ORIGINAL PAPER

NEW FAMILIES OF SOLUTIONS FOR THE SPACE-TIME FRACTIONAL BURGERS' EQUATION

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Manuscript received: 03.08.2021; Accepted paper: 15.10.2021;
Published online: 30.12.2021.

Abstract. *In this paper, the hyperbolic tangent function method is applied for constructing exact solutions for space-time conformal fractional Burgers' equation. Furthermore, the space-time conformal fractional Burgers' equation is tested for the Painlevé property, and consequently, new numerous exact solutions are generated via Bäcklund transform.*

Keywords: *Bäcklund transform; conformal fractional Burgers' equation; fractional differential equations; hyperbolic tangent function method; Painlevé property.*

1. INTRODUCTION

Recently, fractional Calculus has made a very important impact on most fields of science, such as mathematics, engineering, physics, economics, etc. The applications of fractional calculus are contemporary [1-4].

Historically, the fractional derivatives were introduced in different ways, for example, Riemann–Liouville, Riesz, Caputo, Modified Riemann–Liouville [5-7]. A new fractional derivative is defined by the authors [8], named conformal fractional derivative (CFD). The definition and basic concepts of CFD are developed by [9]. Furthermore, the interpretations of CFD in engineering and physics applications are introduced and discussed [10].

The definition of CFD of a function $f: (0, \infty) \to \mathbb{R}$ of order α, where $0 < \alpha \leq 1$ is mainly given by the following limit

$$D^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

The function f is said to be α- differentiable. The fractional derivative at $t = 0$ is given as

$$D^{\alpha} f(0) = \lim_{t \to 0^+} D^{\alpha} f(t)$$

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The advantages of CFD are that satisfying the properties of the classical integer derivatives [8-10]. Assume that f and g are α- differentiable functions and λ, a, b are constants, then the CFD satisfies the following properties:

i. $D^\alpha t^p = pt^{p-\alpha}, p \in \mathbb{R}$.

ii. $D^\alpha \lambda = 0$.

iii. $D^\alpha f(t) = t^{1-\alpha} \dfrac{df}{dt}$.

iv. $D^\alpha (af + bg) = aD^\alpha f + bD^\alpha g$.

v. $D^\alpha (fg) = fD^\alpha g + gD^\alpha f$.

vi. $D^\alpha \left(\dfrac{f}{g}\right) = \dfrac{gD^\alpha f - fD^\alpha g}{g^2}$.

vii. $D^\alpha \Big(f\big(g(t)\big)\Big) = \dfrac{df}{dg} D^\alpha g(t) = t^{1-\alpha} \dfrac{df}{dg}\dfrac{dg}{dt}$.

In the last few years, there are considerable and numerous models in the literature are formulated in terms of conformable fractional derivatives [11-15].

In mathematical physics and many other phenomena in various fields of applied science are described by nonlinear models, particularly by nonlinear partial differential equations (NLPDEs), such as astrophysics, optics, fluid dynamics, mathematical biology, plasma physics, and so on. It is important to search for the solutions of the concerned models to understand and interpret their physical mechanisms and behavior. The search for exact solutions to NLPDE has become of great interest to mathematicians and physicists and they have exerted great efforts for that. There are many powerful and efficient methods for finding exact solutions that have been introduced, here we mention some of them as the Bäcklund transformation [16-18], hyperbolic tangent function method [19, 20], Jacobi elliptic function method [21, 22], truncated Painlevé expansion method [23-25], and there are other various methods in the literature.

The Bäcklund transformations (BT) are considered as powerful tools for integrable systems to relate NLPDEs and their solutions [16-18, 26]. Up to now, the research is still devoted and ongoing for finding the BT, e.g. from the Painlevé property [23-25], the Ablowitz-Kaup-Newell-Segur (AKNS) system [26], the nonclassical symmetries [27].

One of the simple NLPDEs in mathematical physics is Burgers' equation that arises in many areas of science such as Navier-Stokes equations, traffic flow, and acoustics [28, 29]. Consider the space-time conformable fractional Burgers' equation (CFBE)

$$D_t^\alpha u + uD_x^\alpha u = \sigma D_x^{\alpha\alpha} u, \tag{1}$$

where σ is arbitrary constants, $\alpha \in (0,1]$, and $D_x^{\alpha\alpha} u = D_x^\alpha (D_x^\alpha u)$. A few years ago, many researchers are introduced the solutions of CFBE by different methods. The Hopf-Cole transform is applied to a time CFBE, subsequently, the approximate analytical solution is founded by applying a Homotopy Analysis Method [11]. The residual power series method is introduced for finding approximate solutions of a time CFBE [30]. Also, the residual power series method is used in finding the solution of the space-time conformable fractional KdV-Burgers equation [31]. The solution of regular and singular space-time coupled CFBEs is formulated by applying the double Laplace transform [32].

The rest of the paper is organized as follows: In Sec. 2, we show that the space-time CFBE possesses the Painlevé property. In Sec. 3 the exact solutions for the space-time CFBE are constructed based on the hyperbolic tangent function method. In Sec. 4, the BT is used for generating abundant new exact solutions for the space-time CFBE.

2. PAINLEVÉ PROPERTY FOR THE SPACE-TIME CONFORMAL FRACTIONAL BURGERS' EQUATION

In this section, we intend to test the Painlevé property for the space-time CFBE given by Eq. (1) following the approach introduced by [23-25, 33]. The space-time CFBE has the Painlevé property when all the movable singularities are simple poles. For Eq. (1) we let

$$u = u\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\alpha}}{\alpha}\right) = \phi^{n} \sum_{j=0}^{\infty} \phi^{j} u_{j}, \tag{2}$$

where $u_0 \neq 0$, n is an integer, $\phi = \phi\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\alpha}}{\alpha}\right)$ and $u_j = u_j\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\alpha}}{\alpha}\right)$ are analytic functions of $\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\alpha}}{\alpha}\right)$ in a neighborhood of $M = \left\{\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\alpha}}{\alpha}\right) : \phi\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\alpha}}{\alpha}\right) = 0\right\}$.

