n) is convergent and $\lim x_n = f(\lim x_n,,\lim x_{n+1},\dots,\lim x_{n+k-1})$.

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The Banach contraction mapping principle can be obtained in the case $k = 1$. That is, this theorem is a generalization of Banach fixed point theorem. Later many generalizations have been done with Prešić type contraction on ordered metric spaces such as [17-20]. In this work, we extend the results of [15, 16] by combining the ideas of [6, 8, 10, 18] on ordered vector metric spaces. As a result, we extend many other results in the literature. We also give an example on which the previous results can not be applicable and some remarks to emphasize the difference and connections between our work and the ones mentioned earlier.

Now, we give some useful concepts for our work. A lattice is a partially ordered set whose any two elements assume a supremum and an infimum. If a partially ordered set satisfies the vector space conditions, while the order relation is suitable for space operations, then it is called ordered vector space. Any ordered vector space is named Riesz space if it is also a lattice. Let $(E, \leqslant)$ be a Riesz space and $a \in E$, then the element $|a| = a \vee (-a)$ is called module (or absolute value) *of* a. The Riesz space E is Dedekind (σ-)complete if any subset bounded from above (below) has a supremum (infimum). Also, the Riesz space E is called Archimedean if $\frac{1}{n}a \downarrow 0$ for all $a \in E^+$, where the notation $a_n \downarrow a$ used to indicate that (a_n) is order-reversing and a is infimum of the set $\{a_n : n \in \mathbb{N}\}$. Any sequence (a_n) is o-convergent (order convergent) to a if there exists a sequence (b_n) in E such that $b_n \downarrow 0$ and $|a_n - a| \leqslant b_n$ for all n, and it is written as $a_n \overset{o}{\to} a$. For other facts and concepts related with Riesz space we reefer [21].

Now, let us remind some concepts in [8]. Let X be a nonempty set and E be a Riesz space. If a map $\rho: X \times X \to E$ satisfies the conditions

$$\rho(x, y) = 0 \Leftrightarrow x = y, \qquad for\ all\ x, y \in X$$

$$\rho(x, y) \leqslant \rho(x, z) + \rho(y, z), \qquad for\ all\ x, y, z \in X$$

then it is called vector metric (or E-metric). Also (X, ρ, E) (shortly X) is called vector metric space (or E-metric space). In addition to that, if X is an ordered set, then the triple (X, ρ, E) is named as ordered vector metric space. Clearly, every metric space is a vector metric space, since $\mathbb{R}$ is a Riesz space with usual order. For any vector metric space X, a sequence (x_n) vectorially converges (or E-converges) to $x \in X$ if there exists a sequence (b_n) in E such that $b_n \downarrow 0$ and $\rho(x_n, x) \leqslant b_n$ for all n. Also, a sequence (x_n) is E-Cauchy if there is a sequence (b_n) in E such that $b_n \downarrow 0$ and $\rho(x_{n+p}, x_n) \leqslant b_n$ for all n,p. As expected X is said to be E-complete if every E-Cauchy sequence in X is E-convergent.

Note that on a vector metric space X, a sequence (x_n) E-converges to a point x iff $\rho(x_n, x)$ o-converges to 0. Also, (x_n) is E-Cauchy iff $\rho(x_{n+p}, x_n)$ o-converges to 0 for all n, p. Another important concept is vectorial continuity. Let (X, ρ_1, E_1) and (Y, ρ_2, E_2) be two vector metric spaces. A function $f: X \to Y$ satisfying $f(x_n) \overset{\rho_2, E_2}{\to} f(x)$ whenever $x_n \overset{\rho_1, E_1}{\to} x$ is called vectorially continuous [22].

In [18] the ordered Prešić type contraction is defined. Similarly, when we say a function $f: X^k \to X$ is ordered Prešić type contraction on ordered vector metric space we mean that it satisfies the condition

$$\rho\big(f(x_0, \dots, x_{k-1}), f(x_1, \dots, x_k)\big) \leqslant \sum_{i=1}^{k} \beta_i \rho(x_{i-1}, x_i) \qquad (1)$$

where $x_0, \dots, x_{k-1}, x_k \in X$ with $x_0 \preceq x_1 \preceq \cdots \preceq x_{k-1} \preceq x_k$ and $\beta_i \in \mathbb{R}^+$ for $1 \leq i \leq k$, $\sum_{i=1}^{k} \beta_i < 1$.

2. MAIN PART

Now we state our main theorem. Through the rest of the work, we assume $(X, \preceq)$ is an E-complete ordered vector metric space and $(E, \leqslant)$ is an Archimedean Riesz space.

Theorem 2 *Let k be a positive integer and $f: X^k \to X$ is an ordered Presić contraction where $\beta_i \in \mathbb{R}^+$ with $\sum_{i=1}^{k} \beta_i < 1$. Suppose that f is order-preserving and one of the followings is satisfied*

(i) f is vectorially continuous

(ii) for any order-preserving sequence (x_n) in X, if $x_n \overset{\rho,E}{\to} x$ then $x_n \preceq x$ for all n.

If there exist $x_0, \dots, x_{k-1}, x_k \in X$ satisfying $x_0 \preceq x_1 \preceq \cdots \preceq x_{k-1} \preceq x_k = f(x_0, \dots, x_{k-1})$ then the associate operator $F: X \to X$ defined as $F(x) = f(x, \dots x)$ has a fixed point. Moreover, the fixed point is unique iff the set $A = \{x \in X: F(x) = x\}$ is well ordered. Also, the sequence (w_n) defined by $w_n = F(w_{n-1})$ vectorially converges to x, the fixed point of F, for any $w_0 \in X$ satisfying $w_0 \preceq x$.

Proof: Let define the sequence (x_n) as $x_k = f(x_0, \dots, x_{k-1})$ for given elements $x_0, \dots, x_{k-1}$ and $x_0 \preceq x_1 \preceq \cdots \preceq x_{k-1} \preceq x_k = f(x_0, \dots, x_{k-1})$. Since the function f is order-preserving, its clear that $x_k = f(x_0, \dots, x_{k-1}) \preceq f(x_1, \dots, x_k) = x_{k+1}$. By following this procedure we get

$$x_0 \preceq x_1 \preceq \cdots \preceq x_{k-1} \preceq x_k \preceq \cdots \preceq x_{k+n-1} \preceq x_{k+n} \preceq \cdots \tag{2}$$

where

$$x_{k+n} = f(x_n, \dots, x_{n+k-1})$$

for all $n \in \mathbb{N}$. We deduce that (x_n) is an order-preserving sequence.

On the other hand, since f is an order Presić contraction, it satisfies

$$\rho\big(f(x_0, \dots, x_{k-1}), f(x_1, \dots, x_k)\big) \leqslant \textstyle\sum_{i=1}^{k} \beta_i \rho(x_{i-1}, x_i)$$

where $x_0, \dots, x_{k-1}, x_k \in X$ with $x_0 \preceq x_1 \preceq \cdots \preceq x_{k-1} \preceq x_k$ and $\beta_i \in [0,1)$. Similar deductions can be made for any $n \geq k$. That is, for any n $(\geq k)$ we have

$$\rho\big(f(x_{n-k}, \dots, x_{n-1}), f(x_{n-k+1}, \dots, x_n)\big) \leqslant \textstyle\sum_{i=1}^{k} \beta_i \rho(x_{n-k+i-1}, x_{n-k+i}).$$

Now we show that $\rho(x_n, x_{n+1}) \leqslant \mathcal{S}\Omega^n$ where

$$\Omega = \left[\textstyle\sum_{j=1}^{k} \beta_j\right]^{1/k} \text{ and } \mathcal{S} = \sup\left\{\frac{\rho(x_{j-1}, x_j)}{\Omega^j} : 1 \leq j \leq k\right\}.$$

We use mathematical induction for this aim. It is easy to see that $\rho(x_n, x_{n+1}) \leqslant \mathcal{S}\Omega^n$ is satisfied for $n = 1, \dots, k$. Assume the following inequalities are also satisfied;

$$\rho(x_n, x_{n+1}) \leqslant \mathcal{S}\Omega^n, \rho(x_{n+1}, x_{n+2}) \leqslant \mathcal{S}\Omega^{n+1}, \dots, \rho(x_{n+k-1}, x_{n+k}) \leqslant \mathcal{S}\Omega^{n+k-1}.$$