To determine the values of n we consider the ansätz

$$u = \phi^{n} u_0. \tag{3}$$

By using the chain rule on Eq. (3) we obtain

$$\left.\begin{aligned} D_t^{\alpha} u &= n u_0 \phi^{n-1} D_t^{\alpha} \phi + \phi^{n} D_t^{\alpha}\, u_0 \\ D_x^{\alpha} u &= n u_0 \phi^{n-1} D_x^{\alpha} \phi + \phi^{n} D_x^{\alpha}\, u_0 \\ D_x^{\alpha\alpha} u &= n u_0 \phi^{n-1} D_x^{\alpha\alpha} \phi \\ &\quad + n(n-1) u_0 \phi^{n-2} (D_x^{\alpha} \phi)^2 \\ &\quad + 2n\phi^{n-1} D_x^{\alpha} \phi D_x^{\alpha}\, u_0 + \phi^{n} D_x^{\alpha\alpha}\, u_0. \end{aligned}\right\} \tag{4}$$

By substituting Eqs. (3) and (4) into Eq. (1), we get

$$\begin{aligned} & n u_0 \phi^{n-1} D_t^{\alpha} \phi + \phi^{n} D_t^{\alpha}\, u_0 + n u_0^2 \phi^{2n-1} D_x^{\alpha} \phi + u_0 \phi^{2n} D_x^{\alpha}\, u_0 \\ &= n\sigma u_0 \phi^{n-1} D_x^{\alpha\alpha} \phi + n(n-1)\sigma u_0 \phi^{n-2} (D_x^{\alpha} \phi)^2 \\ &+ 2n\sigma \phi^{n-1} D_x^{\alpha} \phi D_x^{\alpha}\, u_0 + \sigma \phi^{n} D_x^{\alpha\alpha}\, u_0. \end{aligned} \tag{5}$$

The dominant terms of Eq. (5) are ϕ^{2n-1} & ϕ^{n-2}. Balancing of these gives $n = -1$. Thus,

$$u_0 = -2\sigma D_x^{\alpha} \phi. \tag{6}$$

Back to Eq. (2), we can consider the following Ansätz with resonance r as

$$u = u_0 \phi^{-1} + b\phi^{r-1} = -2\sigma \phi^{-1} D_x^{\alpha} \phi + b\phi^{r-1}. \tag{7}$$

From Eq. (7) we get

$$\left.\begin{aligned} D_x^{\alpha} u &= 2\sigma\phi^{-2}(D_x^{\alpha}\phi)^2 - 2\sigma\phi^{-1}D_x^{\alpha\alpha}\phi \\ &\quad + b(r-1)\phi^{r-2}D_x^{\alpha}\phi \\ &= 2\sigma\phi^{-2}(D_x^{\alpha}\phi)^2 + b(r-1)\phi^{r-2}D_x^{\alpha}\phi + \cdots \\ D_x^{\alpha\alpha} u &= -4\sigma\phi^{-3}(D_x^{\alpha}\phi)^3 + 4\sigma\phi^{-2}D_x^{\alpha}\phi D_x^{\alpha\alpha}\phi \\ &\quad + 2\sigma\phi^{-2}D_x^{\alpha}\phi D_x^{\alpha\alpha}\phi - 2\sigma\phi^{-1}D_x^{\alpha\alpha\alpha}\phi \\ &\quad + b(r-1)(r-2)\phi^{r-3}(D_x^{\alpha}\phi)^2 \\ &\quad + b(r-1)\phi^{r-2}D_x^{\alpha\alpha}\phi \\ &= -4\sigma\phi^{-3}(D_x^{\alpha}\phi)^3 \\ &\quad + b(r-1)(r-2)\phi^{r-3}(D_x^{\alpha}\phi)^2 + \cdots \end{aligned}\right\} \tag{8}$$

The fractional derivatives are rearranged in terms of powers of $D_x^{\alpha}\phi$. Now by substituting Eqs. (7) and (8) into $uD_x^{\alpha}u = \sigma D_x^{\alpha\alpha}u$, we have

$$(-2\sigma\phi^{-1}D_x^{\alpha}\phi + b\phi^{r-1})(2\sigma\phi^{-2}(D_x^{\alpha}\phi)^2 + b(r-1)\phi^{r-2}D_x^{\alpha}\phi + \cdots)$$

$$= \sigma(-4\sigma\phi^{-3}(D_x^{\alpha}\phi)^3 + b(r-1)(r-2)\phi^{r-3}(D_x^{\alpha}\phi)^2 + \cdots)$$

$$\Rightarrow -2b(r-1)\sigma\phi^{r-3}(D_x^{\alpha}\phi)^2 + 2b\sigma\phi^{r-3}(D_x^{\alpha}\phi)^2$$

$$= \sigma b(r-1)(r-2)\phi^{r-3}(D_x^{\alpha}\phi)^2 + \cdots$$

$$\Rightarrow (-2r + 2 + 2 - r^2 + 3r - 2)\sigma b\phi^{r-3}(D_x^{\alpha}\phi)^2 = \cdots$$

$$\Rightarrow -(r+1)(r-2)\sigma b\phi^{r-3}(D_x^{\alpha}\phi)^2 = \cdots$$

$\Rightarrow r = -1$ and $r = 2$.

If M is a singularity manifold, it is obtained that $n = -1$. By leading order analysis,

$$u = \phi^{-1}\sum_{j=0}^{\infty}\phi^j u_j = \sum_{j=0}^{\infty}\phi^{j-1}u_j. \tag{9}$$

From (9) we get

$$D_t^{\alpha}u = \sum_{j=0}^{\infty}\left[(j-1)\phi^{j-2}u_j D_t^{\alpha}\phi + \phi^{j-1}D_t^{\alpha}u_j\right]$$

$$= \sum_{j=0}^{\infty}\left[(j-2)\phi^{j-3}u_{j-1}D_t^{\alpha}\phi + \phi^{j-3}D_t^{\alpha}u_{j-2}\right],$$

$$D_x^{\alpha}u = \sum_{j=0}^{\infty}\left[(j-1)\phi^{j-2}u_j D_x^{\alpha}\phi + \phi^{j-1}D_x^{\alpha}u_j\right]$$

$$= \sum_{j=0}^{\infty}\left[(j-1)\phi^{j-2}u_j D_x^{\alpha}\phi + \phi^{j-2}D_x^{\alpha}u_{j-1}\right],$$

$$D_x^{\alpha\alpha}u = \sum_{j=0}^{\infty}\begin{bmatrix}(j-1)(j-2)u_j\phi^{j-3}(D_x^{\alpha}\phi)^2 \\ +(j-1)\phi^{j-2}u_j D_x^{\alpha\alpha}\phi \\ +2(j-1)\phi^{j-2}D_x^{\alpha}\phi D_x^{\alpha}u_j \\ +\phi^{j-1}D_x^{\alpha\alpha}u_j\end{bmatrix}$$

$$= \sum_{j=0}^{\infty} \begin{bmatrix} (j-1)(j-2)u_j\phi^{j-3}(D_x^\alpha\phi)^2 \\ +(j-2)\phi^{j-3}u_{j-1}D_x^{\alpha\alpha}\phi \\ +2(j-2)\phi^{j-3}D_x^\alpha\phi D_x^\alpha\, u_{j-1} \\ +\phi^{j-3}D_x^{\alpha\alpha}\, u_{j-2} \end{bmatrix}.$$

The recursion relations for u_j are found to be

$$D_t^\alpha u_{j-2} + (j-2)u_{j-1}D_t^\alpha\phi + \sum_{m=0}^{j} u_{j-m}[D_x^\alpha\, u_{m-1} + (m-1)u_m D_x^\alpha\phi]$$

$$= \sigma \begin{bmatrix} (j-1)(j-2)u_j(D_x^\alpha\phi)^2 + (j-2)u_{j-1}D_x^{\alpha\alpha}\phi \\ +2(j-2)D_x^\alpha\phi D_x^\alpha\, u_{j-1} + D_x^{\alpha\alpha}\, u_{j-2} \end{bmatrix}, \tag{10}$$

where $u_k = 0$ for $k = -1, -2, -3, \ldots$

From the recurrence formula (10) we obtain for:

$$j = 0 \Longrightarrow -u_0^2 D_x^\alpha\phi = 2\sigma u_0(D_x^\alpha\phi)^2$$

$$\Longrightarrow u_0 = -2\sigma D_x^\alpha\phi. \tag{11}$$

Eq. (11) is identical to Eq. (6).

$j = 1 \Longrightarrow$

$$-u_0 D_t^\alpha\phi + \sum_{m=0}^{1} u_{1-m}[D_x^\alpha\, u_{m-1} + (m-1)u_m D_x^\alpha\phi]$$

$$= -\sigma[u_0 D_x^{\alpha\alpha}\phi + 2D_x^\alpha\phi D_x^\alpha\, u_0]$$

$$\Longrightarrow -u_0 D_t^\alpha\phi - u_0 u_1 D_x^\alpha\phi + u_0 D_x^\alpha\, u_0 = 6\sigma^2 D_x^\alpha\phi D_x^{\alpha\alpha}\phi$$

$$\Longrightarrow D_t^\alpha\phi + u_1 D_x^\alpha\phi - \sigma D_x^{\alpha\alpha}\phi = 0. \tag{12}$$

$j = 2$ and considering Eqs. (11, 12), then from Eq. (10), we get

$$D_t^\alpha u_0 + \sum_{m=0}^{2} u_{2-m}[D_x^\alpha\, u_{m-1} + (m-1)u_m D_x^\alpha\phi] = \sigma D_x^{\alpha\alpha}\, u_0$$

$$\Longrightarrow D_x^\alpha(D_t^\alpha\phi + u_1 D_x^\alpha\phi - \sigma D_x^{\alpha\alpha}\phi) = 0. \tag{13}$$

By Eq. (12) the compatibility condition Eq. (13) at $j = 2$ is satisfied identically. For $j = 3$, Eq. (10) yields

$$D_t^\alpha u_1 + u_2 D_t^\alpha\phi + \sum_{m=0}^{3} u_{3-m}[D_x^\alpha\, u_{m-1} + (m-1)u_m D_x^\alpha\phi]$$

$$= \sigma[2u_3(D_x^\alpha\phi)^2 + u_2 D_x^{\alpha\alpha}\phi + 2D_x^\alpha\phi D_x^\alpha\, u_2 + D_x^{\alpha\alpha}\, u_1]$$

$$\Longrightarrow [D_t^\alpha u_1 + u_1 D_x^\alpha\, u_1 - \sigma D_x^{\alpha\alpha}\, u_1]$$

$$+[u_2 D_x^\alpha\, u_0 + 2u_0 D_x^\alpha\, u_2] + 2u_3 u_0 D_x^\alpha\phi = 0. \tag{14}$$

Since the resonances occur at $r = -1, 2$, and (ϕ, u_2) are arbitrary functions of $\left(\frac{x^\alpha}{\alpha}, \frac{t^\alpha}{\alpha}\right)$ in the expansion (14). If we let the arbitrary functions $u_2 = u_3 = 0$, then we get

$$D_t^\alpha u_1 + u_1 D_x^\alpha u_1 = \sigma D_x^{\alpha\alpha} u_1. \tag{15}$$

For $j = 3$ then Eq. (15) is automatically satisfied for u_1.
For $j = 4$, $u_0 = -2\sigma D_x^\alpha \phi$, and $u_2 = u_3 = 0$ then from Eq. (10) we obtain

$$\sum_{m=0}^{4} u_{4-m}[D_x^\alpha u_{m-1} + (m-1) u_m D_x^\alpha \phi] = 6\sigma u_4 (D_x^\alpha \phi)^2$$

$$\Rightarrow u_4 D_x^\alpha \phi (2u_0 - 6\sigma D_x^\alpha \phi) = 0 \Rightarrow u_4 = 0.$$

Thus, we conclude that all

$$u_j = 0, j \geq 2, \tag{16}$$

providing u_1 satisfies space-time CFBE (Eq. (15)). Now, we can conclude that from Eq. (16), the space-time CFBE possesses the Painlevé property and the truncation of the Painlevé expansion Eq. (9) takes the form

$$u = -\frac{2\sigma}{\phi} D_x^\alpha \phi + u_1, \tag{17}$$

which is the Bäcklund transform for the space-time CFBE. When $u_1 = 0$, then from Eq. (17) we obtain

$$u = -\frac{2\sigma}{\phi} D_x^\alpha \phi, \tag{18}$$

Eq. (18) yields the fractional Cole-Hopf transform. When $u_1 = \phi$, then we have

$$u = -\frac{2\sigma}{\phi} D_x^\alpha \phi + \phi, \tag{19}$$

where

$$D_t^\alpha \phi + \phi D_x^\alpha \phi = \sigma D_x^{\alpha\alpha} \phi, \tag{20}$$

Eqs. (19) and (20) represent the Bäcklund transform for the space-time CFBE.