By (1) and (2) we have

$$\rho(x_{n+k}, x_{n+k+1}) = \rho\big(f(x_n, \dots, x_{n+k-1}), f(x_{n+1}, \dots, x_{n+k})\big)$$

$$\leqslant \sum_{i=1}^{k} \beta_i \rho(x_{n+i-1}, x_{n+i})$$

$$\leqslant \sum_{i=1}^{k} \beta_i \mathcal{S}\Omega^{n+i-1}$$

$$= \mathcal{S}\Omega^{n} \sum_{i=1}^{k} \beta_i \Omega^{i-1}$$

$$\leqslant \mathcal{S}\Omega^{n} \sum_{i=1}^{k} \beta_i \qquad \text{(since } \Omega = \big[\textstyle\sum_{j=1}^{k} \beta_j\big]^{1/k} < 1\text{)}$$

$$= \mathcal{S}\Omega^{n+k}$$

that completes the induction.

Now, for any $n, p \in \mathbb{N}$ we have

$$\rho\big(x_n, x_{n+p}\big) \leqslant \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \cdots + \rho\big(x_{n+p-1}, x_{n+p}\big)$$

$$\leqslant \mathcal{S}\Omega^{n} + \mathcal{S}\Omega^{n+1} + \cdots + \mathcal{S}\Omega^{n+p-1}$$

$$\leqslant \mathcal{S}\Omega^{n}[1 + \Omega + \Omega^2 + \cdots + \Omega^{p-1}]$$

$$= \mathcal{S}\Omega^{n} \frac{1-\Omega^p}{1-\Omega}$$

$$\leqslant \mathcal{S} \frac{\Omega^n}{1-\Omega}$$

As $\mathcal{S}\frac{\Omega^n}{1-\Omega} \downarrow 0$, (x_n) is an E-Cauchy sequence. By E-completeness of X, there exists an element x in X such that $x_n \overset{\rho,E}{\to} x$. That is, there exists a sequence (a_n) such that $\rho(x_n, x) \leqslant a_n$ for all n and $a_n \downarrow 0$. Now we define the associate operator $F: X \to X$ as $F(x) = f(x, \dots x)$ and investigate two cases separately.

(i) Let f be vectorially continuous. Hence, there exists a sequence (b_n) satisfying $b_n \downarrow 0$ and $\rho\big(f(x_n, \dots, x_n), f(x, \dots, x)\big) \leqslant b_n$ for all n where $(x_n, \dots, x_n)$ is a sequence in X^k and $(x_n) \overset{\rho,E}{\to} (x)$. Thus

$$\rho(F(x), x) = \rho(f(x, \dots, x), x)$$

$$\leqslant \rho(f(x_n, \dots, x_n), x) + \rho\big(f(x_n, \dots, x_n), f(x, \dots, x)\big)$$

$$= \rho(x_{n+1}, x) + \rho\big(f(x_n, \dots, x_n), f(x, \dots, x)\big)$$

$$\leqslant a_{n+1} + b_n$$

for all n. Since $a_{n+1} + b_n \downarrow 0$ we have $F(x) = x$.

(ii) For any order-preserving sequence (x_n) in X, let $x_n \preceq x$ for all n whenever $x_n \overset{\rho,E}{\to} x$. Since

$$\rho(x, F(x)) \leqslant \rho(x, x_{n+k}) + \rho(x_{n+k}, F(x))$$

$$= \rho(x, x_{n+k}) + \rho(f(x_n, \dots, x_{n+k-1}), f(x, \dots, x))$$

$$\leqslant \rho(x, x_{n+k}) + \rho(f(x_n, \dots, x_{n+k-1}), f(x_{n+1}, \dots, x_{n+k-1}, x))$$

$$+\rho(f(x_{n+1}, \dots, x_{n+k-1}, x), f(x_{n+2}, \dots, x_{n+k-1}, x, x))$$

$$+\rho(f(x_{n+2}, \dots, x_{n+k-1}, x), f(x_{n+3}, \dots, x_{n+k-1}, x, x, x))$$