3. EXACT SOLUTIONS FOR THE SPACE-TIME CONFORMAL FRACTIONAL BURGERS' EQUATION

The hyperbolic tangent function method (tanh-method) is an efficient method used for constructing the exact traveling wave solutions for NLPDEs. The Ansätz is considered as a power series in tanh, where tanh is introduced as a new variable. Moreover, the derivatives of tanh are also given in terms of tanh itself [20]. In the following steps, we seek to describe the general tanh-method for constructing the solutions of the space-time conformal fractional partial differential equations (CFPDEs). Consider a general space-time conformal fractional partial differential equation (space-time CFPDE):

$$H(u, D_t^{\alpha}u, D_x^{\alpha}u, D_t^{\alpha\alpha}u, D_t^{\alpha}(D_x^{\alpha}u), D_x^{\alpha\alpha}u, \dots) = 0. \tag{21}$$

In the following steps, we summarize the tanh-method for solving Eq. (21):

Step 1: Consider the traveling wave solution of Eq. (21) as

$$u(x,t) = u(\xi), \qquad \xi = \frac{k}{\alpha}(x^{\alpha} - \omega t^{\alpha}), \tag{22}$$

where k and ω are the wave number and wave velocity, respectively. Substitution of Eq. (22) into Eq. (21) produces the following ordinary differential equation for $u(\xi)$

$$\widetilde{H}(u, u', u'', \dots) = 0, \qquad u' = \frac{du}{d\xi}, \dots\ etc. \tag{23}$$

Step 2: Assume that the solution of Eq. (23) can be expressed as a finite power series of $F(\xi)$

$$u(\xi) = a_0 + \sum_{j=1}^{s} a_j F^j(\xi), \qquad a_s \neq 0, \tag{24}$$

where $s \in \mathbb{N}$, which is determined by balancing the highest power of the linear term with the highest power of the nonlinear term in Eq. (23), and a_j are constants to be determined. The new exact solutions of the space-time CFPDE can be obtained via the solutions of the Riccati equation that are satisfied by the tanh function [34, 35].

Consider the required Riccati equation to be

$$F' = A + BF + CF^2, \qquad ' \equiv \frac{d}{d\xi}, \tag{25}$$

where A, B and C are constants.

Step 3: Substitution of Eq. (24) into Eq. (23), generates a system of algebraic equations for $a_0, a_1, \dots, a_s, \omega$, and k.

Step 4: Solution of the system obtained in step 3, produces the values of $a_0, a_1, \dots, a_s, \omega$ and k in terms of A, B and C. By substituting these results into Eq. (24), we obtain the general form of traveling wave solution of Eq. (21).

Choosing of each proper value for A, B and C in Eq. (25) corresponds to a solution $F(\xi)$ of Eq. (25) that is could be one of the hyperbolic functions or triangular function as follows.

Case 1: If $A = C = 1$, and $B = 0$, then Eq. (25) has the solutions, $tan\xi$.
Case 2: If $A = C = -1$, and $B = 0$, then Eq. ((25) has the solutions, $cot\xi$.
Case 3: If $A = 1$, $B = 0$, and $C = -1$, then Eq. (25) has the solutions, $tanh\xi, coth\xi$.
Case 4: If $A = \frac{1}{2}$, $B = 0$ and, $C = -\frac{1}{2}$, then Eq. (25) has the solutions, $tanh\xi \pm i\ sech\xi$, $coth\xi \pm csch\xi, \frac{tanh\xi}{1 \pm\ sech\xi}, \frac{coth\xi}{1 \pm i\ csch\xi}, i^2 = -1$.

Case 5: If $A = C = \frac{1}{2}$, and $B = 0$, then Eq. (25) has the solutions, $tan\xi \pm sec\xi$, $csc\xi - cot\xi$, $\frac{tan\xi}{1\pm sec\xi}$.
Case 6: If $A = C = -\frac{1}{2}$, and $B = 0$, then Eq. (25) has the solutions, $cot\xi \pm csc\xi$, $sec\xi - tan\xi$, $\frac{cot\xi}{1\pm csc\xi}$.
Case 7: If $A = 1, C = -4$, and $B = 0$, then Eq. (25) has the solutions, $\frac{tanh\xi}{1+tanh^2\xi}$.
Case 8: If $A = 1, C = 4$, and $B = 0$, then Eq. (25) has the solutions, $\frac{tan\xi}{1-tan^2\xi}$.
Case 9: If $A = -1, C = -4$, and $B = 0$, then Eq. (25) has the solutions, $\frac{cot\xi}{1-cot^2\xi}$.
Case 10: If $A = 1, B = -2$, and $C = 2$, then Eq. (25) has the solutions, $\frac{tan\xi}{1+tan\xi}$.
Case 11: If $A = 1$ and $B = C = 2$, then Eq. (25) has the solutions, $\frac{tan\xi}{1-tan\xi}$.
Case 12: If $A = -1, B = 2$, and $C = -2$, then Eq. (25) has the solutions, $\frac{cot\xi}{1+cot\xi}$.
Case 13: If $A = -1, B = C = -2$, then Eq. (25) has the solutions, $\frac{cot\xi}{1-cot\xi}$.
Case 14: If $A = B = 0$ and $C \neq 0$, then Eq. (25) has the solutions, $\frac{-1}{C\xi+C_0}$.
Case 15: If $A \neq 0$, $C = 0$, and $B \neq 0$, then Eq. (25) has the solutions, $\frac{1}{B}(\exp(B\xi) - A)$.
Case 16: If $A = 0$ and $B = C = 1$, then Eq. (25) has the solutions, $\frac{\exp(\xi)}{1-\exp(\xi)}$.
Case 17: If $A = 0$, and $B = C = \frac{1}{2}$, then Eq. (25) has the solutions, $\frac{\exp\left(\frac{\xi}{2}\right)}{1-\exp\left(\frac{\xi}{2}\right)}$.
Case 18: If $A = -\frac{1}{2}, B = 0$, and $C = \frac{1}{2}$, then Eq. (25) has the solutions, $-tanh\left(\frac{\xi}{2}\right)$, $-coth\left(\frac{\xi}{2}\right)$.
Case 19: If $A = -1$, and $B = C = 2$, then Eq. (25) has the solutions, $-\frac{1}{2} - \frac{\sqrt{3}tanh(\sqrt{3}\xi)}{2}$, $-\frac{1}{2} - \frac{\sqrt{3}coth(\sqrt{3}\xi)}{2}$.
Case 20: If $A = 1$, $B = 1$, and $C = 1$, then Eq. (25) has the solutions, $-\frac{1}{2} + \frac{\sqrt{3}tan\left(\frac{\sqrt{3}}{2}\xi\right)}{2}$, $-\frac{1}{2} - \frac{\sqrt{3}cot\left(\frac{\sqrt{3}}{2}\xi\right)}{2}$.
Case 21: If $A = -4, B = 0$, and $C = 4$, then Eq. (25) has the solutions, $-tanh(4\xi), -coth(4\xi)$.
Case 22: If $A = \frac{1}{2}, B = -1$, and $C = 1$, then Eq. (25) has the solutions, $\frac{1}{2} + \frac{1}{2}tan\left(\frac{\xi}{2}\right)$, $\frac{1}{2} - \frac{1}{2}cot\left(\frac{\xi}{2}\right)$.