$$+ \cdots + \rho(f(x_{n+k-1}, x, \dots, x), f(x, \dots, x))$$

by using (1) we obtain

$$\rho(x, F(x)) \leqslant \rho(x, x_{n+k}) + \left[\sum_{i=1}^{k-1} \beta_i \rho(x_{n+i-1}, x_{n+i}) + \rho(x_{n+k-1}, x)\right]$$

$$+\left[\sum_{i=1}^{k-2} \beta_i \rho(x_{n+i}, x_{n+i+1}) + \rho(x_{n+k-1}, x)\right]$$

$$+\left[\sum_{i=1}^{k-3} \beta_i \rho(x_{n+i+1}, x_{n+i+2}) + \rho(x_{n+k-1}, x)\right] \dots + \rho(x_{n+k-1}, x)$$

$$\leqslant \rho(x, x_{n+k}) + \beta_1 \rho(x_n, x_{n+1}) + [\textstyle\sum_{i=1}^{2} \beta_i] \rho(x_{n+1}, x_{n+2})$$

$$+[\textstyle\sum_{i=1}^{3} \beta_i] \rho(x_{n+2}, x_{n+3}) + \cdots + \left[\textstyle\sum_{i=1}^{k-1} \beta_i\right] \rho(x_{n+k-2}, x_{n+k-1})$$

$$+\left[\textstyle\sum_{i=1}^{k} \beta_i\right] \rho(x_{n+k-1}, x).$$

If we use the fact that $\sum_{i=1}^{k} \beta_i < 1$, then we get

$$\rho(x, F(x)) \leqslant \rho(x, x_{n+k}) + \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2})$$

$$+ \cdots + \rho(x_{n+k-2}, x_{n+k-1}) + \rho(x_{n+k-1}, x)$$

$$\leqslant a_{n+k} + \mathcal{S}\Omega^n + \mathcal{S}\Omega^{n+1} + \mathcal{S}\Omega^{n+2}$$

$$+ \cdots + \mathcal{S}\Omega^{n+k-1} + a_{n+k-1}$$

Since $a_n \downarrow 0$ and $\mathcal{S}\Omega^n \downarrow 0$ we obtain $F(x) = x$.

For uniqueness part let $A = \{x \in X : F(x) = x\}$ be well ordered. So A has a least element and any $s, t \in A$ are comparable. Without loss of generality assume $s \preceq t$. By definition F, $s = f(s, \dots s)$, $f = f(t, \dots, t)$ and we have

$$\rho(s, t) = \rho(f(s, \dots, s), f(t, \dots, t))$$

$$\leqslant \rho(f(s, \dots, s), f(s, \dots s, t)) + \rho(f(s, \dots, s, t), f(s, \dots, s, t, t))$$

$$+\cdots+\rho\big(f(s,t,\dots,t),f(t,\dots,t)\big)$$

$$\leqslant \beta_k\rho(s,t)+\beta_{k-1}\rho(s,t)+\cdots+\beta_1\rho(s,t)$$

$$=\left[\sum_{i=1}^{k}\beta_i\right]\rho(s,t).$$

Since $\left[\sum_{i=1}^{k}\beta_i\right]<1$, we deduce that $\rho(s,t)=0$, which means that $s=t$. Hence, the fixed point is unique. For backward, if F has unique fixed point, then A has only one element. Naturally A is well ordered. For the last part, let x be the fixed point of F. Now define a sequence (w_n) as $w_n=F(w_{n-1})$ for all n and choose an element w_0 satisfying $w_0\preceq x$. Since f is order-preserving, so does F and $w_n\preceq x$ for all n. As

$$\rho(w_n,x)=\rho\big(f(w_{n-1},\dots,w_{n-1}),f(x,\dots,x)\big)$$

$$\leqslant \rho\big(f(w_{n-1},\dots,w_{n-1}),f(w_{n-1},\dots w_{n-1},x)\big)$$

$$+\cdots+\rho\big(f(w_{n-1},x,\dots,x),f(x,\dots,x)\big)$$

$$=\left[\sum_{i=1}^{k}\beta_i\right]\rho(w_{n-1},x)$$

if we repeat this action n times, we get

$$\rho(w_n,x)\leqslant\left[\sum_{i=1}^{k}\beta_i\right]^{n}\rho(w_0,x)=\Omega^{nk}\rho(w_0,x).$$

That means the sequence (w_n) vectorially converges to x. □

We know that Theorem 1 coincides with the Banach contraction mapping principle in the case $k=1$. Similar deductions can be made for theorems in [10] and [6] which are also generalizations of classical Banach theorem.

Remark 1. *Theorem 2 generalizes many results in the literature. For* instance, *if we assume $k=1$ in this theorem, then the results for order-preserving functions in [10] can be obtained. In addition to choice of $k=1$, if we add the condition E is real numbers equipped with usual order then the results for nondecreasing functions in [6] can be obtained.*

Remark 2. *Let X be an E-complete ordered vector metric space. Particularly, if we assume $E=\mathbb{R}$, then X become complete partially ordered metric space. In this case, our result coincides with the results for ordered Prešić type contraction on ordered metric spaces such as Corollary 17 in [19].*

The next example is crucial because it shows that our result is applicable for this example, while neither Theorem 1 nor the results for Prešić type contraction on ordered metric spaces are applicable.

Example 1. *Let $E=\mathbb{R}^2$ be equipped with coordinatewise ordering while $X=\{0,1,2\}$ is equipped with an order relation defined as*

$$x\preceq y\Leftrightarrow x\leq y,\ \ x,y\in\{0,1\}\text{ or x = y = 2}$$

where $\leq$ is usual order. E is Archimedean Riesz space and the set X is E-complete ordered vector metric space with the map $\rho: X \times X \to E$ defined as $\rho(x,y) = (2|x-y|, 3|x-y|)$. Let the function $f: X^2 \to X$ defined as

$$f(x,y) = \begin{cases} 2 & , \quad (x,y) = (1,2) \text{ or } (x,y) = (2,0) \\ 1 & , \quad (x,y) = (2,1) \text{ or } (x,y) = (0,2) \\ 0 & , \quad \text{otherwise} \end{cases}$$

It is clear that f is order-preserving and there are only 5 possible cases for $x_0, x_1, x_2 \in X$ satisfying $x_0 \preceq x_1 \preceq x_2$ .These cases are;

$$x_0 = x_1 = x_2 = 0$$

$$x_0 = x_1 = 0, x_2 = 1$$

$$x_0 = 0, x_1 = x_2 = 1$$

$$x_0 = x_1 = x_2 = 1$$

$$x_0 = x_1 = x_2 = 2$$

In each cases f satisfies

$$\rho\big(f(x_0,x_1), f(x_1,x_2)\big) \leqslant \tfrac{1}{3}\rho(x_0,x_1) + \tfrac{1}{2}\rho(x_1,x_2).$$

Also if (x_n) in X is an order-preserving sequence, then $x_n \preceq x$ for all $n \in \mathbb{N}$. Hence, by taking $k = 2$, $\beta_1 = 1/3$ and $\beta_2 = 1/2$ we see that all hypothesis of Theorem 2 are satisfied and 0 is the only fixed point of f. However, Theorem 1 can not be applicable to this example. Because, if we assume $x_0 = 0$, $x_1 = 1$, $x_2 = 2$ we see that $\rho\big(f(x_0,x_1), f(x_1,x_2)\big) = (4,6)$ while $\frac{1}{3}\rho(x_0,x_1) + \frac{1}{2}\rho(x_1,x_2) = \left(\frac{5}{6}, \frac{5}{6}\right)$. Moreover, we can not apply the results of Prešić type contraction given on ordered metric space such as Corallary 17 in [19] to this example.

4. CONCLUSIONS

To summarize, we presented a fixed point theorem for order-preserving Prešić type operators on ordered vector metric spaces. With our result, we extended many results in literature obtained for Prešić type operators both on metric spaces and partially ordered metric spaces. We also made connections between those and our result to ephasize the differences. Finaly we gave an example showing that previous results in the literature can not be applicable to it.

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