Now, we need to implement the tanh-method technique into the space-time CFBE given by Eq. (1) to generate new exact solutions. Firstly, substituting the traveling wave solution given by Eq. (22) into Eq. (1) we obtain

$$-\omega u' + uu' - k\sigma u'' = 0, \tag{26}$$

where $D_t^{\alpha}u = -\omega ku'$, $D_x^{\alpha}u = ku'$, $D_x^{\alpha\alpha}u = k^2u''$, by balancing u'' with uu' gives $s = 1$. Use $s = 1$ in Eq. (24), then the solution of Eq. (1) can be expressed as

$$u = a_0 + a_1F, \tag{27}$$

substituting Eq. (27) into Eq. (26) and using Eq. (25), then we obtain a set of algebraic equations with respect to $F^i(i=0,1,2,3)$. Equating the coefficients of $F^i(i=0,1,2,3)$ to zero. The solution of the resulting system is given by

$$a_0 = \omega + \sigma Bk, \quad a_1 = 2\sigma Ck, \tag{28}$$

with ω and k are arbitrary constants. Inserting Eq. (28) into Eq. (27) and using the special solutions of Eq. (25), we obtain the following soliton like-solution and triangular periodic solutions of the space-time CFBE:

$$u_1 = \omega + 2\sigma k\ tan\xi, \tag{29}$$

$$u_2 = \omega - 2\sigma k\ cot\xi, \tag{30}$$

$$u_3 = \omega - 2\sigma k\ tanh\xi, \tag{31}$$

$$u_4 = \omega - 2\sigma k\ coth\xi, \tag{32}$$

$$u_5 = \omega - \sigma k(tanh\xi \pm i\ sech\xi), \tag{33}$$

$$u_6 = \omega - \sigma k(coth\xi \pm csch\xi), \tag{34}$$

$$u_7 = \omega + \sigma k(tan\xi \pm\ sec\xi), \tag{35}$$

$$u_8 = \omega - \sigma k(cot\xi \pm\ csc\xi), \tag{36}$$

$$u_9 = \omega - \frac{8\sigma k\ tanh\xi}{1 + tanh^2\xi}, \tag{37}$$

$$u_{10} = \omega + \frac{8\sigma k\ tan\xi}{1 - tan^2\xi}, \tag{38}$$

$$u_{11} = \omega - \frac{8\sigma k\ cot\xi}{1 - cot^2\xi}, \tag{39}$$

$$u_{12} = \omega - \frac{\sigma k\ tanh\xi}{1 + sech\xi}, \tag{40}$$

$$u_{13} = \omega - \frac{\sigma k\ coth\xi}{1 + i\ csch\xi}, \tag{41}$$

$$u_{14} = \omega + \frac{\sigma k\ tan\xi}{1 + sec\xi}, \tag{42}$$

$$u_{15} = \omega - \frac{\sigma k\ cot\xi}{1 + csc\xi}, \tag{43}$$

$$u_{16} = \omega - 2\sigma k + \frac{4\sigma k\ tan\xi}{1 + tan\xi}, \tag{44}$$

$$u_{17} = \omega + 2\sigma k - \frac{4\sigma k\ cot\xi}{1 + cot\xi}, \tag{45}$$

$$u_{18} = \frac{(\sigma k + \omega) + (\sigma k - \omega)e^{\xi}}{1 - e^{\xi}}, \tag{46}$$

$$u_{19} = \frac{(\sigma k/2 + \omega) + (\sigma k/2 - \omega)e^{\xi/2}}{1 - e^{\xi/2}}, \tag{47}$$

$$u_{20} = \omega - \sigma k\ tanh\frac{\xi}{2}, \tag{48}$$

$$u_{21} = \omega - \sigma k\ coth\frac{\xi}{2}, \tag{49}$$

$$u_{22} = \omega - 2\sqrt{3}\sigma k\ tanh\sqrt{3}\xi, \tag{50}$$

$$u_{23} = \omega - 2\sqrt{3}\sigma k\ cothh\sqrt{3}\xi, \tag{51}$$

$$u_{24} = \omega + \sqrt{3}\sigma k\ tan\left(\frac{\sqrt{3}}{2}\xi\right), \tag{52}$$

$$u_{25} = \omega - \sqrt{3}\sigma k\ cot\left(\frac{\sqrt{3}}{2}\xi\right), \tag{53}$$

$$u_{26} = \omega - 8\sigma k\ tanh(4\xi), \tag{54}$$

$$u_{27} = \omega - 8\sigma k\ coth(4\xi), \tag{55}$$

$$u_{28} = \omega + \sigma k\ tan\left(\frac{\xi}{2}\right), \tag{56}$$

$$u_{29} = \omega - \sigma k\ cot\left(\frac{\xi}{2}\right), \tag{57}$$

Remarks:

1- Making the transformation $k \to 2k$ then Eq. (48) and Eq. (49) transformed to Eq. (31) and Eq. (32) respectively, and if $k \to 2ki$ where $i = \sqrt{-1}$, they transformed into Eq. (29) and Eq. (30) respectively.

2- Making the transformation $k \to \frac{k}{\sqrt{3}}$ then Eq. (50) and Eq. (51) transformed to Eq. (31) and Eq. (32) respectively.

3- Making the transformation $k \to \frac{2k}{\sqrt{3}}$ then Eq. (52) and Eq. (53) transformed to Eq. (29) and Eq. (30) respectively.

4- Making the transformation $k \to 2k$ then Eq. (47) transformed to Eq. (46).

5- Making the transformation $k \to \frac{k}{4}$ then Eq. (54) and Eq. (55) transformed to Eq. (31) and Eq. (32) respectively.

6- Making the transformation $k \to 2k$ then Eq. (56) and Eq. (57) transformed to Eq. (29) and Eq. (30) respectively.

Also, we can get two new exact solutions

$$u_{30} = \omega - 2\sigma k\ (tanh\xi + coth\xi), \tag{58}$$

$$u_{31} = \omega + 2\sigma k\ (tan\xi - cot\xi), \tag{59}$$

with $\xi = \frac{k}{\alpha}(x^\alpha - \omega t^\alpha)$.

In the following section, we use the obtained solutions Eqs. (29) - (59) to generate new abundant exact solutions via BT.

4. BÄCKLUND TRANSFORMATIONS AND ABUNDANT EXACT SOLUTIONS

From section 2, we show that the space-time CFBE possesses the Painlevé property and it has a BT in the form

$$u = -\frac{2\sigma}{\phi} D_x^\alpha \phi + w,$$

where $\phi = \phi\left(\frac{x^\alpha}{\alpha}, \frac{t^\alpha}{\alpha}\right)$ is the singular manifold variable, w is a function of $\frac{x^\alpha}{\alpha}$ and $\frac{t^\alpha}{\alpha}$. Also, the function w solves the space-time CFBE given by Eq. (1) and the function ϕ satisfies the FDE $D_t^\alpha \phi + w D_x^\alpha \phi = \sigma D_x^{\alpha\alpha} \phi$.

Now, if we take $w = \phi$ then the function ϕ satisfies also the space-time CFBE

$$D_t^\alpha \phi + \phi D_x^\alpha \phi = \sigma D_x^{\alpha\alpha} \phi,$$

thus the BT for the space-time CFBE takes the following recurrence form

$$u_{n+1} = -\frac{2\sigma}{u_n} \frac{\partial^\alpha u_n}{\partial x^\alpha} + u_n. \tag{60}$$

We turn to the application of BT for the FDEs. Their power lies in that they may be used to generate additional solutions of the FDEs. Here u_{n+1}quantities refer to the new solution and u_n quantities refer to the old solution. This means that, based on a known solution to the space-time CFBE, we can find a new solution for space-time CFBE. To construct the new solution of the space-time CFBE one can start with the solution u_1 obtained in Eq. (29) and using BT given in Eq. (60) we get the following set of new solutions:

$$u_{32} = \frac{-4\sigma^2 k^2 + \omega^2 + 4\sigma k\omega\ tan\xi}{\omega + 2\sigma k\ tan\xi}.$$

Inserting u_{32} into the BT given in Eq. (60) we have

$$u_{33} = \frac{12\sigma^2 k^2 \omega - \omega^3 + 2\sigma k(4\sigma^2 k^2 - 3\omega^2)\ tan\xi}{4\sigma^2 k^2 - \omega^2 - 4\sigma k\omega\ tan\xi},$$

furthermore, using u_{33} and BT given in Eq. (60) we get

$$u_{34} = \frac{-16\sigma^4 k^4 + 24\sigma^2 k^2 \omega^2 - \omega^4 + (32\sigma^3 k^3 \omega - 8\omega^3 \sigma k)\ tan\xi}{12\sigma^2 k^2 \omega - \omega^3 + 2\sigma k(4\sigma^2 k^2 - 3\omega^2)\ tan\xi},$$

and so on, we can get sequences of exact solutions generated by the known tan-function solution Eq. (29) of the space-time CFBE. Starting from u_2 obtained in Eq. (30) and using BT given in Eq. (60) we get

$$u_{35} = \frac{-4\sigma^2k^2 + \omega^2 - 4\sigma k\omega\, cot\xi}{\omega - 2\sigma k\, cot\xi}.$$

Inserting u_{35} into the BT given in Eq. (60) we have

$$u_{36} = \frac{-12\sigma^2k^2\omega + \omega^3 + 2\sigma k(4\sigma^2k^2 - 3\omega^2)\, cot\xi}{-4\sigma^2k^2 + \omega^2 - 4\sigma k\omega\, cot\xi},$$

furthermore, using u_{36} and BT given in Eq. (60) we get

$$u_{37} = \frac{16\sigma^4k^4 - 24\sigma^2k^2\omega^2 + \omega^4 + (32\sigma^3k^3\omega - 8\omega^3\sigma k)\, cot\xi}{-12\sigma^2k^2\omega + \omega^3 + 2\sigma k(4\sigma^2k^2 - 3\omega^2)\, cot\xi},$$

and so on, we can get sequences of exact solutions generated by the cot-function solution Eq. (30) of the space time fractional CFBE. Starting from u_3 obtained in Eq. (31) and using BT given in Eq. (60) we get

$$u_{38} = \frac{-4\sigma^2k^2 - \omega^2 + 4\sigma k\omega \tanh\xi}{-\omega + 2\sigma k \tanh\xi}.$$

Inserting u_{38} into the BT given in Eq. (60) we have

$$u_{39} = \frac{12\sigma^2k^2\omega + \omega^3 - 2\sigma k(4\sigma^2k^2 + 3\omega^2) \tanh\xi}{4\sigma^2k^2 + \omega^2 - 4\sigma k\omega \tanh\xi},$$

furthermore, using u_{39} and BT given in Eq. (60) we get

$$u_{40} = \frac{16\sigma^4k^4 + 24\sigma^2k^2\omega^2 + \omega^4 - (32\sigma^3k^3\omega + 8\omega^3\sigma k) \tanh\xi}{12\sigma^2k^2\omega + \omega^3 - 2\sigma k(4\sigma^2k^2 + 3\omega^2) \tanh\xi},$$

and so on, we can get sequences of exact solutions generated by the tanh-function solution Eq. (31) of the space-time CFBE. Starting from u_4 obtained in Eq. (32) and using BT given in Eq. (60) we get

$$u_{41} = \frac{-4\sigma^2k^2 - \omega^2 + 4\sigma k\omega \coth\xi}{-\omega + 2\sigma k \coth\xi}.$$

Inserting u_{41} into the BT given in Eq. (60) we have

$$u_{42} = \frac{12\sigma^2k^2\omega + \omega^3 - 2\sigma k(4\sigma^2k^2 + 3\omega^2) \coth\xi}{4\sigma^2k^2 + \omega^2 - 4\sigma k\omega \coth\xi},$$

furthermore, using u_{42} and BT given in Eq. (60) we get

$$u_{43} = \frac{16\sigma^4k^4 + 24\sigma^2k^2\omega^2 + \omega^4 - (32\sigma^3k^3\omega + 8\omega^3\sigma k) \coth\xi}{12\sigma^2k^2\omega + \omega^3 - 2\sigma k(4\sigma^2k^2 + 3\omega^2) \coth\xi},$$

and so on, we can get sequences of exact solutions generated by the coth-function solution Eq. (32) of the space-time CFBE. Starting from u_5 obtained in Eq. (33) and using BT given in Eq. (60) we get

$$u_{44}=\frac{\sigma^2k^2+\omega^2-2\sigma k\omega\,(\tanh\xi\pm i\,\mathrm{sech}\,\xi)}{\omega-\sigma k\,(\tanh\xi\pm i\,\mathrm{sech}\,\xi)},$$

and so on. Starting from u_6 obtained in Eq. (34) and using BT given in Eq. (60) we get

$$u_{45}=\frac{\sigma^2k^2+\omega^2-2\sigma k\omega\,(\coth\xi\pm\mathrm{csch}\,\xi)}{\omega-\sigma k\,(\coth\xi\pm\mathrm{csch}\,\xi)},$$

and so on. Starting from u_7 obtained in Eq. (35) and using BT given in Eq. (60) we get

$$u_{46}=\frac{\sigma^2k^2-\omega^2-2\sigma k\omega\,(\tan\xi\pm\sec\xi)}{-\omega-\sigma k\,(\tan\xi\pm\sec\xi)},$$

and so on. Starting from u_8 obtained in Eq. (36) and using BT given in Eq. (60) we get

$$u_{47}=\frac{-\sigma^2k^2+\omega^2-2\sigma k\omega\,(\cot\xi\pm\csc\xi)}{\omega-\sigma k\,(\cot\xi\pm\csc\xi)},$$

and so on. Starting from u_9 obtained in Eq. (37) and using BT given in Eq. (60) we get

$$u_{48}=\frac{(16\sigma^2k^2+\omega^2)\tanh^2\xi-16\sigma k\omega\tanh\xi+16\sigma^2k^2+\omega^2}{\omega\tanh^2\xi-8\sigma k\tanh\xi+\omega}.$$

Inserting u_{48} into the BT given in Eq. (60) we have

$$u_{49}=\frac{(48\sigma^2k^2\omega+\omega^3)\tanh^2\xi-(24\sigma k\omega^2+128\sigma^3k^3)\tanh\xi+48\sigma^2k^2\omega+\omega^3}{(16\sigma^2k^2+\omega^2)\tanh^2\xi-16\sigma k\omega\tanh\xi+16\sigma^2k^2+\omega^2},$$

and so on. Starting from u_{10} obtained in Eq. (38) and using BT given in Eq. (60) we get

$$u_{50}=\frac{(16\sigma^2k^2-\omega^2)\tan^2\xi+16\sigma k\omega\tan\xi-16\sigma^2k^2+\omega^2}{\omega-\omega\tan^2\xi+8\sigma k\tan\xi}.$$

Inserting u_{50} into the BT given in Eq. (60) we have

$$u_{51}=\frac{(48\sigma^2k^2\omega-\omega^3)\tan^2\xi+(24\sigma k\omega^2-128\sigma^3k^3)\tan\xi-48\sigma^2k^2\omega+\omega^3}{(16\sigma^2k^2-\omega^2)\tan^2\xi+16\sigma k\omega\tan\xi-16\sigma^2k^2+\omega^2},$$

and so on. Starting from u_{11} obtained in Eq. (39) and using BT given in Eq. (60) we get

$$u_{52}=\frac{(16\sigma^2k^2-\omega^2)\cot^2\xi-16\sigma k\omega\cot\xi-16\sigma^2k^2+\omega^2}{\omega-\omega\cot^2\xi-8\sigma k\cot\xi}.$$

Inserting u_{52} into the BT given in Eq. (60) we have

$$u_{53}=\frac{(\omega^3-48\sigma^2k^2\omega)\cot^2\xi+(24\sigma k\omega^2-128\sigma^3k^3)\cot\xi+48\sigma^2k^2\omega-\omega^3}{(\omega^2-16\sigma^2k^2)\cot^2\xi+16\sigma k\omega\cot\xi+16\sigma^2k^2-\omega^2},$$

and so on. Starting from u_{12} obtained in Eq. (40) and using BT given in Eq. (60) we get

$$u_{54}=\frac{\sigma^2k^2+\omega^2+(\sigma^2k^2+\omega^2)\,sech\,\xi-2\sigma k\omega\tanh\xi}{\omega+\omega\,sech\,\xi-\sigma k\tanh\xi},$$

and so on. Starting from u_{13} obtained in Eq. (41) and using BT given in Eq. (60) we get

$$u_{55} = \frac{\sigma^2 k^2 + \omega^2 + (\sigma^2 k^2 + \omega^2)\, i\, csch\, \xi - 2\sigma k\omega \coth \xi}{\omega + \omega i\, csch\, \xi - \sigma k \coth \xi},$$

and so on. Starting from u_{14} obtained in Eq. (42) and using BT given in Eq. (60) we get

$$u_{56} = -\frac{(\sigma^2 k^2 - \omega^2)\, cos\, \xi + \sigma^2 k^2 - \omega^2 - 2\sigma k\omega \sin \xi}{\omega + \omega\, cos\, \xi + \sigma k \sin \xi},$$

and so on. Starting from u_{15} obtained in Eq. (43) and using BT given in Eq. (60) we get

$$u_{57} = \frac{(\sigma^2 k^2 - \omega^2)\, sin\, \xi + \sigma^2 k^2 - \omega^2 + 2\sigma k\omega \cos \xi}{-\omega - \omega\, sin\, \xi + \sigma k \cos \xi},$$

and so on. Starting from u_{16} obtained in Eq. (44) and using BT given in Eq. (60) we get

$$u_{58} = \frac{\omega^2 - 4\sigma^2 k^2 - 4\sigma k\omega + (\omega^2 - 4\sigma^2 k^2 + 4\sigma k\omega) \tan \xi}{(\omega + 2\sigma k) \tan \xi + \omega - 2\sigma k}.$$

Inserting u_{58} into the BT given in Eq. (60) we have

$$u_{59} = \frac{\omega^3 + 8\sigma^3 k^3 - 6\sigma k\omega^2 - 12\sigma^2 k^2 \omega + (\omega^3 - 8\sigma^3 k^3 + 6\sigma k\omega^2 - 12\sigma^2 k^2 \omega) \tan \xi}{\omega^2 - 4\sigma^2 k^2 - 4\sigma k\omega + (\omega^2 - 4\sigma^2 k^2 + 4\sigma k\omega) \tan \xi},$$

and so on. Starting from u_{17} obtained in Eq. (45) and using BT given in Eq. (60) we get

$$u_{60} = \frac{\omega^2 - 4\sigma^2 k^2 + 4\sigma k\omega + (\omega^2 - 4\sigma^2 k^2 - 4\sigma k\omega) \cot \xi}{(\omega - 2\sigma k) \cot \xi + \omega + 2\sigma k}.$$

Inserting u_{60} into the BT given in Eq. (60) we have

$$u_{61} = \frac{\omega^3 - 8\sigma^3 k^3 + 6\sigma k\omega^2 - 12\sigma^2 k^2 \omega + (\omega^3 + 8\sigma^3 k^3 - 6\sigma k\omega^2 - 12\sigma^2 k^2 \omega) \cot \xi}{\omega^2 - 4\sigma^2 k^2 + 4\sigma k\omega + (\omega^2 - 4\sigma^2 k^2 - 4\sigma k\omega) \cot \xi},$$

and so on. Starting from u_{18} obtained in Eq. (46) and using BT given in Eq. (60) we get

$$u_{62} = \frac{(\sigma k + \omega)^2 - (\sigma k - \omega)^2 e^{\xi}}{\sigma k + \omega + (\sigma k - \omega) e^{\xi}}.$$

Inserting u_{62} into the BT given in Eq. (60) we have

$$u_{63} = \frac{(\sigma k + \omega)^3 + (\sigma k - \omega)^3 e^{\xi}}{(\sigma k + \omega)^2 - (\sigma k - \omega)^2 e^{\xi}}.$$

Furthermore, using u_{63} and BT given in Eq. (60) we get

$$u_{64} = \frac{(\sigma k + \omega)^4 - (\sigma k - \omega)^4 e^{\xi}}{(\sigma k + \omega)^3 + (\sigma k - \omega)^3 e^{\xi}},$$

and so on, we can get a new sequence of the exact solution of the space-time CFBE.

Starting from u_{30} obtained in Eq. (58) and using BT given in Eq. (60) we get

$$u_{65} = \frac{-(\omega^2 + 16\sigma^2 k^2) \cosh \xi \sinh \xi + 8\sigma k\omega \cosh^2 \xi - 4\sigma k\omega}{-\omega \cosh \xi \sinh \xi + 4\sigma k \cosh^2 \xi - 2\sigma k},$$

and so on, we can get a sequence of solutions generated by the addition of two functions tanh and coth-function solution of the space-time CFBE. Starting from u_{31} obtained in Eq. (59) and using BT given in Eq. (60) we get

$$u_{66} = -\frac{(\omega^2 - 16\sigma^2 k^2)\cos\xi\sin\xi - 8\sigma k\omega\cos^2\xi + 4\sigma k\omega}{-\omega\cos\xi\sin\xi + 4\sigma k\cos^2\xi - 2\sigma k},$$

and so on, we can get a sequence of solutions generated by the addition of two functions tan and cot-function solution of the space-time CFBE.

By means of the variational iteration method (Inc, 2008) and the Adomian decomposition method [36] the solution of the space-time CFBE in closed form is

$$u_{67} = \frac{(\omega + k) + (\omega - k)\exp(\xi/\sigma)}{1 + \exp(\xi/\sigma)}.$$

From u_{67} we can get a new sequence of the exact solution for the space-time CFBE by using BT given in Eq. (60) in the form

$$u_{68} = \frac{(\omega + k)^2 + (\omega - k)^2\exp(\xi/\sigma)}{(\omega + k) + (\omega - k)\exp(\xi/\sigma)},$$

$$u_{69} = \frac{(\omega + k)^3 + (\omega - k)^3\exp(\xi/\sigma)}{(\omega + k)^2 + (\omega - k)^2\exp(\xi/\sigma)},$$

$$u_{70} = \frac{(\omega + k)^4 + (\omega - k)^4\exp(\xi/\sigma)}{(\omega + k)^3 + (\omega - k)^3\exp(\xi/\sigma)},$$

$$u_{71} = \frac{(\omega + k)^5 + (\omega - k)^5\exp(\xi/\sigma)}{(\omega + k)^4 + (\omega - k)^4\exp(\xi/\sigma)},$$

and so on, we can get a new sequence of the exact solution of the space-time CFBE.

5. CONCLUSIONS

In this work, we discuss the Painlevé property for non-linear conformal fractional differential equations for the first time. We apply the desired method to the space time conformal fractional Burgers' equation. Also, we derive the Bäcklund transform. The general solutions of the space-time conformal CFDEs are described based on the tanh-method, accordingly, the method is successively implemented to space-time CFBE. Moreover, the space-time CFBE is found to possess the Painlevé property and then Bäcklund transform. Also, we introduced a new recurrence formula based on the Bäcklund transform, that enables us to derive an analytical solution from a known solution or old solution to give the new solution. New numerous exact solutions are generated based on the Bäcklund transform.

Acknowledgments: *The authors gratefully acknowledge the approval and the support of this research study by grant no. SCI-2019-1-10-F-8306 from the Deanship of Scientific Research at Northern Border University, Arar, Saudi Arabia.*

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